BIRS 10w5069: Test problems for the theory of finite dimensional algebras

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1 Overview of the Field

The roots of representation theory go far back into the history of mathematics: the study of symmetry, starting with the Platonic solids and the development of group theory; the study of matrices and the representation theory of groups by Klein, Schur and others which led to the development of the concepts of rings, ideals and modules; the study of normal forms in analysis, in the work of Weierstrass, Jordan and Kronecker, among others; the development of Lie theory. Some of the famous Hilbert's problems relate representation theory with fundamental geometric concepts.

Starting in the middle 60's of last century, the 'modern' Representation Theory of finite dimensional algebras had a very fast start with three main driving forces: The categorical point of view, represented by Maurice Auslander and his school, leading to the concepts of almost-split sequences, Auslander-Reiten duality, and Auslander-Reiten quivers. The introduction of the concept of quiver representations by Pierre Gabriel, which is now a main tool in the analysis of the representation theory of finite dimensional algebras. The reformulation of problems from representation theory as matrix problems, associated to the Ukrainian school of A. Roiter lead to classification results in certain representation-infinite situations and the conceptual dichotomy of algebras according to their representation type as tame (including representation-finite) or wild.

This 'modern' Representation Theory of finite dimensional algebras, typically over an algebraically closed field, is thus characterized in its early stages by the dominance of linear methods, functor categories and homological theory. A specific flavor, in that form not present in any other subject, is constituted by Auslander-Reiten theory (almost-split sequences, Auslander-Reiten quivers, Auslander-Reiten duality), an aspect reflecting a deep combinatorial structure of homological nature. The area has by now reached a highly mature stage, leaving behind the original motivation of determining (if possible) all the indecomposable representations relating to the concepts of representation-finite, tame and wild type, respectively. The present situation is best described by strong interactions of representation theory with other mathematical subjects, like Graph Theory, Combinatorics, Lie Theory, Algebraic and Differential Geometry, Singularity Theory, Quantum Groups, and Mathematical Physics. Moreover, as reflected by conferences like the biannual series of ICRAs (International Conferences on the representations of algebras and related topics), the vitality of the subject is characterized by continuously conquering new topics of neighboring areas.

2 Recent Developments and Open Problems

In a sketchy fashion we list some important recent developments and open problems:

2.1 Cluster algebras and categories.

Cluster algebras were introduced by S. Fomin and A. Zelevinski [18, 19] at the start of this decade by formalizing the concept of *mutations*. Cluster algebras are interesting and difficult subalgebras of rational function fields reflecting deep combinatorial properties; correspondingly they enjoy a ubiquitous appearance in mathematics. By work of A. Buan, R. Marsh, M. Reineke, I. Reiten, and T. Todorov [8], finite dimensional representation theory enters the scene by achieving a categorification (of central classes) of cluster algebras through the *cluster category* of a finite quiver Q, interpreting mutations as (generalized) reflections, a subject closely related to tilting theory. The cluster category $\mathcal{C}$ is constructed as the orbit category of the derived category $D^b(\mathrm{kQ})$ of the path algebra of Q, more generally of a hereditary category. By a central result of B. Keller [31] the cluster category is itself triangulated. Much recent work has been invested by C. Amiot, O. Iyama, B. Keller, I. Reiten and collaborators in extending the context in various directions (a) by forming cluster categories for finite dimensional algebras of global dimension two [1], (b) by introducing and investigating cluster mutations for 2-Calabi-Yau categories [1], (c) by developing 'higher' cluster categories, see [28, 2], (c) introducing a categorification for quivers with potential [1]. In this Workshop the lectures by B. Keller, see section 3.1, and O. Iyama, see section 4.6, dealt with aspects of cluster and higher cluster theory.

2.2 Higher Auslander-Reiten and higher cluster theory.

This topic has some overlap with the previous one, but has a different and wider scope. A major recent development is O. Iyama's re-investigation of Auslander-Reiten theory, see [27], leading to what is now called *higher Auslander-Reiten theory*, see also [26, 28, 2]. We thus have n-almost-split sequences, n-representation-finiteness, n-preprojective algebras, n-Calabi-Yau triangulated categories, n-cluster categories etc., where the case $n = 2$ corresponds to the 'classical' situation. For instance, a triangulated category is called n-Calabi-Yau if there are functorial isomorphisms $\mathrm{Hom}(X, Y) = \mathrm{DHom}(Y, X[n])$ where D stands for the formation of the dual vector space. The subject is judged to have a high mathematical potential, correspondingly we expect it to have a large impact on the further development of representation theory. O. Iyama's Workshop lecture was about joint work with C. Amiot and I. Reiten [2] with the focus on stable categories of n-Cohen-Macaulay modules. Interpreting higher preprojective algebras as coordinate algebras of non-commutative projective schemes, the subject also produces challenging examples for non-commutative algebraic geometry, see section 4.4.

2.3 Wild algebras, and link to Algebraic Geometry.

Wild algebras, roughly speaking, don't allow an *explicit* classification of all their indecomposable modules. Though, much information (for instance growth behavior, shape of Auslander-Reiten quivers, certain aspects of module varieties) is now available for many classes of wild algebras, additional structural information on the set of all (or suitable families of) representations is missing, though in some cases moduli spaces for small dimension vectors have been studied successfully. Clearly the matter is geometric in spirit, and conjecturally related to non-commutative algebraic geometry. In the Workshop the matter was addressed in sections 4.4 and 4.7 for the class of wild hereditary algebras, yielding significant progress but leaving major questions still open.

