Beta Ensembles: Universality, Integrability, and Asymptotics (16w5076)

Peter Forrester (University of Melbourne),
Alice Guionnet (Massachusetts Institute of Technology),
Brian Rider (Temple University),
Benedek Valkó (University of Wisconsin - Madison)

April 11–April 15, 2016

1 Overview

Random matrix theory is a vibrant area of probability theory, with applications across mathematics, physics and engineering. The physically motivated β-ensembles (which can initially be viewed as models of a Coulomb gas) provide one-parameter families of particle systems that interpolate between the eigenvalue distributions of several of the classical models of random matrix theory (realized at $\beta = 1, 2, 4$). In recent years, the introduction of a range of new tools led to a period of intense research activity on the general beta ensembles, and our understanding of their properties continues at a fast pace. The wide range of new results naturally raise many open problems.

This meeting brought together 34 researchers from various groups working on different aspects of β-ensembles and related phenomena. Among the invited participants there were 7 postdocs and 3 Ph.D. students.

2 Scientific overview

The Gaussian invariant ensembles are one of the most studied models in random matrix theory. They were introduced in the 1950s by Wigner with the goal of modeling the energy levels of heavy atomic nuclei. The main idea was that the energy levels correspond to eigenvalues of a very complicated self-adjoint (or symmetric) operator, which we can be modeled by considering a large hermitian (or symmetric) matrix with iid complex or real standard normals (observing the appropriate symmetry). These models are called the Gaussian unitary (respectively orthogonal) ensembles (GUE and GOE), corresponding to the underlying invariance under conjugations by these symmetries. Dyson found that the joint eigenvalue density of the Gaussian ensembles is given by

$$\frac{1}{Z_{n,\beta}}\, e^{-\beta \sum_{k=1}^{n} \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}, \tag{1}$$

where β is 2 in the complex (Hermitian), and 1 in the real (symmetric) case. There is also a classical model for $\beta = 4$ with self-dual quaternion entries. Although the formula (1) is obtained as a joint eigenvalue density function of a random matrix, the density itself makes sense for any $\beta > 0$ with the appropriate normalizing constant $Z_{n,\beta}$. This family of distributions is called the Gaussian (or Hermite) β-*ensemble*. Note the structure of the density function: it contains a Vandermonde determinant raised to the power β and another component which can be thought of as a reference measure of iid normal distributions. It turns out that several other classical matrix models have joint eigenvalue densities with the same structure (Wishart, MANOVA, Haar unitary) for the special values $\beta = 1, 2, 4$. These models also have β-generalizations and so give rise to other beta-ensembles (e.g., the Laguerre, Jacobi, circular ensembles).

Dyson observed that these density functions can be thought of as a Boltzmann factor of a one-dimensional log-gas with logarithmic interaction and a certain background potential. The interaction corresponds to the Vandermonde term, while the background potential comes from the reference measure. Then the parameter β becomes the inverse temperature in the system. This observation lead to the study of β-ensembles using the methods of statistical mechanics. Several results about the classical $\beta = 1, 2$ and 4 cases were first discovered via this method by treating the general β case. A comprehensive overview of the field is covered in the recent monograph [18].

Starting with the work of Dyson, Gaudin and Mehta in the 60s and 70s and then with the ground breaking results of Tracy and Widom in the 90s building on the work of the Kyoto group in the 80s, the classical β cases turned out to be exactly solvable. The asymptotic local behavior of the eigenvalues of the Gaussian (and other classical) invariant ensembles has been fully described. (See [1], [28], [18] for an overview of these results.) Moreover, many of the limiting objects were soon after understood to capture the scaling limits of other integrable systems from combinatorics and statistical physics, most famously of various models of random growth composing the KPZ universality class [31]. In the meantime, the picture for the general beta case remained unclear.

The 2002 paper of Dumitriu and Edelman [15] introduced a key new tool into the study of general β-ensembles. There the authors showed that for any $\beta > 0$ one can construct a family of symmetric random tridiagonal matrices with independent entries so that the joint eigenvalue density is exactly the Gaussian β-ensemble (or β-Wishart/Laguerre ensemble). Similarly sparse structured matrix models were later found for the circular β-ensembles, including the Jacobi case [22].

Edelman and Sutton [17] then proposed that these tridiagonal models should be viewed as discretizations of random differential operators, and that the large dimensional limit of various local spectral statistics of the β ensembles could be obtained via the natural continuum limit of the given operator. This program was eventually carried out rigorously in [34], [32], [36], and [24]. The first two references provide random (Airy or Bessel type) differential operator descriptions of the "general β" Tracy-Widom edge distributions. The latter two gave (seemingly different) characterizations of the general β bulk process (for the Gaussian and circular ensemble respectively).

