Supercharacters and Hopf Monoids

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Before the workshop, we suggested that the following reading list [1, 2, 3, 4, 5, 6] would be a good refreshment for all of us. At the start of the workshop, we identified the following problems that were worth considering given the recent progress in the area.

1. Higman's conjecture.
2. Compute the antipode of the monoid $\mathbf{scf}(U)$ in the supercharacter basis.
3. Explicitly describe the multiplication in the supercharacter basis of type D.
4. Find a Hopf theoretic interpretation for the inner product in $\mathbf{scf}(U)$.
5. Find a categorification of NCQSym and of other Hopf monoid structures (like $\mathbf{\Pi}$).
6. Construct a Hopf monoid of Young tableaux.

For each question we considered what we knew and what we could do. We worked on some questions more than others. Below, we recount the week's progress on each question.

1 Higman's conjecture (see [2] for more background).

Consider the group $UT_n(q)$ of unipotent upper triangular matrices over the finite field $\mathbb{F}_q$. Higman conjectured that the number $k_n(q)$ of conjugacy classes of $UT_n(q)$ is a polynomial in q. This problem has been open for more than 50 years. A refinement of this conjecture is given in [2] as follows. Define the constants $c_n(q)$ satisfying the generating series

$$\sum_{n\geq 1} c_n(q)x^n = 1 - \frac{1}{\sum_{n\geq 0} k_n(q)x^n}.$$

It is conjectured in [2] that the $c_n(q)$ should also be a polynomial in q, and moreover, if we let $t = q-1$, the polynomial $c_n(t)$ should have positive integer coefficients. At BIRS, we proved the following propositions:

Proposition 1.1. *The Hopf monoid* $\mathbf{cf}(U)$ *is free. In particular, the integers* $c_n(q)$ *are nonnegative for every* $n \geq 1$ *and prime power* q*. They count the number of free generators of* $\mathbf{cf}(U)$ *of degree* n*.*

We remark that in the case of $\mathbf{f}(U)$ and $\mathbf{scf}(U)$ it is straightforward to show that these monoids are free. In these cases we are able to show that the numbers $c^f_n(q)$ and $c^s_n(q)$ of free generators of degree n in each monoid, respectively, are polynomials. Moreover we understand the coefficients of the polynomials $c^f_n(t)$ and $c^s_n(t)$ when $t = q-1$ as counting certain free generators with given properties. We have some conjectures about the coefficients of the (alleged) polynomial $c_n(q)$. Denoting by $[t^r]P(t)$ the coefficient of t^r in a polynomial $P(t)$ we state the following conjecture.

Conjecture 1.2. *For any n, q, r, we expect $[t^r]c_n^f(t) \geq [t^r]c_n(t) \geq [t^r]c_n^s(t)$. Moreover* $[t]c_n(t) = 1, \quad [t^2]c_n(t) = (n-2)^2, \quad [t^3]c_n(t) = \frac{1}{12}(n-3)(n-2)(5n^2 - 29n + 48)$

At this time we do not have a concrete plan to prove these conjectures. A remark about Proposition (1.1) has been included in [2].

2 Compute the antipode of the monoid $\mathrm{scf}(U)$ in the supercharacter basis.

There are two results known for the antipode on the supercharacter basis in $\mathbf{scf}(U)$. Using some partial order on the indexing set, we know the leading term of the image under the antipode map using this ordering. This gives us a triangularity relation. Also, we understand explicitly the antipode for the supercharacters indexed by a single arc. These results have not been published yet but are in the process of being written by Nat and Nantel. It was suggested that computing the antipode for supercharacters indexed by two intersecting arcs would be a good problem for students, so we did not work on this problem any further.

3 Explicitly describe the multiplication in the supercharacter basis of type D.

In type A, the Hopf algebra constructed in [1] is defined in terms of representation-theoretic operations. In particular, the multiplication is defined in terms of inflation maps between special subgroups of $UT_n(q)$ and the group $UT_n(q)$ itself. For the Hopf algebra of type D, constructed in [4], the multiplication is defined in a way that mimics the type A definition even though no inflation maps are known to exist for type D. One can multiply supercharacters of type D with this new multiplication, but nothing guarantees that the result would be positive linear combination of supercharacters. During her visit at BIRS, Carolina showed the following

Proposition 3.1. *The multiplication (as defined in [4]) of two supercharacters of type D is positive when expanded in terms of supercharacters of type D. Moreover, a dimension counting argument suggests that it should be coming from some sort of type D inflation map.*

This is added in her paper [4] and opens the door to a new (super)representation problem:

Problem 3.2. *Find a representation-theoretic map from $UT_n^D(q)$ to $UT_k^D(q) \times UT_{n-k}^D(q)$ that can be used to construct the multiplication as defined in [4].*

4 Find a Hopf theoretic interpretation for the inner product in $\mathrm{scf}(U)$.

The Hopf algebra of symmetric functions is known to be a Hopf ring if we take into account the "internal multiplication" induced by the inner product of characters. It is natural to ask if a similar structure exists for the Hopf algebra of supercharacters. A representation-theoretic inner product certainly exists, but the compatibility relation with the other two operations is not fully satisfied (the distributivity law fails). We are interested in knowing whether these three operations exhibit any additional structural relations in a broader context. We explored some avenues, including a possible module structure on Hopf monoids, but there is nothing very concrete to report at this time.

