Recent Advances in Discrete and Analytic Aspects of Convexity

Karoly Bezdek (University of Calgary),
Alexander Koldobsky (University of Missouri),
Dmitry Ryabogin (Kent State University),
Vladyslav Yaskin (University of Alberta),
Artem Zvavitch (Kent State University)

May 21 - May 26, 2017

1 Overview of the Field

Convexity is a very old topic which can be traced back at very least to Archimedes. It shares ideas and methods from many fields of Mathematics, including Differential Geometry, Discrete Geometry, Functional Analysis, Harmonic Analysis, Geometric Tomography, Combinatorics, Probability, and it has numerous applications. The aim of the meeting was to discuss most recent developments in the area and interrelate new discrete and analytic methods.

2 Presentation Highlights

The topics of the workshop included convex geometry, discrete geometry, theory of valuations, geometric inequalities, probability, and geometric functional analysis.

Grigoris Paouris presented his work "Affine isoperimetric inequalities on flag manifolds" joint with S. Dann and P. Pivovarov. Let K be a compact set in $\mathbb{R}^n$ (and convex, depending on the context) and k be an integer $1 \leq k \leq n-1$. The following are known as affine quermassintegrals and dual affine quermassintegrals correspondingly:

$$\Phi_k(K) = \left(\int_{G(n,k)} |K|F|^{-n} \right)^{-1/(nk)} dF,$$

$$\Psi_k(K) = \left(\int_{G(n,k)} |K \cap F|^{n} \right)^{1/(nk)} dF,$$

where $G(n,k)$ is the Grassmanian of the k-dimensional subspaces in $\mathbb{R}^n$.

The results of Furstenberg-Tzkoni and Grinberg show that Ψ_k is invariant under linear volume-preserving transformations, and Φ_k is invariant under affine volume-preserving transformations. In this talk affine and dual affine quermassintegrals were extended from the Grassmanian to the flag manifolds. The corresponding invariant properties were also established.

Busemann-Strauss and Grinberg established the following extremizing inequality for Ψ_k. Let K be of volume 1 and D_n be the Euclidean ball of volume 1, then $\Psi_k(K) \leq \Psi_k(D_n)$. Lutwak asked whether the following inequality is true: $\Phi_k(K) \geq \Phi_k(D_n)$. Paouris and Pivovarov earlier proved that the latter holds with an absolute multiplicative constant. In this talk the authors extended these results to the setting of flag manifolds. Some functional forms of these constructions were also discussed.

Peter Pivovarov spoke about his joint work "On a quantitative reversal of Alexandrov's inequality" with G. Paouris and P. Valettas. Alexandrov's inequality implies that for a convex body K we have the following:

$$\left(\frac{V_n(K)}{V_n(B)}\right)^{1/n} \leq \left(\frac{V_{n-1}(K)}{V_{n-1}(B)}\right)^{1/(n-1)} \leq \cdots \leq \frac{V_1(K)}{V_1(B)},$$

where B is the unit Euclidean ball and V_1,..., V_n are the intrinsic volumes.

Milman's random version of Dvoretzky's theorem shows that a large initial segment of this sequence is essentially constant, up to a critical parameter called the Dvoretzky number. The authors showed that this near-constant behavior actually extends further, up to a different parameter associated with K. Namely, set

$$\beta_* = \beta_*(K) = \frac{Var(h_K(g))}{(\mathbb{E}h_K(g))^2},$$

where h_K is the support function of K and g is a standard Gaussian random vector in $\mathbb{R}^n$. It is shown that there exists a constant $c > 0$ such that if K is a symmetric convex body in $\mathbb{R}^n$ and $1 \leq k \leq c/\beta_*(K)$, then

$$\frac{V_1(K)}{V_1(B)} \leq \left(1 + c\sqrt{k\beta_* \log\left(\frac{e}{k\beta_*}\right)}\right)\left(\frac{V_k(K)}{V_k(B)}\right)^{1/k}.$$

This yields a new quantitative reverse inequality that sits between the approximate reverse Urysohn inequality, due to Figiel–Tomczak-Jaegermann and Pisier, and the sharp reverse Urysohn inequality for zonoids, due to Hug–Schneider.

Liran Rotem gave a talk "Powers of convex bodies" based on a joint work with V. Milman. The main question discussed in the talk is the following: given a convex body K in $\mathbb{R}^n$ and a number $\alpha \in \mathbb{R}$, is there a natural definition for the power K^α. If $\alpha = -1$, then the natural definition of K^{-1} is K^o, the polar of K. This is due to the fact that polarity satisfies properties analogous to those of the function $x \mapsto 1/x$. The authors suggested a construction of K^α for $0 < \alpha < 1$ (again the motivation was that this operation should be similar to the function $x \mapsto x^\alpha$). This is done by first defining this operation for ellipsoids, in a natural way, and then passing to general bodies by using ellipsoidal envelopes.