Importantly, these random operator and diffusion descriptions were novel even at the classical $\beta = 1, 2,$ and 4 values of the parameter. Furthermore, they have found basic applications in problems where direct analytic methods had appeared insufficient. To name a few examples: the proof of (a corrected version of) Dyson's conjecture for the bulk spacing distribution [37], sharp tail asymptotics of the edge laws [14] and unified small deviation bounds [27], and the resolution of the limit of the principal component for real spiked covariance matrices [6, 7].

The promise of these new methods prompted the 2009 AIM meeting *Brownian motion and random matrices* (*http://aimath.org/pastworkshops/brownianrmt.html*) organized by two members from the present list along with Bálint Virág (University of Toronto). Several of the problems and ideas proposed at that meeting have been taken up and in some cases successfully resolved, yet many basic problems remain and new directions have opened. In part, the BIRS workshop provided an important opportunity to see just how far the field has come in these few years and offered a forum to discuss the key open problems.

Recap of objectives

The main lines of proposed discussion for the BIRS workshop were the following.

1. **Universality**

   Spectacular progress has been made on the question of universality. That is, various of the general β limit distributions discovered via the random operator method have been proved to be universal for a wide class of Coulomb gas models on the line (where the classical Gaussian weight is replaced by a generic potential). The method of Krishnapur-Rider-Virág pushes forward the operator approach [23], while the robust proofs of Bourgade-Erdős-Yau [4, 5] realize in part Dyson's original vision. The transportation of measure approach of Bekerman-Figalli-Guionnet [2] introduces yet additional techniques. Still, basic questions regarding universality remain. As just one example, one imagines the "higher-order" Tracy-Widom laws (which arise in the case of non-regular potentials and have as yet only been rigorously addressed for $\beta = 2$ in [11]) have general β extensions, which are in turn universal in their context.

2. **Properties of the limit laws**

   Currently, the limiting point processes of the general β-ensembles are described via their counting function using coupled systems of differential equations, or as the spectra of certain random differential operators. These characterizations can be used to study various asymptotic properties of these processes (e.g. central limit theorem for the number of points, large deviation estimates of various quantities). Again though, there are many basic questions which are unanswered. The connection between the determinantal and Pfaffian descriptions of the classical limit processes and the characterizations of the general β processes is far from understood. In a second direction, the infinite systems of interacting diffusions introduced by Osada [30] and collaborators offer another setting to study the qualitative properties of the local laws. Last, new work on high-order expansions of the general β partition function and moment functionals (see e.g. [10], [19]) can provide insight given the topological information carried by the allied expansions at the classical values of β.

3. **Integrable systems**

   Another deep question is whether the general beta limit laws share a link to integrable systems as is famously the case for many of the classical random matrix limit laws. On one hand, the operator approach provides new partial differential description of many of these distribution functions (simultaneously for all beta). Whether or not the well known Painlevé property at $\beta = 1, 2, 4$ continues to hold is a tantalizing issue, while Rumanov has very recently made important progress in the case $\beta = 6$ [35]. In an entirely different direction, the theory of "Integrable Probability", in particular as introduced through the Macdonald Processes of Borodin-Corwin [8], are now known to include certain families of β-Jacobi ensembles [9].

4. **Numerics**

   Advances in software encompassing relevant special functions (such as multiple orthogonal polynomials and hypergeometric functions of matrix arguments) provide powerful new method for experimenting with beta ensemble distributions. Perhaps more importantly, the numerics have often guided new identities among these special functions (leading to exact finite dimensional formulas for certain beta function distributions) as well as introduced yet new sparse matrix models [13]. Finally, Edelman's method of "ghosts" [16] (with allied ideas appearing in the work of Forrester-Rains, see [20] for an application) offers the potential of a new algebraic framework in which to understand the beta ensembles.

3 Presentations

The workshop centered around 19 talks (of 50 minutes). Here we provide an overview of each, arranged loosely according to topic.

1. **Operator limits**

   Again, one of the primary tools for identifying the distributional limits for the beta-ensemble eigenvalues rests on proving the matrix models themselves have continuum (random) operator limits. We had three talks relating to quite different aspects of progress along these lines.

   José Ramírez spoke on the limiting smallest eigenvalue distributions (or hard edge laws) for sample covariance type matrices drawn from a spiked population [33]. Spiking here refers to a certain double scaling limit in which a fixed number of the coordinates of the (diagonal) population matrix tend to zero as the dimension goes to infinity. This complements (and through an additional limit recaptures) the spiked β soft edge laws due to Bloemendal-Virág [6, 7]. The hard edge limit laws are described in terms of a random integral operators, and partial differential equations satisfied by the corresponding distribution functions are derived as corollaries.