5 Find a categorification of NCQSym and of other Hopf monoid structures (like Π).

This is a very vast question. In [1], we categorified $NCSym$ showing that it is isomorphic to the Grothendieck group of the category of super-representations of $UT_n(q)$ with inflation and restriction functors defining the Hopf structure. In [2], we lifted these notions to the Hopf monoid level (constructing $\mathbf{scf}(U)$ which maps to

NCSym under the $\overline{\mathcal{K}}$ functor of [3] when $q = 2$). It is still open to find a category of representations of groups (or even algebras) that would be linked to NCQSym. We still have no candidate to propose.

A second question is to find other family of groups with (super)representations and functors that defines a Hopf monoid. For example we would like to have this for the Hopf monoid $\mathbf{\Pi}$. We did not work on this general question, but we agreed to have it in the back of our minds going forward. Section 6 below is our attempt to find a Hopf monoid more suitable for the family of symmetric groups.

6 Construct a Hopf monoid of Young tableaux.

Our general motivation is to better understand the interaction between the representation theory of towers of groups (or algebras) and Hopf monoid structures. For the tower of symmetric groups, the Grothendieck group of the category of S_n-representations (for $n \geq 0$) is isomorphic to the Hopf algebra Sym of symmetric functions. We presently do not have a Hopf monoid equivalent of this statement. To better understand the general theory, we felt that the following question is central to our program: what is a Hopf monoid constructed from the representation theory of the symmetric groups that projects to Sym under the $\overline{\mathcal{K}}$ functor of [3]? The monoid $\mathbf{\Pi}$ of [3] does project to Sym under $\overline{\mathcal{K}}$ but we do not have a categorification of $\mathbf{\Pi}$ from a tower of groups (or algebras). In this direction, we constructed the following Hopf monoid of Young tableaux with the hope that a certain quotient of this monoid has the desired structure (see [3] for most of the notation used here). Given a finite set J, let $L[J]$ denote the set of all linear orders on J. Given $\sigma \in L[J]$, we say that T is a standard tableau on J with respect to σ if $T\colon \lambda \to J$ is a bijection from a Ferrer diagram λ of a partition of the integer $|J|$ to the set J which is increasing along row and column with respect to the linear order σ. Let $SYT_\sigma[J]$ denote the set of all standard tableaux with respect to the linear order σ. Inspired by the work of [7] we defined a multiplication (μ) and a comultiplication (δ) on such objects. The following was proven during the Banff week.

Theorem 6.1. *Let $\mathcal{T}$ be the species such that for any finite set J, we have $\mathcal{T}[J] = Span\{(T,\sigma) | \sigma \in L[J], T \in SYT_\sigma[J]\}$. With the operations (μ, δ), $\mathcal{T}$ is a Hopf monoid. Moreover we have an embedding of Hopf monoids $\Psi\colon \mathcal{T} \hookrightarrow (L^* \times L)$.*

Given this, we have the following commutative diagram of Hopf monoids

$$\begin{array}{ccccc}
\overrightarrow{\Sigma} & \xhookrightarrow{\Theta} & \mathcal{T} & \xhookrightarrow{\Psi} & (L^* \times L) \\
\downarrow & & \downarrow & & \downarrow \Psi^* \\
C & \hookrightarrow & A & \hookrightarrow & \mathcal{T}^* \\
\downarrow & & \downarrow & & \downarrow \Theta^* \\
B & \hookrightarrow & D & \hookrightarrow & \overrightarrow{\Sigma}^*
\end{array}$$

where $A = \mathrm{Im}(\Psi^* \circ \Psi)$ and $B = \mathrm{Im}(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$. We constructed explicitly all the maps above and showed that under the functor $\overline{K}$ all four Hopf monoids A, B, C and D map to Sym. Furthermore, A and B are self-dual by construction. Both A and B are interesting candidates for our quest, but we need to understand the structure better. To this end, we studied $\ker(\Psi^* \circ \Psi)$ and $\ker(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$. We will write our findings in an upcoming paper. Our preliminary exploration suggests the following.

Speculation 6.2. *The Hopf monoid $\mathcal{T}$ seems to be isomorphic to the Hadamard product $\mathcal{I} \times L$, where $\mathcal{I}$ is the species of involutions.*

Speculation 6.3. *Generators of the kernel of $\Theta^* \circ \Psi^* \circ \Psi \circ \Theta$ seems to be related to flows on n-spheres.*

Item (6.3) is fascinating and, if true, the space B will have a structure related to a very interesting geometry on n-spheres.

Final remarks: In the course of the workshop, we had some extensive computation programmed in SAGE for parts Section 1 and Section 6. The Hopf monoid of Young tableaux $\mathcal{T}$ has been implemented by Franco and may find its way to SAGE/Combinat if the community shows some interest. Algorithms for computing $\ker(\Psi^* \circ \Psi)$ and $\ker(\Theta^* \circ \Psi^* \circ \Psi \circ \Theta)$ were also implemented, which allowed us to generate interesting data and discover the speculative geometry in (6.3).

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