Extremal Problems in Combinatorial Geometry (18w5058)

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1 Field overview and objectives

The workshop on Extremal problems in combinatorial geometry focused on the algebraic and combinatorial properties of points, lines, and other simple geometric objects in Euclidean space. The goal was to gather experts and promising young researchers in combinatorial geometry and related areas, to discuss the recent developments of algebraic and combinatorial methods used in the field. Over the past 6 years in particular, several ground-breaking results have been discovered, answering some of the oldest problems in the field. This workshop was a timely event to capitalize on this momentum.

Problems in combinatorial geometry are often simply described, and usually involve finite set of points, lines, convex sets, and other geometric objects in Euclidean space. Extremal problems in the field asks how large or small a finite set of geometric objects can be under certain restrictions. For example, given a set of n points in the plane, how often can the unit distance occur among them, or what is the size of the largest subset in convex position? Many questions in the field are very natural and worth studying for their own sake, while others are fueled by the more recent development of computational geometry. Over the past 6 years, in particular, combinatorial geometry has seen tremendous growth, and numerous unexpected connections to other fields of mathematics are being discovered. Two notable examples using algebraic geometry are the works of Guth and Katz [6], who solved the Erdos distinct distances problem, and of Green and Tao [5], who solved the long-standing conjecture of Dirac and Motzkin on the number of ordinary lines.

The workshop focused on several areas in combinatorial geometry, such as incidence geometry, intersection graphs, graph drawing, Erdos-Szekeres-type theorems, and combinatorial number theory. One of the major goals was to study the interplay between algebraic and combinatorial methods used in the field. Some of the key topics that the workshop covered are:

1. **The polynomial method and its applications in incidence geometry and combinatorial number theory**. Guth and Katz [6] invented, as a step in their nearly complete solution of Erdős's distinct distances problem, a new method for partitioning finite point sets in $\mathbb{R}^d$, based on the Stone-Tukey polynomial ham-sandwich theorem. This method has had numerous applications in incidence geometry. Even more applications were discussed during the workshop.

2. **The container method and its applications in discrete and computational geometry**. This useful method was recently introduced independently by Balogh, Morris and Samotij, and by Saxton and Thomason [1, 10]. Roughly speaking, it says that if a hypergraph H has a uniform edge distribution, then one can find a relatively small collection of sets, containers, covering all independent sets in H.

One can also require that the container sets span few edges only. More recently, Balogh and Solymosi [2] realized that this machinery is useful in discrete geometry. More precisely, the used the hypergraph container method to tackle several longing standing problems for points sets in the plane, including epsilon nets. Further possible applications were discussed during the workshop.

3. **The theory of semi-algebraic hypergraphs: regularity lemmas, VC-dimension arguments, and applications**. Famous Ramsey, Turán, and Szemerédi-type results prove the existence of certain patterns in graphs and hypergraphs under mild assumptions. Recently, several authors have shown that much stronger results hold for semi-algebraic hypergraphs, that is hypergraphs whose vertices are points in $\mathbb{R}^d$ and edges are defined by a semi-algebraic set in $\mathbb{R}^{dk}$ of bounded complexity. These results had several applications in discrete geometry, and more applications and results were discussed at the workshop.

With all of the recent developments and exciting breakthrough results in the field, it was important to bring together prominent experts in the field from all over the world. We believe it is very important to bring together and foster interactions between senior and junior researchers in combinatorial geometry. The workshop had a number of key lectures by international experts, which will survey the state-of-the-art of several long-standing open problems in the field. Other participants had the opportunity to give a 30 minute talk to present their work. The workshop schedule also provided ample time and opportunity for participants to interact and engage in mathematical discussion.