2.4 Existence of tilting objects in triangulated categories.

This is a vast subject with many open problems, where only partial results are known, the initial results mainly due to work of Russian mathematicians, for instance A. Rudakov and collaborators. One fundamental question is: '*Which smooth projective varieties (stacks, orbifolds) admit a tilting object, more generally a full exceptional sequence for their derived category of coherent sheaves?*. At the Workshop significant progress in the surface case was reported by L. Hille and K. Ueda, see section 4.1 for further details. Another instance of the problem is the existence of tilting objects (full exceptional sequences) in the singularity category à la Buchweitz and Orlov [9, 45]. For Kleinian and Fuchsian singularities this has been solved by work of H. Kajiura, K. Saito, A. Takahashi [29, 30] or Lenzing, de la Peña [40]. For triangle singularities there are independent solutions by A. Takahashi and D. Kussin, H. Meltzer and H. Lenzing [36], respectively. The subject is still a matter of much current research. For further aspects we refer to sections 3.2 and 4.3.

2.5 Homological techniques and conjectures.

During the last 10 to 15 years finite dimensional representation theory experienced an important shift in emphasis. Instead of module categories their bounded derived categories have moved into the center of interest, allowing to prove that many representation theoretic concepts are actually invariant under derived equivalence.

It is customary to claim that triangulated categories are the natural framework for homological techniques. To confirm this claim still a lot of fundamental work has to be done. As an example we mention *Hochschild cohomology* another is the (graded) center of a triangulated category. By results of B. Keller one knows that the Hochschild cohomology of a finite dimensional algebra A is a derived invariant, hence only depends on the derived category $D^b(\mathrm{mod}(A))$. However, a direct interpretation of Hochschild cohomology for a triangulated category is missing, though in case of a tilting object T, one can use Keller's result resorting to the Hochschild cohomology of the endomorphism ring of T. The Hochschild cohomology ring is an essential tool, for instance to investigate *support varieties* for suitable classes of finite dimensional algebra. This approach, yielding a geometric insight for modules follows the group-theoretic example by L. Evens and J.F. Carlson and is the basis of much recent work by Ø. Solberg, K. Erdmann, N. Snashall, and Benson-Iyengar-Krause [5].

Typically, important homological conjectures for representations of a finite dimensional algebra A are still unsolved, though being around for several decades and being verified in many special cases. Here, we just mention the *Auslander conjecture*, dealing with bounds on the vanishing of $\mathrm{Ext}^n(X, -)$, the (generalized) *Nakayama conjecture* dealing with the structure of the minimal injective resolution of the regular A-module A, and the *finitistic dimension conjecture* which asks whether a bound of the finite projective dimensions of (finite dimensional) A-modules exists.

An exception to this general rule is M. Auslander's question on the possible values for the *representation dimension* which measures how far a finite dimensional algebra is from being representation-finite. Here, O. Iyama's proof [25] that this dimension is always finite came as a surprise and stimulated a lot of still ongoing activity to determine the exact value for special classes of algebras. (It is meanwhile known that this dimension can get arbitrarily large, a non-trivial fact, whose proof involves significant input from algebraic geometry.)

3 Presentation Highlights

The following two Workshop talks were video-taped.

3.1 Quantum Dilogarithm identities from quiver mutations

A highlight of the Workshop, yielding substantial mathematical progress, was the talk by B. Keller *Quantum Dilogarithm identities from quiver mutations*. Such identities for the quantum dilogarithm first occurred in work of Faddeev-Kashaev [17]; they were generalized by Kontsevich-Soibelman [34] and Reineke [47] using ideas from algebraic geometry, for instance stability structures and Donaldson-Thomas type invariants. The Faddeev-Kashaev identities and their generalizations imply corresponding identities for the classical dilogarithm, a fact relevant to number theory. Another application is to discrete dynamical systems with applications to mathematical physics.

In his talk, Keller presented a transparent and systematic method to derive a whole bunch of such identities from the theory of cluster algebras and categories by using sequences of socalled 'green quiver mutations'. Assume that the quiver Q in question is "good"; for instance Dynkin quivers or box products of two Dynkin quivers (corresponding to the tensor product of their path algebras) are good. First, one forms an enlargement $\tilde{Q}$ of Q by adding to each vertex v of Q a new vertex v' together with a new arrow $v \to v'$. The new vertices are considered to be frozen: it is not allowed to mutate there. Each time two such mutation sequences for $\tilde{Q}$ yield isomorphic quivers (the isomorphisms need to fix the frozen vertices), there results a quantum dilogarithm identity where two products of quantum dilogarithms turn out to be equal. The number of factors on each side of the identity equals the length of the corresponding mutation sequence, yielding new non-trivial identities. Moreover, the two sides may have different numbers of factors. The Faddeev-Kashaev identity corresponds that way to the Dynkin quiver A_2. Since for more complicated (good) quivers the combinatorics of quiver mutations is quite delicate, Keller has enhanced his Java applet for quiver mutations (obtainable from `www.math.jussieu.fr/ keller/quivermutation/`) to deal with this specific application of cluster theory. The Dynkin quiver $Q_1 = E_6$, the box product $Q_2 = A_5 \square A_2$ of two Dynkin quivers, and the corresponding identities, were presented in detail. Since diagrams like Q_1 and Q_2 play a prominent role in singularity theory, this potential relationship needs to investigated further.

Summary: Cluster mutations, more generally cluster algebras and categories, are a powerful mathematical tool related to many mathematical subjects. We should expect to see further instances of spectacular applications of cluster techniques to many mathematical questions.

3.2 Sequences of triangulated categories

H. Lenzing's talk *Sequences of triangulated categories with focus on ADE-chains* did address two of the Workshop's main objectives, viz. the two test problems *Sequences of algebras* and *Nilpotent operators*.

There is much experimental evidence that sequences (A_n) of finite dimensional algebras obeying the 'same building law', enjoy a close relationship between representation-theoretic properties of their bounded derived categories $D^b(A_n)$ and the spectral properties of A_n (Coxeter transformation, Coxeter polynomial, etc.). An attempt to formalize 'nice' building laws is the concept of sequences (A_n) of *accessible algebras* [42] which are obtained from the base field by forming successive 1-point-extension by exceptional modules. An important example is formed by the Nakayama algebras $A(n, r)$, given by the linear quiver $1 \to 2 \to 3 \cdots \to n-1 \to n$ with n vertices, where the composition of any r consecutive arrows is assumed to be zero. Note that D. Happel and U. Seidel [21] determined all such algebras which are piecewise hereditary, i.e. derived equivalent to the path algebra of a finite acyclic quiver.