Igor Rumanov's talk showed how these partial differential equations can be used to give a Painlevé representation for the Tracy-Widom laws at $\beta = 6$, producing the first result of this kind for a β value which is not 1, 2 or 4.

Bálint Virág described recent results with Benedek Valkó on the Sine_β operator, a first order vector-valued self-adjoint differential operator with spectrum given by the Sine_β process, the bulk limit of the Gaussian β-ensembles. This work also provides a natural proof for the equivalence of the previous different-looking descriptions of the bulk limit through the Brownian carousel of Valkó-Virág and the diffusion type process discovered by Kilip-Stociu in the context of beta generalizations of unitary matrices. As an important side note, the resulting paper [38] was completed and posted on the ArXiv during the workshop.

2. **Log-correlated fields**

   Over the last several years there has been considerable interest in the properties of log-correlated random fields (the Gaussian Free Field and the Branching Random Walk providing two prime example), and in particular on the limiting distribution of the maximum which is supposed to have certain universal features. An important conjecture of Fyodorov and Simm is that the characteristic polynomial of various random matrix ensembles falls into this universality class.

   Both Elliot Paquette and Joseph Najnudel reported on progress on this conjecture, Paquette for the Gaussian Unitary Ensemble ($\beta = 2$) and Najnudel for the general beta circular ensembles.

   Christian Webb discussed a Stein's method approach to a central limit theorem for linear statistics of the beta circular ensembles [39].

3. **β ensembles in higher dimensions**

   While most of the progress thus far has focussed on β ensembles on the line (and their corresponding matrix models), it is natural to consider particle systems in two or higher dimensions with logarithmic (or similar) interaction. This is particularly relevant in dimension two in which case the ensemble is an honest Coulomb gas.

   Thomas Leblé gave an overview of recent results ([25, 26]) on large deviation principles of spatially averaged logarithmic and Riesz gases - a detailed asymptotics of the partition function being one approach to fluctuation results for linear statistics which is completely open in this context.

   Miika Nikula discussed an approach to rigidity estimates and local limit laws for 2-dimensional models using what is commonly referred to as the loop equations. This work has subsequently appeared in print [3].

4. **Sine_β process**

   A major success of the operator (or diffusion) limit approach in random matrix theory is that it provides a tool to understand the qualitative features of the limiting spectral point processes. Laure Dumaz showed how the diffusion connected to the Brownian carousel can be used to prove that the high temperature ($\beta \to 0$) limit of the bulk β process is a homogeneous Poisson process. Diane Holcomb gave an overview of asymptotic results on low probability events (such as overcrowding of eigenvalues) connected to the Sine_β process.

   In a different direction, Fumihiko Nakano showed how one can obtain the Sine_β as the scaling limit of the spectrum of certain Schrödinger operators, and how this leads to another proof of the equivalence of the limit of the circular β-ensembles and the Sine_β.

5. **Numerics**

Numerical analysis has long been a motivating source of random matrix theory. Indeed, the tridiagonal models of Dumitriu and Edelman are tied to the well known Householder transformations.

In the opening talk of the workshop Alan Edelman revisited these now classical beta ensembles but through his idea of "ghosts". Ghosts offer the possibility of full matrix models for the beta ensembles. While not rigorously defined, Edelman demonstrated their effectiveness in computations.

Dumitriu meanwhile discussed the importance of the extreme eigenvalues of the beta Jacobi ensembles in various numerical applications, and also presented certain closed formulas for their distributions at finite dimension.

In a different direction, Govind Menon gave an overview on recent empirical studies [12] which suggest universality in the convergence times of various standard algorithms to compute eigenvalues. The beta ensembles provide important test cases in this work.

6. **Integrable systems**

   From the solvability of the unitary, orthogonal and symplectic ensembles to the appearance of the Painlevé functions in the limiting gap probabilities, there are intimate connections between random matrix theory and integrable systems.

   Alex Moll spoke about a law of large numbers and central limit theorem for the profile of partitions under Jack measures. The analysis rests on a connection to the quantum Benjamin-Ono equation [29].

   Karol Kozlowski discussed a Bethe Ansatz approach to the XXZ spin chain which has correlation functions that bare structural similarities to those of β ensembles.

7. **Dynamics, generalizations, and large deviations**

   As mentioned above, one of the most basic tools in random matrix theory is Dyson's Brownian motion: the eigenvalues of a Hermitian random matrix of independent Brownian motions form their own diffusion. While we do not have a stochastic matrix model for all beta (see the open problems below), one can write down a stochastic differential equation (with interaction) that has the correct eigenvalue law (as its stationary measure). Hirofumi Osada gave an overview of the theory required to construct these diffusions for infinitely many particles (eigenvalues).