In her talk "Recent results on approximation of convex bodies by polytopes", Elisabeth Werner discussed two results. The first one, joint with J. Grote, generalizes a theorem by Ludwig, Schuett and Werner on approximation of a convex body K in the symmetric difference metric by an arbitrarily placed polytope with a fixed number of vertices. Namely, let K be a convex body in $\mathbb{R}^n$, $n \geq 2$, that is C^2_+. Let $f : \partial K \to \mathbb{R}_+$ be a continuous strictly positive function with

$$\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1,$$

where $\mu_{\partial K}$ is the surface area measure on ∂K. Then for sufficiently large N there exists a polytope P_f in $\mathbb{R}^n$ having N vertices such that

$$\mathrm{vol}_n(K \Delta P_f) \leq a N^{-2/(n-1)} \int_{\partial K} \frac{\kappa_K(x)^{1/(n-1)}}{f(x)^{2/(n-1)}} d\mu_{\partial K}(x),$$

where $a > 0$ is an absolute constant, and κ_K is the Gaussian curvature.

The second recent result, joint with S. Hoehner and C. Schuett, gives a lower bound, in the surface deviation, on the approximation of the Euclidean ball by an arbitrary positioned polytope with a fixed number of k-dimensional faces.

Dan Florentin spoke about "New Prékopa Leindler Type Inequalities and Geometric Inf-Convolution of Functions". The talk is based on a joint work with S. Artstein and A. Segal. Consider the class $Cvx_0(\mathbb{R}^n)$ of non-negative convex functions on $\mathbb{R}^n$ vanishing at the origin (these are called geometric convex functions). The authors define a geometric inf-involution $\phi \boxdot_\lambda \psi$ by first providing a geometric interpretation and then showing the precise formula:

$$(\phi \boxdot_\lambda \psi)(z) = \inf_{0<t<1} \inf_{z=(1-t)x+ty} \max\left\{\frac{1-t}{1-\lambda}\phi(x), \frac{t}{\lambda}\psi(y)\right\}.$$

Then they prove that for $\lambda \in (0,1)$ and $\phi, \psi \in Cvx_0(\mathbb{R}^n)$ one has

$$\int e^{-\phi \boxdot_\lambda \psi} \geq \left(\frac{1-\lambda}{\int e^{-\phi}} + \frac{\lambda}{\int e^{-\psi}}\right)^{-1}.$$

They also discuss other forms of this inequality and explain why this is an analogue of the Prékopa-Leindler inequality.

Bo'az Klartag presented his work on "Convex geometry and waist inequalities". The spherical waist inequality states that any continuous function f from the unit sphere S^n to $\mathbb{R}^l$ has a large fiber, i.e., the $(n-l)$-dimensional volume of some fiber $f^{-1}(t)$ is at least as large as that of S^{n-l}.

Here the author proves the following result. Let K be a convex body in $\mathbb{R}^n$ and $1 \leq 1 \leq n$. Then for any continuous function $f : K \to \mathbb{R}^l$,

$$\sup_{t\in\mathbb{R}^l} \mathrm{vol}_{n-l}(f^{-1}(t)) \cdot \sup_{E\in AG(n,l)} \mathrm{vol}_l(K \cap E) \geq \mathrm{vol}_n(K).$$

Furthermore, if $K \subset \mathbb{R}^n$ is a convex body of volume 1, then there exists a volume-preserving linear map T_K such that $\tilde{K} = T_K(K)$ has the following property. Let $1 \leq l \leq n$ and $f : \tilde{K} \to \mathbb{R}^l$ be a continuous map. Then there exists $t \in \mathbb{R}^l$ with

$$\mathrm{vol}_{n-l}(f^{-1}(t)) \geq c^{n-l},$$

where $c > 0$ is a universal constant.

The theme of waist inequalities is continued in the talk of Arseniy Akopyan "Waists of balls in different spaces", based on a joint work with R. Karasev and A. Hubard. The speaker starts with the following curious question. Can one map a shape of width 1 into a shape inside the strip of width 0.99 in such a way that the distances do not decrease? The answer is "No", as follows from the following result of the authors. For any convex body $K \in \mathbb{R}^n$ and a continuous map $f : K \to \mathbb{R}^{n-1}$ there exists a fiber $f^{-1}(y)$ of 1-Hausdorff measure at least the width of K.

Further the speaker presents a number of waist inequalities for balls in spaces of constant curvature, tori, parallelepipeds, projective spaces and other metric spaces.

Christos Saroglou spoke about his joint work with S. Myroshnychenko and D. Ryabogin on "Star bodies with completely symmetric sections". The starting point of this research was the question: Let K be a convex body in R^n, $n \geq 3$. If all orthogonal projections of K are 1-symmetric, is it true that K must be a Euclidean ball? (A body is said to be 1-symmetric if its symmetry group contains the symmetry group of a cube of the same dimension).