Of particular importance is the sequence $A_n = A(n, 3)$ since it 'extends' the derived types of the Dynkin diagrams E_6, E_7 and E_8. (We say that the sequence (A_n) forms an ADE-chain.) A workshop at Bielefeld University, directed by C.M. Ringel, October 31-November 1, 2008 on the "ADE-chain problem" dealt with the ubiquity (and surprising similarity) of such ADE-chains arising in many different mathematical subjects, among others in

1. the *bounded derived category* of the Nakayama algebras $A(2(n-1), 3)$,
2. the triangulated *singularity category* à la Orlov [45] and Buchweitz [9] of the triangle singularity $f_n = x^2 + y^3 + z^n$. This category has alternative incarnations as *stable category of vector bundles* $\underline{\mathrm{vect}}(\mathbb{X})$ on the weighted projective line $\mathbb{X}(2, 3, n)$, or as the *stable category of CM-modules* (graded sense) over $S = k[x, y, z]/(f_n)$ or, alternatively, as the (stable) category of matrix factorizations of f_n.
3. The triangulated category of *invariant subspaces of nilpotent operators* of nilpotency degree n (graded sense) of Ringel-Schmidmeier [52].

It came as a surprise [35], see also [10], that the problems, listed above, yield identical triangulated categories: notation $\mathcal{T}_{2(n-1)}$, 'showing' a remarkable uniqueness of such ADE-chains. Major features of the category $\mathcal{T} = \mathcal{T}_{2(n-1)}$ are:

1. $\mathcal{T}$ has 'natural' tilting objects T_1, T_2 whose endomorphism rings are, respectively, the Nakayama algebra $A(2(n-1), 3)$ or the tensor product $k\vec{A}_2 \otimes k\vec{A}_{n-1}$ of two linear Dynkin quivers. (There is related work of Ladkani [37]).
2. The category $\mathcal{T}$ is fractionally Calabi-Yau of CY-dimension $1 - 2\chi_{\mathbb{X}}$, where $\chi_{\mathbb{X}}$ is the (orbifold) Euler characteristic of the weighted projective line $\mathbb{X}(2, 3, n)$. (It then follows that the Coxeter transforma-

tion is periodic, typically of period $\mathrm{lcm}(3, n)$), and the Coxeter polynomial factors into cyclotomic polynomials.

The matter — in a more general setting — is taken up again in the Workshop lecture of D. Kussin, see section 4.2; additional aspects are treated in the Workshop lecture of A. Takahashi, see section 4.5.

Summary. The talk reported on significant recent progress on ADE-chains, and further sequences of algebras and triangulated categories, revealing surprising links between representation theory, singularity theory and operator theory. Arising Calabi-Yau properties, existence and shape of tilting objects were highlighted.

4 Scientific Progress Made

4.1 Varieties with a tilting object

The two lectures by L. Hille on *"Tilting Bundles on Rational Surfaces and Quasi-Hereditary Algebras"* and K. Ueda on *"Dimer models, exceptional collections, and non-commutative crepant resolutions"* dealt with similar questions from a somewhat orthogonal perspective. They constitute significant progress and, potentially, allow the representation theory of finite dimensional algebras to deal with specific three-dimensional singularities. (Sofar, the link to singularity theory is mainly in dimensions one and two).

The existence of tilting objects (in different terminology, of full strong exceptional sequences/collections) in derived categories of coherent sheaves over a smooth projective variety (more generally in a weighted, or stacks, or orbifold version) is central for linking finite dimensional representation theory, algebraic geometry and singularity theory. Concerning the link to singularity theory, the introduction of Orlov's singularity category [45], see also former work by R. Buchweitz [9], has given the subject a strong additional momentum.

Categories of coherent sheaves $coh(X)$ on a smooth projective variety X share many important properties with categories $mod(A)$ of finite dimensional representations over a finite dimensional algebra A (of finite global dimension): they are abelian categories whose homomorphism and extension spaces are finite dimensional over the base field k (mostly assumed to be algebraically closed). Moreover, the bounded derived categories $D^b(X)$ and $D^b(A)$ satisfy Serre duality, and consequently admit Auslander-Reiten theory, thus enabling to speak of their Auslander-Reiten quiver and Auslander-Reiten components. While this analogy is merely on the formal level, it gets very specific once the (derived) category of coherent sheaves on X admits a *tilting object*, that is, an object T without self-extensions and generating the derived category: In this case $D^b(X)$ and $D^b(A)$ are actually equivalent, thus allowing a direct comparison of complexes of modules or sheaves, and establishing the endomorphism ring of T as a kind of coordinate system for the category $D^b(X)$. Since the first appearance of this effect in work of Beilinson's influential paper [4], there has been much interest in the question for which smooth projective varieties/schemes/stacks the derived category has a tilting object. For a one-dimensional variety, Riemann-Roch implies that this occurs only for the projective line (if the base field k is algebraically closed). This result extends to a weighted (=stacks, orbifold) setting: only the weighted projective lines allow a tilting object, see [39] and [49].