2 Presentation Highlights

The first plenary talk was given by Frank De Zeeuw, who talked about ordinary lines in space. The Sylvester-Gallai theorem states that if a finite set of points in the real plane is not contained in a line, then it spans at least one ordinary line, i.e. a line containing exactly two of the points. In fact, such a set always spans a linear number of ordinary lines, and that is best possible because there are point sets on cubic curves that determine only a linear number of ordinary lines. One can consider the same question in three-dimensional space. By projection, the results in the plane hold word-for-word in space. But note that the known constructions with a linear number of ordinary lines are contained in a plane. I will show that if one assumes that the point set does not have too many points on a plane, then it spans a quadratic number of ordinary lines. More precisely, for any $a < 1$ there is a $c > 0$ such that if we have n points in space with at most an points on a plane, then there are at least cn^2 ordinary lines. The proof uses projection, Becks theorem, and a variant of the Sylvester-Gallai theorem. More details can be found in his paper [12].

Joshua Zahl gave a talk on his new result which gives a new bound on the unit distances in three dimensions. In his talk, he talked about a more general problem about cutting curves into pseudo segments and its applications. More details can be found in his paper [11].

Misha Rudnev gave a review on some better or less known results about incidences of sufficiently small noncollinear point sets with straight lines. The two best known and in some sense sharp results are the theorem on distinct directions proved in the 90s by T. Szonyi and the recent Szemeredi-Trotter type incidence theorem by S. Stevens and F. de Zeeuw.

The second plenary talk was given by Orit Raz, who talked about a generalization of Elekes-Rónyai theorem to d variables. This was joint work with Zvi Shemtov.

Balázs Keszegh presented his new result, where he proved that the intersection hypergraph of a finite family of pseudo-disks with respect to another family of pseudo-disks admits a proper coloring with 4 colors. His result serves as a common generalization and strengthening of many earlier results, including ones about coloring points with respect to pseudo-disks, coloring pseudo-disks with respect to points and coloring disks with respect to disks. More details can be found in his paper [7].

The third plenary talk was given by Natan Rubin, who improved the upper bound on weak epsilon nets in the plane. This is the first improvement in over 25 years. More precisely, he showed that for any set P of n points in the plane and $\epsilon > 0$ there exists a set of $O(1/\epsilon^{1.7})$ points in the plane that pierce every convex set K with $|K \cap P| \geq \epsilon|P|$. This improves the previously known upper bound (from 1992) of $O(1/\epsilon^2)$ by Alon, Brny, Füredi, and Kleitman.

The fourth plenary talk was given by Nathan Linial, who talked about Hypertrees. In a seminal paper, Kalai (1983) extended the notion of a tree to higher dimensions. Formally, an n-vertex d-dimensional hypertee is a Q-acyclic simplicial complex with a full $(d-1)$ dimensional skeleton and $\binom{n-1}{d}$ d-dimensional faces. In his talk, Nati discussed several of the many open problems that arise here and describe some of the more recent new discoveries.

Radoslav Fulek talked about Z_2-genus drawings of graphs. A drawing of a graph on a surface is independently even if every pair of independent edges in the drawing crosses an even number of times. The Z_2-genus of a graph G is the minimum g such that G has an independently even drawing on the orientable surface of genus g. He showed that the Z_2-genus of graphs in these families is unbounded in t; in fact, equal to their genus. Together, this implies that the genus of a graph is bounded from above by a function of its Z_2-genus, solving a problem posed by Schaefer and Štefankovič, and giving an approximate version of the Hanani-Tutte theorem on surfaces. These results were joint work with Jan Kyncl [3, 4].

Bartosz Walczak talked about a new upper bound on the chromatic number of intersection graphs of L-figures in the plane, which is asymptotically tight. This was joint work with Alexandre Rok.

Lena Yuditsky talked about a structure theorem conjectured by Janson and Uzzell: The vertex set of *almost all* string graphs on n vertices can be partitioned into *five* cliques such that some pair of them is not connected by any edge. As a corollary, *almost all* string graphs on n vertices are intersection graphs of plane convex sets. This was joint work with Bruce Reed and János Pach [8].

3 Scientific Progress Made

The workshop provided ample free time for participants to work together on joint research projects. Moreover, there were a number of new research projects were initiated during the workshop, while some other researchers used the opportunity to continue to work on projects started earlier. All talks were well received.

References

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