The authors gave a positive answer to this problem, as well as other closely related questions. For, example the corresponding problem for sections also has a positive answer. In fact, a more general statement is true. If f is an even function on the sphere whose restriction to every equator is isotropic, then f is a constant.

Jaegil Kim presented a joint work with S. Dann and V. Yaskin titled "Busemann's intersection inequality in hyperbolic and spherical spaces". A version of Busemann's intersection inequality says that ellipsoids in $\mathbb{R}^n$ are the only maximizers of the quantity

$$\int_{S^{n-1}} |K \cap \xi^\perp|^n d\xi \qquad (1)$$

in the class of star bodies of a fixed volume.

The authors study this question in the hyperbolic space $\mathbb{H}^n$ and the sphere $\mathbb{S}^n$. It is shown that in $\mathbb{H}^n$ the centered balls are the unique maximizers of (1) in the class of star bodies of a fixed volume. However, on the sphere the situation is completely different. It is surprising that in $\mathbb{S}^n$ with $n \geq 3$ the centered balls are neither maximizers nor minimizers of (1), even in the class of origin-symmetric convex bodies.

Ning Zhang spoke about his work "On bodies with congruent sections by cones or non-central planes". Let K and L be convex bodies in $\mathbb{R}^3$ such that the projections $K|H$ and $L|H$ are congruent for every subspace H. Does this imply that K is a translate of $\pm L$. This question is open. Ryabogin solved a version of this problem when congruency is replaced by rotation. In the language of functions his result can be stated as follows. Let f and g be continuous functions on S^2 such that for every 2-dimensional subspace H there is a rotation ϕ_H in H such that $f(\theta) = g(\phi_H(\theta))$ for all $\theta \in S^2 \cap H$. Then $f(\theta) = g(\theta)$ or $f(\theta) = g(-\theta)$ for all $\theta \in S^2$.

The speaker considered a similar question for small circles on the sphere, instead of great circles. Namely, consider a fixed $t \in (0, 1)$ and let f and g be continuous functions on S^2 such that for every 2-dimensional affine subspace H that is distance t from the origin there is a rotation ϕ_H in H such that $f(\theta) = g(\phi_H(\theta))$ for all $\theta \in S^2 \cap H$. Is it true that $f(\theta) = g(\theta)$ for all $\theta \in S^2$. In this talk it is shown that the answer to this question is affirmative if f and g are of class $C^2(S^2)$.

Petros Valettas gave a talk about "A Gaussian small deviation inequality", a joint work with G. Paouris. The Gaussian concentration phenomenon states that for any L-Lipschitz map $f : mathbbR^n \to \mathbb{R}$ one has

$$\mathbb{P}\left(|f(Z) - M| > t\right) \leq \exp\left(-\frac{1}{2}t^2/L^2\right),$$

for all $t > 0$, where Z is an n-dimensional standard Gaussian random vector and M is the median for $f(Z)$.

The authors are interested in refining (a one-sided version of) this inequality by replacing L with the variance $\mathrm{Var} f(Z)$. It is known that $\mathrm{Var} f(Z) \leq L^2$. However, there are many Lipschitz maps for which $\mathrm{Var} f(Z) \ll L^2$.

Their main result reads as follows. For any convex map $f \in L_2(\gamma_n)$ one has

$$\mathbb{P}\left(f(Z) - M < -t\right) \leq \exp\frac{1}{2}\left(-\frac{\pi}{1024}t^2/\mathrm{Var} f(Z)\right),$$

for all $t > 0$.

Monika Ludwig presented her joint work with L. Silverstein "Valuations on lattice polytopes". The study of valuations on convex bodies is a classical area. There are well-known classifications of such valuations. In this talk the authors study valuations on the space $\mathcal{P}(\mathbb{Z}^n)$ of lattices polytopes (these are convex hulls of finitely many points from $\mathbb{Z}^n$). A natural question is to classify $SL_n(\mathbb{Z})$ and translation invariant valuations on $\mathcal{P}(\mathbb{Z}^n)$. Some work in this direction was done by Betke and Kneser in the case of real-valued valuations. Here the authors prove the following result about vector-valued valuations. $Z : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ is an $\mathrm{SL}_n(\mathbb{Z})$ equivariant, translation covariant valuation if and only if there exist c_1,..., $c_{n+1} \in \mathbb{R}$ such that

$$Z = c_1 l_1 + \cdots + c_{n+1} l_{n+1}.$$

Here, l_i's are defined by the formula

$$l(kP) = \sum_{i=1}^{n+1} l_i(P)k^i,$$

where P is a lattice polytope, k is a positive integer, and

$$l(P) = \sum_{x \in P \cap \mathbb{Z}^n} x$$

is the discrete moment vector of P.