Passing to projective varieties X of higher dimension, the question of the existence of a tilting object (more generally, of a full exceptional sequence) constitutes a severe restriction since it forces the Grothendieck group to be finitely generated free. And the existence problem constitutes an open problem: While it is frequently possible to disprove the existence of full exceptional sequences by general arguments (for instance, in case of big size or the existence of non-trivial torsion for the Grothendieck group, or by the non-existence of any exceptional objects for Calabi-Yau varieties), any existence proof affords an explicit construction and thus a highly explicit understanding of the (derived) category of coherent sheaves and its exceptional objects. The conjecture has been around for about 20 years that such tilting objects (resp. full exceptional sequences) exist exactly for rational varieties (always assumed to be smooth projective). While there had not been much progress in this matter for a long time, recent development did create a strong momentum: In 1997, A. D. King [33] conjectured that any complete smooth toric variety has a tilting sheaf which is a direct sum of line bundles. In this form, the conjecture has been disproved by L. Hille and M. Perling [22] already for (Hirzebruch) surfaces; but with a proper modification King's conjecture is basically correct, as was reported in L. Hille's Workshop talk *Tilting Bundles on Rational Surfaces and Quasi-Hereditary Algebras*. Based on previous work [23], Hille and Perling (2010) have shown how to construct tilting bundles for any rational surface X: They first construct a full exceptional sequence E of line bundles; in a second step, by killing higher extensions through *universal extensions* they obtain a *tilting bundle* T which, in general, is no longer a direct sum of line bundles and such that, moreover, the endomorphism algebra of T is *quasi-hereditary*. (Quasi-hereditary algebras are of finite global dimension with significant extra structure; their main structural properties have been established by V. Dlab and C.M. Ringel, see for instance [14]).

In dimension two, the weighted (stack, orbifold) version still remains open, however important results exist. K. Ueda in his Workshop talk "*Dimer models, exceptional collections, and non-commutative crepant resolutions*" reported on joint work [24] with A. Ishii using *dimer models*. A dimer model is a bicolored graph on a real two-torus which encodes the information of a quiver with relation. Ueda showed how to use dimer models to establish a *cyclic* tilting object of line bundles for two-dimensional weak Fano stacks, thus establishing King's conjecture for such stacks, a result also proved by Borisov and Hua [7] by different methods. The significant improvement by Ueda and Ishii concerns the combinatorial strategy to use dimer models in constructing tilting objects of a special kind.

Summary: The new and significant results reported by Hille and Ueda constitute a breakthrough and provide substantial additional insight. Moreover, they pave the way for a representation-theoretic treatment of three-dimensional singularities. (Two-dimensional singularities are related to the weighted projective curves.)

4.2 Nilpotent operators

This topic constitutes one of the test problems of the Workshop. As mentioned in section 3.2 it is related to the test problem on sequences of algebras and deals with the classification of invariant subspaces of nilpotent operators (on finite dimensional vector spaces). This problem has a long history that can be traced back to Birkhoff's problem [6], dealing with the classification of subgroups of finite abelian p-groups. Despite earlier work by D. Simson [54], the major breakthrough was was through recent work by Ringel and Schmidmeier [52]. Their work includes an explicit classification for nilpotency degree six, yielding *tubular type*, related to the representation theory of tubular algebras, a problem initiated and accomplished by Ringel in [51].

The three Workshop lectures

1. C.M. Ringel: *What is known about invariant subspaces of nilpotent operators? A survey. I*
2. M. Schmidmeier: *What is known about invariant subspaces of nilpotent operators? A survey. II*
3. M. Schmidmeier: *Three slides.*

reported the current status of the invariant subspace problem for nilpotent operators. Then D. Kussin: *The two-flag invariant subspace problem for nilpotent operators.* reported on the link to singularities and generalizations to more complicated systems of invariant subspaces, yielding interesting new classes of exact categories and triangulated categories.

For this test problem representation theory has passed the test: It was already mentioned in Lenzing's lecture that the invariant subspace problem is related to the investigation of triangle singularities of type $(2, 3, n)$ and the stable categories of vector bundles on a weighted projective line of the same type. Kussin, reporting on unpublished joint work with Lenzing and Meltzer, described an even more general setting, constituting substantial progress: Given a triple (a_1, a_2, b) of integers ≥ 2, the invariant two-flag problem of nilpotency degree b (graded version) with flag lengths $a_1 - 1$ and $a_2 - 1$ yields an exact category which is *almost Frobenius*, in general no longer Frobenius, whose associated stable category is equivalent to the triangulated category of stable vector bundles on a weighted projective line $\mathbb{X}$ with weight triple (a_1, a_2, b). This way, a complete classification for the indecomposables for Euler characteristic $\chi_{\mathbb{X}} \geq 0$ is achieved. In general, these triangulated categories are all fractional Calabi-Yau (of CY-dimension $1 - 2\chi_{\mathbb{X}}$) and all have tilting objects whose endomorphism rings are arising in singularity theory. This last issue is also related to A. Takahashi's Workshop talk and work by Xiao-Wu Chen [10]. Despite this enormous progress, new problems pop up: Schmidmeier reported on the interesting concept of *curvature* for the Frobenius categories of invariant subspaces (of fixed nilpotency degree), a concept, needing further clarification: in particular it's relationship to the curvature (orbifold sense) of the corresponding weighted projective line is presently unclear. Schmidmeier compared subspace problems for growing nilpotency degree, thus raising the question about the nature of the (direct) limit of the subspace categories.

Summary: The representation theory has passed the test concerning the test problem *Nilpotent operators*. Substantial, and in this form unexpected, progress was achieved. As a result a firm link between the topics representation theory, singularity theory and operator theory was established. New, and probably important, questions about the shape of corresponding limit categories did evolve.

4.3 Sequences of finite dimensional algebras, ADE-chain problem

The test problem "*Sequences of finite dimensional algebras*" was somehow the core of the meeting; a large number of Workshop contributions was devoted to this topic. For the general scope we refer to section 3.2, for further aspects to section 4.2. By the cumulative efforts of Workshop participants, significant progress could be achieved, and a large number of aspects has now been clarified. The main achievements are:

1. A firm link has been established between finite dimensional representation theory, singularity theory and operator theory,
2. The role of ADE-chains, in relationship to Nakayama algebras, weighted projective lines, matrix factorizations, singularities, and nilpotent operators has obtained much clearer contours,
3. The central role of being *fractional Calabi-Yau* for the sequence problem has been recognized.
4. As a new problem, the formation of limits of triangulated categories, determined by a sequence (A_n) of finite dimensional algebras, has emerged.