   Moving away from the exact set up of the beta ensembles it is of interest to consider particle systems whose pairwise interaction is only "locally" coulombic. Starting with his thesis work (see [21]), Martin Venker has been one of the main researchers considering these questions. His talk summarized the current state of the art.

   Rounding out the talks, Alain Rouault showed how various sum rules (or "trace formulas") from the theory of orthogonal polynomials can be derived via large deviations estimates on standard beta ensembles.

4 Open problems

The workshop featured an open problem session Wednesday evening. All participants were invited to present a problem or topic for discussion. At the end of the of the session we also reviewed some the questions that were posed in the 2009 AIM workshop *Brownian Motion and Random Matrices* (and can be found at *http://www.aimath.org/WWN/brownianrmt/brownianrmt.pdf*) on which there has been notable progress. The following is a sample of the open problems discussed.

1. M. Katori: Is there a stochastic (dynamic) version of the β-ghosts introduced by Edelman? Said differently, is it possible to generate a stochastic differential equation on tridiagonals whose fixed time marginals reproduce matrix models of Dumitriu and Edelman for say the β Hermite ensembles? This would in principle allow for a beta generalization of the important Airy process.

2. P. Koev: What algebraic properties should a "β-orthogonal matrix" satisfy? Granted that there is an honest object behind Edelman's ghosts, one imagines this can be diagonalized via conjugation by what could be viewed as a beta generalization of an orthogonal transformation. Can these β-orthogonal matrices be characterized in some independent way?

3. B. Virág: Consider the infinite version of the tridiagonal representation of the Gaussian β-ensemble and its spectral measure. Is the spectral measure absolutely continuous with respect to the Gaussian Chaos with the same dimension? (The Haussdorff dimension is expected to be $(1-\frac{2}{\beta})_+$.) The spectral measure can be obtained as the limit of random measures with densities $\frac{1}{p_k(x)^2+p_{k-1}(x)^2}$ where p_k is the k^{th} orthogonal polynomial corresponding to the random three-term recursion determined by the entries of the tridiagonal matrix.

   J. Najnudel: one can ask the same question for the circular β-ensemble and Verblunsky coefficients.

   The presumed connection to the Gaussian chaos is a natural and deep question, and was already raised at the 2009 AIM meeting.

4. T. Leblé: What is the phase diagram of the Sine$_\beta$ process? It is known that as $\beta \to 0$ the process converges to a homogeneous Poisson process while as $\beta \to \infty$ the process converges to the clock (or picket fence) process. Is there a phase transition for certain observables as β changes? A natural quantity to study would be the two-point correlation function.

5. I. Rumanov: Are there Painlevé representations for all Tracy-Widom(β)? Rumanov's progress on $\beta = 6$ is connected to an appropriate ansatz for the solution of the Lax pair of associated Quantum Painlevé II equation. The basic structure of this ansatz naturally extends to even integer values of beta. While pushing this idea forward remains a challenge, for other values of beta we still do not have a starting point for the analysis.

5 Conclusion

In all the workshop was a great success. The breadth of the lectures and open problems indicates a range of current activities with many questions remaining to be explored. We hope to establish regularly held workshops in the area.

References

[1] Anderson, G., Guionnet, A., Zeitouni, O. *Introduction to random matrices.* Cambridge University Press, 2009

[2] Bekerman, F., Figalli, A., Guionnet, A. Transport maps for β-matrix models and universality. *Comm. Math. Physics* **338** (2015), 589–619.

[3] Bauerschmidt, R., Bourgade, P., Nikula, M., Yau, H-T. Local density for two-dimensional one-component plasma. *Preprint, arXiv:1510.02074*

[4] Bourgade, P., Erdős, L., Yau, H.T. Universality of general β-ensembles. *Duke. Math. J.* **163** (2014), 1127–1190.

[5] Bourgade, P., Erdős, L., Yau, H.T. Edge universality of beta ensembles. *Comm. Math. Phys.* **332** (2014), 261–353.

[6] Bloemendal, A., Virág, B. Limits of spiked random matrices I. *Prob. Theory Rel. Fields* **156** (2013), 795–825.

[7] Bloemendal, A., Virág, B. Limits of spiked random matrices I. *To appear, Ann. Probab.*

[8] Borodin, A., Corwin, I. Macdonald processes. *Probab. Theory. Rel. Fields* **158** (2014), 225-400.