The next natural step is to look at what happens for tensor-valued valuations. They find a classification of $\mathrm{SL}_n(\mathbb{Z})$ equivariant, translation covariant valuations with values in the space of symmetric tensors of rank at most 8. The case of tensors of rank 9 and higher appears more complicated.

Franz Schuster talked about "Even $SO(n)$ Equivariant Minkowski Valuations – An Update". The main theme is finding Hadwiger-type theorems for Minkowski valuations. Recall that a map Φ from the set $\mathcal{K}^n$ of convex bodies in $\mathbb{R}^n$ to itself is called a Minkowski valuation if

$$\Phi(K \cap L) + \Phi(K \cup L) = \Phi(K) + \Phi(L),$$

whenever K, L, and $K \cup L$ are in $\mathcal{K}^n$.

The first problem is to describe the set $\mathrm{MVal}^{SO(n)}$ of continuous Minkowski valuations, which are translation invariant and $SO(n)$ equivariant.

The second problem is to find a classification/description of $\mathrm{MVal}_i^{SO(n)}$. Here, $\mathrm{MVal}_i^{SO(n)} = \{\Phi \in \mathrm{MVal}^{SO(n)} : \Phi(\lambda K) = \lambda^i \Phi(K)\}$.

The speaker discussed recent progress and insights on these problems. In particular, Wannerer and Schuster obtained the following result. If $\Phi_i \in \mathrm{MVal}_i^{SO(n),\infty}$ (∞ refers to smooth valuations), $1 \le i \le n-1$, then there exists a unique function $g_{\Phi_i} \in C^\infty([-1,1])$ such that

$$h(\Phi_i K, u) = \int_{S^{n-1}} g_{\Phi_i}(\langle u, v\rangle) dS_i(K, v), \qquad u \in S^{n-1},$$

where dS_i is the i-th surface area measure.

Of course, the natural goal is to remove smoothness in the above theorem. Recently, Dorrek proved the following. If $\Phi_i \in \mathrm{MVal}_i^{SO(n)}$, then there exists a unique function $g_{\Phi_i} \in L^1([-1,1])$ such that

$$h(\Phi_i K, u) = \int_{S^{n-1}} g_{\Phi_i}(\langle u, v\rangle) dS_i(K, v), \qquad u \in S^{n-1}.$$

The description of such functions g_{Φ_i} is still an open problem.

Wolfgang Weil spoke about "Integral representations of mixed volumes", a joint work with D. Hug and J. Rataj. Recall that mixed volumes arise as coefficients in the following expansion:

$$\mathrm{vol}(t_1K_1 + \cdots + t_mK_m) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} t_{i_1} \cdots t_{i_n} V(K_{i_1}, ..., K_{i_n}),$$

where K_1,...,K_m are convex bodies and t_1,...,t_m are positive numbers.

It would be important to represent mixed volumes by integrals of local quantities of K_1,...,K_m. In the case when there are only two bodies such a formula was obtained earlier by the same authors. There is a function $f_{j,n-j}$ such that for all K, M (in suitable general position)

$$V(K[j], M[n-j]) = \int_{F(n,n-j+1)} \int_{F(n,j+1)} f_{j,n-j}(u_1, L_1, u_2, L_2) \psi_j(K, d(u_1, L_1)) \psi_{n-j}(M, d(u_2, L_2)),$$

where $\psi_j(K,\cdot)$ and $\psi_{n-j}(M,\cdot)$ are flag measures of K and M and the integration is over the manifolds of flags $F(n, n-j+1)$ and $F(n, j+1)$ respectively, which are defined as follows:

$$F(n, j+1) = \{(u, L) : L \in G(n, j+1), u \in S^{n-1} \cap L\},$$

$$\psi_j(K, \cdot) = \int_{G(n,j+1)} 1((u, L) \in \cdot) S'_j(K|L, du) dL.$$

Recently the authors extended this result to the case of more than two bodies. The formula is similar to the one above.

Martin Henk gave a talk about "The even dual Minkowski problem", based on joint works with K. Böröczky and H. Pollehn. The Minkowski-Christoffel problem asks for characterizations of area measures $S_i(K,\cdot)$ of a convex body K among all finite Borel measures on the sphere. When $1 < i < n-1$ the problem is still open.

Lutwak initiated the dual Brunn-Minkowski theorem. Instead of the Minkowski addition as in the classical setting, here one uses the radial addition. In this theory there is a local dual Steiner formula, which allows to define ith dual curvature measures $\tilde{C}_i(K,\cdot)$. Moreover, there are explicit formulas for these measures that allow an extension from an integer i to a real number q. The dual Minkowski problem asks for necessary and sufficient conditions for a given finite Borel measure on the sphere to be the qth dual curvature measure of some convex body K. The authors found the following necessary condition. Let K be an origin-symmetric convex body, $q \in (1, n)$, and L be a proper subspace. Then

$$\tilde{C}_i(K, S^{n-1} \cap L) < \min\left\{1, \frac{\mathrm{dim}L}{q}\right\} \tilde{C}_i(K, S^{n-1}).$$

Later Zhao, Böröczky, Lutwak, Yang, Zhang showed that this condition is also sufficient.