In addition to the lectures mentioned in sections 3.2 and 4.2 the following ones concern fundamental aspects of the sequence problem: J.A. de la Peña: "*Accessible algebras*" dealt with the spectral properties of accessible towers algebras. D. Happel: "*Piecewise hereditary Nakayama algebras* reported on general properties of the algebras $A(n, r)$, see [21], in particular settled the question when such an algebra is piecewise hereditary, i.e. derived-equivalent to a hereditary category. S. Oppermann: "*Implications of fractional Calabi-Yau for derived categories*" and S. Ladkani: "*On fractional Calabi-Yau algebras*" collected important evidence that Calabi-Yau properties are essential for the sequences problem. Moreover, Ladkani did present an impressive list of examples. H. Krause: "*Expansions of abelian categories*" reported on joint work with Xiao-Wu Chen [11] to construct sequences of abelian categories by insertion of new simple objects. This was to some extent continued by Chen: "*Recollement of vector bundles and homomorphism chains*" by studying sequences of categories of coherent sheaves on weighted projective lines by changing a single weight and investigating the effect on their categories of vector bundles (resp. stable categories of vector bundles). A.D. King: "Observations on Grassmannian cluster algebras" presented a conjectural relationship between the series $D_5, E_6, E_7, E_8, ...$ and $A_2, D_4, E_6, E_8, ...$ that arises in the context of Grassmannian cluster algebras, and further is related to the study of frieze categories.

Summary: In all respects, the represenation theory has passed the test concerning the test problem *Sequences of finite dimensional algebras*. Because its relationship to many other subjects (singularity theory, matrix factorizations, operator theory, non-commutative algebraic geometry) it is fair to predict that it has a large potential to further influence developments in representation theory. A particular promising topic is the formation of limit categories suggested by the Workshop.

4.4 Kerner's exotic space, and link to non-commutative algebraic geometry

A main objective of the Workshop was to explore "Kerner's exotic space" $\Omega(A)$, a hypothetical mathematical structure parametrizing the Auslander-Reiten components for a wild hereditary algebra A. We always assume A to be finite dimensional and connected, defined over an algebraically closed field; such that we may restrict to the case where A is the path algebra kQ of a finite wild connected quiver Q. A particular aim was to understand $\Omega(A)$ as an object of non-commutative Algebraic Geometry (NCAG for short), and to identify its 'geometric' properties. There were four talks specifically addressing this topic.

1. O. Kerner: *The category of regular modules over wild hereditary algebras*,
2. H. Minamoto: *Ampleness of two-sided tilting complexes*,

3. I. Mori : *Quantum Beilinson algebras*,

4. P. Smith : *Non-commutative spaces and finite dimensional algebras*.

Kerner's talk was introductory, and addressed also to non-experts of representation theory. In the focus of his talk was the surprising existence of 'natural bijections' between any two such 'spaces' $\Omega(A)$, for A a wild, hereditary as above, see [32, 12] The talks by Minamoto and Mori formed a unit. H. Minamoto's talk based on [43, 44] mainly attacked Kerner's exotic spaces by analyzing the projective spectrum of the graded *preprojective algebra* $\Pi(A)$ of a wild hereditary algebra A and of the graded noncommutative *Beilinson algebra* $k\langle x_1, \ldots, x_r\rangle/(\sum_{i=1}^{r} x_i^2)$; he further presented basic features of *Fano algebras*, a new and promising class of finite dimensional algebras inspired from the homological properties of Fano varieties. I. Mori extended the setting by investigating *quantum Beilinson algebras* and their attached projective spectrum. He then continued the investigation of Fano algebras and their NCAG, started by Minamoto. It was very interesting to observe that the *higher preprojective algebras*, as introduced by Iyama, enter the scene, thus creating a new contact surface between NCAG and the representation theory for finite dimensional algebras A of finite global dimension $d \geq 2$. (In the follow-up it was considered very important thus to wave the initial restriction to hereditary algebras). P. Smith finally gave an overview of core techniques of NCAG in a talk specifically addressed to representation theorists. Smith also raised a test problem for NCAG: "*Can NCAG help us to understand finite dimensional algebras?*", a question including the Kerner space test problem. Smith further discussed challenging examples of NCAG, some very remote from classical algebraic geometry. In some detail he treated the projective spectrum of a free non-commutative algebra as a 'baby example' for exotic properties of NCAG.

We now describe the consequences for Kerner's exotic space(s) $\Omega(A)$ as they result from these talks and corresponding follow-up discussions. Let A be a wild hereditary algebra as above. Up to Morita-equivalence A is the path algebra kQ of a finite, connected, wild quiver Q. Typical examples of wild quivers are the r-Kronecker quiver $\circ \Rrightarrow \circ$, $r \geq 3$ arrows, and a wild hereditary star $[p_1, p_2, \ldots, p_t]$ with t branches of lengths $p_1, \ldots, p_t$ and such that $2 - \sum_{i=1}^{t}(1 - 1/p_i)$ is < 0. Then $\Omega(A)$ is a virtual 'space' which parametrizes the set of all Auslander-Reiten components of (indecomposable) regular A-modules. Recall for that purpose that the indecomposable representations of a wild hereditary algebra $A = kQ$ as above are either preprojective, preinjective or regular. The preprojective (resp. preinjective) modules have a somewhat discrete flavor, and each form a single Auslander-Reiten component. By contrast, the regular modules are arranged in a 'continuous' family Auslander-Reiten components. By a fundamental result of C.M. Ringel all these components have shape $\mathbb{Z}A_\infty$, see [50]. For a wild hereditary algebra let $\Omega(A)$ be the set of its regular Auslander-Reiten components. Let A and A' be two wild, connected, hereditary, finite dimensional algebras over an algebraically closed field k. By a surprising result of O. Kerner [32, 12] there exist unexpected 'natural' bijection between the sets $\Omega(A)$ and $\Omega(A')$. In his talk, Kerner included the remarkable result of his student J.C. Reinhold, see [48], who established another bijection extending the setup of the Crawley-Boevey-Kerner result, crossing the border between hereditary algebras and coherent sheaves on a weighted projective line. Let $\mathbb{X}$ be a weighted projective line [20] of wild type $(p_1, \ldots, p_t)$, and let L be any line bundle in $\mathrm{coh}\mathbb{X}$. Then the perpendicular category $L^\perp$ is naturally equivalent to the category $\mathrm{mod}(C)$, where C is the path algebra of the star $[p_1, \ldots, p_t]$ with standard orientation. Let E be the Auslander bundle given as the extension term of the almost-split sequence $0 \to \tau L \to E \to L \to 0$. Then Reinhold shows that the factor category $\mathrm{reg}(C)/[\tau^{\mathbb{Z}}E]$ is equivalent to $\mathrm{vect}\mathbb{X}$, and hence inducing a natural bijection between $\Omega(C)$ and the set of AR-components of $\mathrm{vect}\mathbb{X}$. [Note in this context that 'stabilization' from $\mathrm{reg}C$ to $\mathrm{vect}\mathbb{X}$ leaves the Auslander-Reiten components almost intact, but changes the category structure drastically since it reduces exponential growth for $\mathrm{reg}C$, see [13], to linear growth for $\mathrm{vect}\mathbb{X}$, see [41], when dealing with the sequences $\dim_k \mathrm{Hom}(X, \tau^n Y)$.]