[9] Borodin, A., Gorin, V. General beta Jacobi corners process and the Gaussian free field. *Comm. Pure Appl. Math.* **68**, 1774–1844.

[10] Borot, G., Guionnet, A., Kozlowski, K. Large-N asymptotic expansion for mean field models with Coulomb gas interaction *IMRN* **2015** (2015), 10451–10524.

[11] Claeys, T., Its, A., Krasovsky, I. Higher order analogues of the Tracy-Widom distribution and the Painlevé II hierarchy. *Comm. Pure Appl. Math.* **63** (2010) 362-412.

[12] Deift, P., Menon, G., Olver, S., Trogdon, T. Universality in Numerical Computations with Random Data. Case Studies. *Preprint, arXiv:1407.3829*

[13] Dubbs, A., Edelman, A., Koev, P., Venkaramana, P. The beta-Wishart ensemble *J. Math. Phys.* **54** (2013).

[14] Dumaz, L., Virág, B. The right tail exponent of the Tracy-Widom-beta distribution. *Ann. de l'Institut Henri Poincaré* **49** (2013), 915-933.

[15] Dumitriu, I., Edelman, A. Matrix models for beta ensembles. *J. Math. Phys.* **43** (2002), 5830–5847.

[16] Edelman, A. The random matrix technique of ghosts and shadows. *Markov Proc. Rel. Fields* **16** (2010), 783–790.

[17] Edelman, A., Sutton, B. From random matrices to stochastic operators. *J. Stat. Phys.* **127** (2007), 1121–1165.

[18] Forrester, P.J. *Log-gases and random matrices*. Princeton University Press, Princeton-Oxford, 2010.

[19] Forrester, P.J., Witte, N.S. Moments of the Gaussian β ensembles and the large N expansion of the densities. *J. Math. Phys.* **55**, 083302 (2014).

[20] Forrester, P.J. Probability densities and distributions for spiked and general variance Wishart β-ensembles. *RMTA* **2**, 1350011 (2013).

[21] Götze, F., Venker, M. Local universality of repulsive particle systems and random matrices *Ann. Probab.* **42** (2014), 2207-2242.

[22] Killip, R., Nenciu, I. Matrix models for circular ensembles. *IMRN* **50** (2004), 2665-2701.

[23] Krishnapur, M., Rider B., Virág, B. Universality of the Stochastic Airy Operator. *Comm. Pure Appl. Math.* **69** (2016), 145–199.

[24] Killip, R., Stociu, M. Eigenvalue statistics for CMV matrices: From Poisson to clock via random matrix ensembles. *Duke Math. J.* **146** (2009), 361–399.

[25] Leblé, T., Serfaty, S. Large Deviation Principle for Empirical Fields of Log and Riesz Gases. *Preprint, arXiv:1502.02970.*

[26] Leblé, T. Logarithmic, Coulomb and Riesz energy of point processes. *Preprint, arXiv:1509.05253.*

[27] Ledoux, M., Rider, B. Small deviations for beta ensembles. *Elect. J. Probab.* **15** (2010), 1319–1343.

[28] M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

[29] Moll, A. Random Partitions and the Quantum Benjamin-Ono Hierarchy *Preprint, arXiv:1508.03063*

[30] Osada, H. Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. *Ann. Probab.* **41** (2013), 1–49.

[31] Quastel, J. Introduction to KPZ. *Current developments in mathematics, 2011*, 125-194. Int. Press, Somerville, MA, 2012.

[32] Ramírez, J., Rider, B. Diffusion at the random matrix hard edge. *Comm. Math. Phys.* **288** (2009), 887–906.

[33] Ramírez, J., Rider, B. Spiking the random matrix hard edge. *To appear, PTRF (arXiv:1506.04988).*

[34] Ramírez, J., Rider, B., Virág, B. Beta ensembles, stochastic Airy spectrum and a diffusion. *J. Amer. Math. Soc.* **24** (2011), 919–944.

[35] Rumanov, I. Painlevé representation of Tracy-Widom$_\beta$ distrbution for $\beta = 6$ *Comm. Math. Phys.* **342**. 843–868.

[36] Valkó, B., Virág, B. Continuum limits of random matrices and the Brownian carousel. *Inven. Math.* **177** (2009) 463–508.

[37] Valkó, B., Virág, B. Large gaps between random eigenvalues. *Ann. Probab.* **38** (2010) 1263–1279.

[38] Valkó, B., Virág, B. The Sine_β operator. *Preprint, arXiv:1604.04381*.

[39] Webb, C. Linear statistics of the circular β-ensemble, Stein?s method, and circular Dyson Brownian motion. *EJP* **21** (2016), paper no. 25.