What happens for other values of q? Zhao found a necessary and sufficient condition when $q < 0$. Henk and Pollehn gave a necessary condition for $q \geq n+1$. Let K be an origin-symmetric convex body in $\mathbb{R}^n$. Then for every proper subspace $L \subset \mathbb{R}^n$,

$$\tilde{C}_i(K, S^{n-1} \cap L) < \frac{q - n + \dim L}{q} \tilde{C}_i(K, S^{n-1}),$$

and the bound is best possible.

An open question is whether this condition is also sufficient.

Hermann König presented his work "Submultiplicative operators in C^k-spaces", joint with D. Faifman and V. Milman. Multiplicative operators have been studied by many authors and the corresponding characterization were obtained. In this talk the goal is to look at the stability properties. Can we go from multiplicative operators to submultiplicative?

The first main result is the following. Let $I \subset \mathbb{R}^n$ be an open set and $k \in \mathbb{N}_0$. Consider a map $T : C^k(I) \to C^k(I)$ that is bijective and submultiplicative, meaning that for all $f, g \in C^k(I)$ one has

$$T(f \cdot g) \leq T(f) \cdot T(g).$$

T is also assumed to be pointwise continuous and satisfying the property $f \geq 0$ if and only if $Tf \geq 0$. Then there exist functions p, $A \in C(I)$, satisfying $p > 0$, $A \geq 1$, and a C^k-diffeomorphism u so that

$$\begin{aligned} &\text{for } k = 0, \quad Tf(u(x)) = \begin{cases} f(x)^{p(x)}, & f(x) \geq 0, \\ -A(x)|f(x)|^{p(x)}, & f(x) < 0, \end{cases} \\ &\text{for } k > 0, \quad Tf(u(x)) = f(x). \end{aligned}$$

The second main result, which is joint with V. Milman, reads as follows. Let $\phi : \mathbb{R} \to \mathbb{R}$ be submultiplicative, i.e.

$$\phi(xy) \leq \phi(x)\phi(y), \qquad x, y \in \mathbb{R}.$$

ϕ is also assumed to be measurable, continuous at 0, 1, and satisfying $\phi(-1) < 0 < \phi(1)$. Then there exist numbers $p > 0$, $A \geq 1$ so that

$$\phi(x) = \begin{cases} x^p, & x \geq 0, \\ -A|x|^p, & x < 0. \end{cases}$$

Konstantin Tikhomirov spoke about "Superconcentration, and randomized Dvoretzky's theorem for spaces with 1-unconditional bases". For an origin-symmetric convex body B in $\mathbb{R}^n$ and a linear operator $U : \mathbb{R}^n \to \mathbb{R}^n$ define

$$\ell(B, U) = \left(\mathbb{E}\|U(G)\|_B^2\right)^{1/2},$$

where G is the standard Gaussian vector in $\mathbb{R}^n$. The body B is said to be in ℓ-position if $\ell(B, \mathrm{Id}_n) = 1$ and

$$1 = \det \mathrm{Id}_n = \sup\left\{|\det U| : U \in \mathbb{R}^{n \times n}, \ell(B, U) \leq 1\right\}.$$

The main result is the following. Let B be an origin-symmetric convex body in $\mathbb{R}^n$ in the ℓ-position, and such that the space $(\mathbb{R}^n, \|\cdot\|_B)$ has a 1-unconditional basis. Further, let $\epsilon \in (0, 1/2]$ and $k \leq c\epsilon \log n / \log \frac{1}{\epsilon}$. Then for a random k-dimensional subspace $E \subset \mathbb{R}^n$ uniformly distributed according to the Haar measure, one has

$$\mathbb{P}\{B \cap E \text{ is } (1+\epsilon)\text{-spherical}\} \geq 1 - 2n^{-c\epsilon},$$

where $c > 0$ is a universal constant.

This shows that the "worst-case" dependence on epsilon in the randomized Dvoretzky theorem in the ell-position is significantly better than in John's position.

Alexander Litvak presented his joint work with K. Tikhomirov titled "Order statistics of vectors with dependent coordinates". In 2000 Mallat and Zeitouni posed the following question. Let $X = (X_1, ..., X_n)$ be an n-dimensional random Gaussian vector with independent centered coordinates (possibly with different variances). Let T be an orthogonal transformation of $\mathbb{R}^n$ and $Y = T(X)$. Is it true that for every $k \le n$ one has

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\le n} X_i^2 \le \mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\le n} Y_i^2?$$

Here, for a sequence of numbers a_1,..., a_n and $j \le n$, $j\text{-}\min_{1\le i\le n} a_i$ denotes the j-th smallest element of the sequence.