The existence of Kerner bijections raises some obvious questions:

1. What is the *mathematical nature* of the hypothetical 'spaces' $\Omega(A)$ parametrizing the regular components for wild hereditary algebras? Are they objects of NCAG?

2. Should we think of just one (universal) space Ω or of several spaces $\Omega(A)$, linked by Kerner bijections, and then all having a comparable structure?

As an outcome of the workshop, the answer to the first question is 'yes', more specifically $\Omega(A)$ *should be viewed as an exotic curve whose category of coherent sheaves is derived equivalent to the category of finite* *dimensional representations of A.* Concerning the second question, the idea to have a single universal space was refuted; from the structural point of view, the relationship between the various $\Omega(A)$, however, still needs clarification. In more detail, $\Omega(A)$ should be the virtual non-commutative space, whose category of coherent sheaves is the projective spectrum of the (graded) preprojective algebra $\Pi(A)$: Let A be a representation-infinite, hereditary, connected finite dimensional algebra. The *preprojective algebra* $\Pi(A)$, which has infinite k-dimension, has a simple combinatorial definition in terms of the quiver of A due to I.M. Gelfand and V.A. Ponomarev. For the present purpose, however, the following two $\mathbb{Z}$-graded (isomorphic) incarnations of $\Pi(A)$ are relevant, see [3]. Namely, $\Pi(A)$ can be considered as the *tensor algebra* $\mathbb{T}(M)$ of the (A, A)-bimodule $M = \mathrm{Tr}\mathrm{D}A = \mathrm{Ext}^1_A(DA, A)$ or alternatively as the *orbit algebra* of $\tau_A^{-1} = \mathrm{TrD}$, where TrD is Auslander's transpose-dual, that is, $\Pi(A) = \bigoplus_{n\geq 0} \mathrm{Hom}_A(A, \tau^{-n}A)$. Applying *Serre's construction* [53] to $\Pi(A)$ yields a category $\mathcal{H}(A) = \mathrm{mod}^{\mathbb{Z}}\Pi(A)/\mathrm{mod}_0^{\mathbb{Z}}\Pi(A)$ with the following properties, independently due to several authors: [38, 3, 43, 44]: (1) $\mathcal{H}(A)$ is abelian, Hom-finite, and hereditary. (2) $\mathcal{H}(A)$ admits *Serre duality* in the form $\mathrm{DExt}^1(X, Y) = \mathrm{Hom}(Y, \tau X)$ for an auto-equivalence τ of $\mathcal{H}(A)$, resembling Serre duality for smooth projective curves. (3) By means of the decomposition $\mathrm{mod}(A) = \mathrm{preproj}(A) \vee \mathrm{reg}(A) \vee \mathrm{preinj}(A)$ into preprojective, regular, resp. preinjective modules, we have

$$\mathcal{H}(A) = (\mathrm{preinj}(A)[-1] \vee \mathrm{prepr}(A)) \vee \mathrm{reg}(A),$$

viewed as a full subcategory of the bounded derived category $D^b(\mathrm{mod}(A))$. We write $\mathcal{H}_l$ for $(\mathrm{preinj}(A)[-1]\vee \mathrm{prepr}(A))$ and $\mathcal{H}_r$ for $\mathrm{reg}(A)$ such that $\mathcal{H}(A) = \mathcal{H}_l \vee \mathcal{H}_r$. (4) $\mathcal{H}_l(A)$ is equivalent to the category $\mathrm{proj}^{\mathbb{Z}}$ of finitely generated projective graded $\Pi(A)$-modules. (5) $\mathcal{H}(A)$ has a tilting object T with endomorphism ring A. (6) $\mathcal{H}(A)$ has no simple objects.

Note that $\mathrm{mod}^{\mathbb{Z}}\Pi(A)$ denotes the category of all *finitely presented*, $\mathbb{Z}$-graded right $\Pi(A)$-modules. Due to coherence of $\Pi(A)$ this category is abelian. The subscript 0 then refers to the full subcategory of finite length modules. It offers more flexibility to interpret $\mathrm{mod}^{\mathbb{Z}}\Pi(A)$, like in [38], as the category of finitely presented abelian-group valued functors on the category $\mathrm{prepr}(A)$ of preprojective A-modules. This also shows that for A the r-Kronecker algebra, $r \geq 3$ we may replace $\Pi(A)$ by the (graded) generalized Beilinson algebra $k\langle x_1, \ldots, x_r\rangle/(\sum_{i=1}^r x_i^2)$ without changing the projective spectrum. Further relevant research includes J.J. Zhang's classification of Artin-Schelter regular algebras of dimension two [55], D. Piantkovski [46], and most specifically H. Minamoto [43, 44]. That $\mathcal{H}(\mathcal{A})$ has no simple objects, is an obstacle to understand the geometry of $\Omega(A)$. This absence of simples suggests the following interpretation. In the category of coherent sheaves $\mathcal{H}(A)$ on $\Omega(A)$ we don't see the points of $\Omega(A)$. Simple objects, and then points, will however exist in the category of quasi-coherent sheaves, to be thought of as the closure of $\mathcal{H}(A)$ under direct limits. It affords further research to clarify how such 'points' relate to the regular Auslander-Reiten components of A.