In their work the authors solve this problem in the affirmative (up to an absolute multiplicative constant). That is, there is an absolute constant $C > 0$ such that

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\le n} X_i^2 \le C\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{i\le n} Y_i^2.$$

Matthew Stephen presented his joint work with N. Zhang "Grünbaum's inequality for projections". Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $\xi \in S^{n-1}$ and denote $\xi^+ = \{x \in \mathbb{R}^n : \langle x, \xi\rangle \ge 0\}$. Grünbaum's inequality says that

$$\frac{\mathrm{vol}_n(K \cap \xi^+)}{\mathrm{vol}_n(K)} \ge \left(\frac{n}{n+1}\right)^n,$$

with equality at the cone.

Another closely related inequality is due to Minkowski and Radon. If K is a convex body with centroid at the origin and h_K is its support function, then

$$\frac{h_K(\xi)}{h_K(\xi) + h_K(-\xi)} \ge \frac{1}{n+1},$$

with equality at the cone.

Stephen and Zhang realized that there should be a connection between these two inequalities. Moreover, they found a general inequality that has these two as particular cases. They proved the following. Fix $1 \le k \le n$ and take $E \in G(n, k)$, $\xi \in S^{n-1} \cap E$. If K is a convex body with centroid at the origin, then

$$\frac{\mathrm{vol}_k((K|E) \cap \xi^+)}{\mathrm{vol}_k(K|E)} \ge \left(\frac{k}{n+1}\right)^k.$$

The equality condition is also characterized.

Susanna Dann talked about "Flag area measures", a joint work with J. Abardia and A. Bernig. Let V be an Euclidean vector space of dimension n and let $\mathcal{K}(V)$ be the space of non-empty compact convex subsets in V. A valuation on V is a map $\mu : \mathcal{K}(V) \to \mathbb{R}$ satisfying

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$. We say that μ is continuous if it is so with respect to the topology on $\mathcal{K}(V)$ induced by the Hausdorff metric. A flag area measure on V is a continuous translation-invariant valuation with values in the space of signed measures on the flag manifold consisting of a unit vector v and a $(p+1)$-dimensional linear subspace containing v where $0 \le p \le n-1$.

Using local parallel sets Hinderer constructed examples of $\mathrm{SO}(n)$-covariant flag area measures. There is an explicit formula for his flag area measures evaluated on polytopes involving the squared cosine of the angle between two subspaces. The authors construct a more general space of $\mathrm{SO}(n)$-covariant flag area measures via integration of appropriate differential forms. They also compute the dimension of this space, discuss their properties and provide explicit formulas on polytopes, which are similar to the formulas for Hinderer's examples, however with an arbitrary elementary symmetric polynomial in the squared cosines of the principal angles between two subspaces. Hinderer's flag area measures correspond to special cases where the elementary symmetric polynomial is just the product. Moreover, they construct an explicit basis for this space, which gives a classification result in the spirit of Hadwiger's theorem.

Boaz Slomka gave a talk “On convex bodies generated by Borel measures”, a joint work with H. Huang. Their goal is to introduce a natural way of generating convex bodies from Borel measures. They suggest the following construction. Given a Borel measure μ on $\mathbb{R}^n$, define

$$M(\mu) = \bigcup_{\substack{0\le f\le 1\\ \int_{\mathbb{R}^n} f d\mu = 1}} \left\{ \int_{\mathbb{R}^n} y f(y) d\mu(y) \right\},$$

where the union is taken over all measurable function $f : \mathbb{R}^n \to [0,1]$ with $\int_{\mathbb{R}^n} f d\mu = 1$. The set $M(\mu)$ is called the metronoid generated by μ. In particular, if μ is a finite combination of δ-measures, then $M(\mu)$ is a polytope. The latter suggests that some classical quantities related to approximation of convex bodies by polytopes can be extended to the class of metronoids.

For example, one such quantity is

$$d_R(K) = \inf \left\{ N \in \mathbb{N} : \exists P = \operatorname{conv}(x_1, \dots, x_N) \subset \mathbb{R}^n, \frac{1}{R} P \subset K \subset P \right\},$$

which naturally extends in the case of metronoids as follows:

$$d_R^*(K) = \inf \left\{ \mu(\mathbb{R}^n) : \frac{1}{R} M(\mu) \subset K \subset M(\mu) \right\}.$$

Similarly, the vertex index introduced by Bezdek and Litvak:

$$\operatorname{vein}(K) = \inf \left\{ \sum_{i=1}^N \|x_i\|_K : K \subset P = \operatorname{conv}(x_1, \dots, x_N) \right\}$$

in the setting of metronoids can be modified as follows:

$$\operatorname{vein}^*(K) = \inf \left\{ \int_{\mathbb{R}^n} \|x\|_K d\mu(x) : K \subset M(\mu) \right\}.$$

Next, the authors establish some bounds for the quantities they introduced. For example, they show that for an origin-symmetric convex body K in $\mathbb{R}^n$ one has

$$c\sqrt{n} \le c \operatorname{vein}^*(B_2^n) \le \operatorname{vein}^*(K) \le C_1 \operatorname{vein}^*(B_1^n) \le C_2 n.$$

Matt Alexander presented his work done with M. Fradelizi and A. Zvavitch “Polytopes of Maximal Volume Product”. Let K be an origin-symmetric convex body in $\mathbb{R}^n$, and let

$$K^o := \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \ \forall y \in K\}$$

be its polar body. Denote by

$$P(K) = \operatorname{vol}_n(K)\operatorname{vol}_n(K^o)$$

the volume product of the body K.

In the non-symmetric case the volume product is defined as follows. The polar body of a convex body K in $\mathbb{R}^n$ with respect to a point z is

$$K^z := \{x \in \mathbb{R}^n : \langle x - z, y - z\rangle \le 1 \ \forall y \in K\}.$$

For a convex body K the Santaló point is the unique point $s(K)$ such that

$$\operatorname{vol}(K^{s(K)}) = \min_{z \in \operatorname{int} K} \operatorname{vol}(K^z).$$

Define the volume product of K to be

$$P(K) = \operatorname{vol}_n(K)\operatorname{vol}_n(K^{s(K)}).$$

The maximum of the volume product in the class of convex bodies is known: the maximizers are centered ellipsoids. However, it is interesting to find the maximum in some specific classes of convex bodies.

The authors examine the volume product for classes of restricted polytopes. Let $\mathcal{P}_m^n$ be the set of all polytopes in $\mathbb{R}^n$ with non-empty interior having at most m vertices. Denote

$$M_n^m = \sup_{K \in \mathcal{P}_m^n} P(K).$$

They prove that this supremum is achieved at some polytope with exactly m vertices and the sequence M_m^n is strictly increasing in m.

A polytope is called simplicial if every facet is a simplex. The authors show that if K is of maximal volume product among polytopes with at most m vertices, then K is a simplicial polytope.

They also investigate some particular classes $\mathcal{P}_m^n$. In particular, if K is an origin-symmetric convex body in $\mathcal{P}_8^3$, then the maximal volume product of such bodies is given by the double cone on a regular hexagonal base. They also find the maximizers of the volume product for convex hulls of $n+2$ points in $\mathbb{R}^n$.

Gideon Schechtman presented his joint work with A.Naor on lower bounds for the distortion of bi-Lipschitz embeddings of metric spaces. A metric space (X, d_X) is said to admit a bi-Lipschitz embedding into a metric space (Y, d_Y) if there exist $s \in (0, \infty), D \in [1, \infty)$ and a mapping $f : X \to Y$ such that

$$sd_X(x, y) \le d_Y(f(x), f(y)) \le Dsd_X(x, y), \qquad \forall x, y \in X.$$

When this happens it is said that (X, d_X) embeds into (Y, d_Y) with distortion at most D. The authors have previously done some work on bounding from below the distortion of embedding certain metric spaces into L_p.

In this talk the speaker considered embedding certain grids in Schatten p-classes S_p into L_p. In particular, let $M_n[m]$ be the grid of all $n \times n$ matrices, whose entries have values in the set $[m] = \{-m, -(m-1), ..., m-1, m\}$, equipped with the S_1 norm. The authors prove that for any n and m large enough with respect to n, the distortion of embedding $M_n[m]$ into a Banach space X is at least of order $n^{1/2}/\alpha(X)$. Here $\alpha(X)$ is the smallest constant K satisfying the so-called linear upper α inequality

$$\mathbb{E}_{\epsilon_{ij}=\pm1} \| \sum_{i,j=1}^{n} \epsilon_{ij} x_{ij} \| \le K \mathbb{E}_{\epsilon_i, \delta_j = \pm 1} \| \sum_{i,j=1}^{n} \epsilon_i \delta_j x_{ij} \|$$

for all $x_{ij} \in X$. Much of the talk was devoted to α inequalities in different settings.

Carsten Schuett spoke about his joint work with O.Giladi, J.Prochno, N.Tomczak-Jaegermann and E.Werner on the geometry of triple tensor products of ℓ_p^n-spaces. Let X be an n-dimensional normed space with the unit ball B_X . The volume ratio of X is defined by

$$vr(X) = \inf_{E \subset B_X} \left(\frac{vol_n(B_X)}{vol_n(E)} \right)^{1/n},$$

where the infimum is taken over all ellipsoids contained in B_X, and vol_n is volume in $\mathbb{R}^n$. In their earlier work, Schuett and Tomczak-Jaegermann established the exact behavior of the volume ratio of tensor products of the spaces ℓ_p^n and ℓ_q^n. In the talk, Schuett explained the extension of this result to triple tensor products of the spaces ℓ_p^n, ℓ_q^n and ℓ_r^n for all choices of $p, q, r \in [1, \infty]$. The authors established the exact behavior of the volume ratio of these triple tensor products.