There is a lot of evidence that Kerner's exotic spaces classify the set of Auslander-Reiten components in many further instances and thus play a role also in singularity and operator theory.

Summary: (1) Many aspects of Kerner's exotic spaces $\Omega(A)$ could be clarified during the Workshop. By the properties of their categories of coherent sheaves it is reasonable to consider them as *non-commutative projective non-noetherian (exotic) curves*. (2) A good knowledge of $\Omega(A)$ is of relevance for many problems in Representation Theory or Singularity Theory. (3) The relationship between 'virtual' points of $\Omega(A)$ and regular Auslander-Reiten components of A still needs clarification by further research. In particular, such a study has to invoke the category of quasi-coherent sheaves. (4) It is desirable to extend the range of the theory to finite dimensional algebras A of higher global dimension ≥ 2 by means of Iyama's construction of higher preprojective algebras, that is, the tensor algebras of higher extension spaces $\mathrm{Ext}^d_A(DA, A)$, $d \geq 2$. For the particular situation that A is Fano, interesting new results are expected, yielding higher dimensional (exotic) spaces.

4.5 Strange duality and mirror symmetry

This topic concerns the talks by A. Takahashi "*Mirror symmetry, strange duality and matrix factorization*" and W. Ebeling "*Strange duality and monodromy*". The two talks concern recent results based on their joint paper [16] and additional collaboration between Ebeling and Gusein-Zade [15]. These investigations were motivated in part by the outcome of the BIRS Workshop "*Spectral Methods in Representation Theory of Algebras and Applications to the Study of Rings of Singularities* (08w5060)", where a successful interaction between representation theory and singularity theory was initiated.

Mirror symmetry is now understood as a categorical duality between algebraic geometry and symplectic geometry. Takahashi and Ebeling apply ideas of mirror symmetry to singularity theory in order to understand various mysterious correspondences among isolated singularities, root systems, Weyl groups, Lie algebras, discrete groups, finite dimensional algebras and so on. In the two talks they generalize Arnold's strange duality for the 14 exceptional unimodal singularities to a specific class of weighted homogeneous polynomials in three variables called *invertible polynomials*.

For the base field of complex numbers, Ebeling and Takahashi obtain a complete classification of such invertible polynomials. In a condensed form, the main result of Ebeling and Takahashi, explaining strange duality, states the following. Let $f = f(x, y, z)$ be such an invertible, weighted homogeneous polynomial, and let f^t denote its Berglund-Hübsch transpose. Then (1) If $R_f = \mathbb{C}[x, y, z]/(f)$ equipped with the maximal L_f-grading, then the projective spectrum of R_f, obtained by Serre construction [53], is a weighted projective line $\mathbb{X}$ with three weights, and whose weight triple $(\alpha_1, \alpha_2, \alpha_3)$ yields the *Dolgachev numbers* for the pair (f, L_f). (2) $f^t(x, y, z)$ deforms into the cusp singularity $T_{\gamma_1,\gamma_2,\gamma_3} = (x')^{\gamma_1} + (y')^{\gamma_2} + (z')^{\gamma_3} - x'y'z'$. (3) The *Gabrielov numbers* $(\gamma_1, \gamma_2, \gamma_3)$ for f^t coincide with the Dolgachev numbers of f. (4) The singularity category $D^{L_f}_{Sg}(R_f)$ in the sense of [45, 9] admits a semi-orthogonal decomposition into the derived category $D^b(\mathrm{coh}\mathbb{X})$ and the derived category of a representation-finite quiver. (5) The singularity category is *fractional Calabi-Yau*, and has a *tilting object*.

The above properties of the singularity category $D^{L_f}_{Sg}(R_f)$ are related to, and for the case $f = x^{\alpha_1} + y^{\alpha_2} + z^{\alpha_3}$ indeed identical to, results obtained by Kussin-Meltzer-Lenzing [35]. They are thus directly related to the discussion on sequences of triangulated categories in section 4.3. W. Ebeling, in his talk, additionally gave a crash-course how to obtain the Coxeter-Dynkin diagram of an isolated singularity f by morsification of f, and to get hold of its monodromy (Coxeter transformation). He also included an account on the spectral properties (Poincaré series, Coxeter polynomials, and zeta functions) and the monodromy of weighted homogeneous polynomials.

Summary: The approach by Ebeling-Takahashi embeds Arnold's strange duality, which is initially an isolated phenomenon dealing with 17 cases, into a larger mathematical set-up, where 'strange duality' finds a natural explanation. It is, moreover, satisfying to witness in this case a particularly high degree of interaction between representation theory and singularity theory with significant impact on a main objective of the Workshop (sequences of algebras and triangulated categories).

4.6 Higher Auslander-Reiten theory, higher cluster categories

In his Workshop lecture "*Stable categories of Cohen-Macaulay modules and cluster algebras*" O. Iyama reported on joint work with C. Amiot and I. Reiten on certain aspects of higher Auslander-Reiten theory and higher cluster theory with the focus on the higher theory of Cohen-Macaulay modules.

A very important recent development is O. Iyama's re-investigation of Auslander-Reiten theory, see [27], leading to what is now called *higher* Auslander-Reiten theory, see also [26, 28, 2]. We thus have n-almost-split sequences, n-representation-finiteness, n-preprojective algebras, n-Calabi-Yau triangulated categories, n-cluster categories etc., where the case $n = 2$ corresponds to the 'classical' situation. For instance, a triangulated category is called n-Calabi-Yau if there are functorial isomorphisms $\mathrm{Hom}(X, Y) = \mathrm{DHom}(Y, X[n])$. The subject is judged to have a high mathematical potential, correspondingly the organizers expect it to have a major impact on the further development of representation theory. O. Iyama's Workshop lecture was about joint work with C. Amiot and I. Reiten [2] with the focus on stable categories of n-Cohen-Macaulay modules. Interpreting higher preprojective algebras as coordinate algebras of non-commutative projective schemes, the subject also produces challenging examples for non-commutative algebraic geometry.

In a certain degree related to the topic was the lecture by R. Takahashi "*Some classifications of resolving subcategories*" dealing, in particular, with aspects of Cohen-Macaulay modules.

Summary: This subject will certainly have a major influence on the further development of representation theory. A particular promising aspect is the *interplay with non-commutative algebraic geometry* through the concept of higher preprojective algebras, as subject, still in its early stages.

4.7 Covering theory

Inclusion of this subject into the Workshop was motivated by the objective to get more insight in the representation theory of wild Kronecker algebras. Covering theory is an essential tool to classify indecomposable situations explicitly, and to determine the representation type. This works well in 'good' situations (always representation-finite or tame) but has a major disadvantage that it does not coexist with degenerations of algebras, another useful tool in determining representation types. There were two related talks on the subject, one by A. Hajduk on "*On the different types of degenerations for algebras*" the other by P. Dowbor on "*Coverings and degenerations*". All these methods are geometric in spirit. Hajduk presented a new type of degenerations, called GCB-degenerations, allowing dimension change during degeneration. Hajduk showed a remarkably complete theory. His main result is that the representation type of a GCB-degeneration is at least as complicated as the representation type of the original algebra. In his subsequent talk, P. Dowbor did introduce degenerations of covering functors and related 'covering degenerations of algebras', thus extending the range of classical covering methods by Bongartz-Gabriel.

Summary: In itself, these investigations constitute a significant progress. The hope, however, using covering theory to obtain a better understanding of the representations of wild Kronecker quivers did not materialize. It seems that further new ideas are necessary to obtain progress in this direction.

5 Outcome of the Meeting

The mere formulation of the test problems for the Workshop has triggered a lot of exciting development, much of it already happening before the Workshop even started. In the judgement of the organizers this fact, and the corresponding success of the Workshop, is due to a clear and predefined focus of workshop topics where, on the other hand, the central workshop topics (Test Problems)

1. The representation theory of wild Kronecker algebras,
2. The unknown nature of Kerner's exotic space,
3. Sequences of algebras,
4. Nilpotent operators,

were sufficiently open to encourage leading experts to participate and contribute. As in the previous BIRS Workshop *Spectral Methods in Representation Theory of Algebras and Applications to the Study of Rings of Singularities (08w5060)* the Workshop was supported by a careful composition of experts, coming from different areas, able and interested to work together across mathematical boundaries. From our point of view, this particular scheme has proven to be incredibly successful; accordingly the response of participants to such a specific setup was very positive, sometimes even enthusiastic.

A workshop at Bielefeld University, directed by C.M. Ringel, October 31-November 1, 2008, was instrumental in sharpening the focus: as reported in section 3.2, it there became clear — still on a conjectural level — that test problems 3. and 4. should be very closely related through the concept of an ADE-chain. The conjectured links are meanwhile mathematical theorems that were reported at the BIRS Workshop, see sections 3.2 and 4.3. As an outcome of problems discussed at this Workshop, C.M. Ringel organized a further workshop at Bielefeld University with the title *Projective dimension two* on October 8-9, 2010, for details see the web-site http://www.math.uni-bielefeld.de/ sek/dim2/.

The status achieved with test problems 3 and 4 has been spectacular, in particular the established link between singularity theory and nilpotent operators was originally absolutely unexpected. For the specific achievements see sections 3.2, 4.3 and 4.2; the shape and the completeness of the results and corresponding completion of research by Ringel and Schmidmeier, see section 4.2, exploiting the tool of weighted projective lines by Kussin-Lenzing-Meltzer, see sections 3.2 and 4.2 was not be expected at all when the organizers did propose the Workshop scheme. It is satisfying, in particular, that through these studies the link between representation theory and singularity theory has gotten very strong; the explanation of Arnold's strange duality by Ebeling and Takahashi, see section 4.5, through invertible polynomials was another highlight of the meeting. It was also satisfying to see an emerging strong cooperation between representation theory and noncommutative algebraic geometry, a link, very worthwhile to be continued.

Concerning test problem 1 the expected progress on representations of wild Kronecker quivers via covering theory did not materialize. Apparently new techniques and insights are needed. On the other hand, a new conceptual understanding of the representation theory of wild Kronecker (and other wild) quivers via methods of noncommutative algebraic geometry (cf. test problem 2, section 4.4) appears to be quite promising; but still much work has to be done there.

Another promising subject emerged with links to the other topics discussed: the higher Auslander-Reiten theory developed by O. Iyama and collaborators, see sections 4.6 and 4.4. Its apparent links to cluster categories (also in a higher version) and noncommutative algebraic geometry makes it particularly challenging.

Though originally not triggered by the particular scope of the meeting, the contribution by B. Keller on dilogarithm identities was another instance of an unexpected and indeed, very powerful, application of representation theory (quiver mutations) to problems outside representation theory. On a similar level, the breakthrough by L. Hille and K. Ueda on the existence and construction of tilting objects could not be expected beforehand, though they were perfectly fitting in the general scheme of the Workshop by enforcing existing links to singularity theory, and pointing to new directions of research.

Summary: The organizers are very happy with the outcome of a meeting that met standards much above our original (high) expectations. The scientific progress achieved through the meeting itself and the preparations in front of the meeting are significant and partly even spectacular. In particular, the organizers feel confirmed and encouraged by the very positive, partly even enthusiastic, response from participants of the meeting.

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