Károly Bezdek has given a talk under the title "From dual bodies to the Kneser-Poulsen conjecture". Let $\mathbb{M}^d$ denote the d-dimensional Euclidean, hyperbolic, or spherical space. The r-dual set of a given set in $\mathbb{M}^d$ is the intersection of balls of radii r centered at the points of the given set. As a Blaschke–Santaló-type inequality for r-duality it was shown in the talk that for any set of given volume in $\mathbb{M}^d$ the volume of the r-dual set becomes maximal if the set is a ball. As an application also the following was proved. The Kneser–Poulsen Conjecture states that if the centers of a family of N congruent balls in Euclidean d-space is contracted, then the volume of the intersection does not decrease. A uniform contraction is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. Finally, the talk presented an outline of the proof of the Kneser–Poulsen conjecture for uniform contractions (with N sufficiently large) in $\mathbb{M}^d$.

Márton Naszódi spoke about his joint work with Károly Bezdek under the title “The Kneser–Poulsen conjecture for special contractions”. Recall that the Kneser–Poulsen Conjecture in general states that if the centers of a family of N unit balls in $\mathbb{E}^d$ is contracted, then the volume of the union (resp., intersection) does not increase (resp., decrease). They consider two types of special contractions. First, a *uniform contraction* is a contraction where all the pairwise distances in the first set of centers are larger than all the pairwise distances in the second set of centers. The authors obtain that a uniform contraction of the centers does not decrease the volume of the intersection of the balls, provided that $N \geq (1+\sqrt{2})^d$. Their result extends to intrinsic volumes. They prove a similar result concerning the volume of the union. Second, a *strong contraction* is a contraction in each coordinate. They show that the conjecture holds for strong contractions. In fact, the result extends to arbitrary unconditional bodies in the place of balls.

Igors Gorbovickis spoke about “The central set and its application to the Kneser-Poulsen conjecture”. He has given new results about central sets of subsets of a Riemannian manifold and applied those results to prove new special cases of the Kneser-Poulsen conjecture in the two-dimensional sphere and the hyperbolic plane.

Robert Connelly talked about ”The isostatic conjecture”, which is a joint work of him with Evan Solomonides and Maria Yampolskaya. They show that a jammed packing of disks with generic radii, in a generic container, is such that the minimal number of contacts occurs and there is only one dimension of equilibrium stresses. They also point out some connections to packings with different radii and results in the theory of circle packings whose graph forms a triangulation of a given topological surface.

Oleg Musin has given a talk under the title “Representing graphs by sphere packings”. His talk surveyed recent major advances on the following three topics: Euclidean and spherical graph representations as two–distance sets; Euclidean and spherical graph representations as contact graphs of congruent sphere packings; generalizations of Steiner’s porism and Soddy’s hexlet.

János Pach talked about ”Disjointness Graphs”, which is a joint work of him with Gábor Tardos and Géza Tóth. The disjointness graph $G = G(S)$ of a set of segments S in $\mathbb{R}^d$, $d \geq 2$, is a graph whose vertex set is S and two vertices are connected by an edge if and only if the corresponding segments are disjoint. They prove that the chromatic number of G satisfies $\kappa(G) \leq (\omega(G))^4 + (\omega(G))^3$, where $\omega(G)$ denotes the clique number of G. It follows , that S has $\Omega(n^{\frac{1}{5}})$ pairwise intersecting or pairwise disjoint elements. Stronger bounds are established for lines in space, instead of segments. They show that computing $\omega(G)$ and $\kappa(G)$ for disjointness graphs of lines in space are NP–hard tasks. However, they can design efficient algorithms to compute proper colorings of G in which the number of colors satisfies the above upper bounds. One cannot expect similar results for sets of continuous arcs, instead of segments, even in the plane. They construct families of arcs whose disjointness graphs are triangle-free ($\omega(G) = 2$), but whose chromatic numbers are arbitrarily large.

3 Outcome of the Meeting

The meeting was very successful, we were lucky to bring together mathematicians from many countries and many research areas, such as convex geometry, discrete geometry, probability, functional analysis. Besides the leading scientists, we also had 6 graduate students and 12 postdocs or recent PhDs participating in the workshop. Female participation was above 21%. The friendly atmosphere created during the workshop helped many participants not only to identify the promising ways to attack the old problems but also to get acquainted with many open new ones.