FINAL REPORT FOR "TOPOLOGICAL METHODS IN TORIC GEOMETRY, SYMPLECTIC GEOMETRY AND COMBINATORICS"

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1. Objectives of the workshop

The subject of toric topology and in particular the study of generalized moment-angle complexes has experienced a wide range of activity with connections among different mathematical areas including algebraic geometry, homotopy theory, symplectic geometry, combinatorics, dynamical systems and robotics.

This workshop brought together some of the leading experts to Banff in these different, but significantly overlapping areas. The main purpose of this conference is to communicate the current state of the art and to facilitate interactions in order to foster new developments. A second purpose is to encourage the development of younger mathematicians and for them to make contact with the diverse frontiers in these subjects.

The focus of the conference was on the investigation of toric manifolds, generalized moment-angle complexes and their applications, the equivariant cohomology and K-theory of toric orbifolds, and related topics in topological and combinatorial geometry.

2. Schedule of talks

Monday	
	Session chair: Martin Bendersky
9:00–10:00	Graham Denham: Topological aspects of partial product spaces: a survey
10:30–11:30	Santiago Lopez de Medrano: Intersections of quadrics and the polyhedral product functor
	Session chair: Alexander Suciu
14:00–15:00	Kiumars Kaveh: Convex bodies associated to reductive group actions
15:30–16:30	Julianna Tymoczko: Computational tools in torus-equivariant cohomology with applications to Schubert calculus
Tuesday	
	Session chair: Mario Salvetti
9:00–10:00	Henry Schenck: Cohomology and Chow rings of toric varieties
10:30–11:30	Mikiya Masuda: Cohomological rigidity problem, topological toric manifolds and face numbers of simplicial cell manifolds
	Session chair: Dietrich Notbohm
14:00–15:00	Michael Davis: Generalized moment angle complexes, graph products of groups and related constructions
15:30–16:30	Nigel Ray: Applications of toric methods to cobordism theory
Wednesday	
	Session chair: Alberto Verjovsky
9:00–10:00	Victor Batyrev: On topological invariants of Calabi–Yau 3-folds constructed by toric geometry
10:30–11:30	Tom Braden: Geometry and representation theory of hypertoric varieties

Thursday	
	Session chair: Volker Puppe
9:00–10:00	Corrado De Concini: Infinitesimal index and some cohomology computations
10:30–11:30	Susan Tolman: The integral cohomology of GKM spaces
	Session chair: Dong Youp Suh
14:00–15:00	Carl Lee: Sweeping the cd-index and the toric h-vector
15:30–16:00	Suyoung Choi: Torus actions on cohomology complex Bott manifolds
16:00–16:30	Jelena Grbić: Higher Whitehead products and polyhedral product functors
20:00–20:30	Dave Anderson: Transversality in equivariant cohomology
20:30–21:00	Volker Puppe: Equivariant cohomology and orbit structure
Friday	
	Session chair: Jim Carrell
9:00–10:00	Taras Panov: Complex-analytic structures on moment-angle manifolds
10:30–11:30	Nickolai Erokhovets: Buchstaber invariant of simple polytopes

3. Lectures

The following are concise synopses of lectures given during the workshop.

Dave Anderson: Transversality in equivariant cohomology

It is often useful to perturb given subvarieties of a smooth complex variety X so that they meet transversally, especially if one is interested in formulas in the cohomology ring of X. If a torus T acts on X, and the subvarieties are T-invariant, it's not obvious that this can be done while preserving the T-invariance: consider the self-intersection of the origin inside a vector space. Anderson explained a technique for producing such an "equivariant perturbation" which is based on "mixing" actions on the fibre and the base of (an approximation to) the Borel construction.

Anderson also sketched an approach via arc spaces, and he mentioned several applications: positivity theorems in equivariant cohomology and K-theory, positivity of Thom polynomials, moving lemmas for quantum cohomology, and a connection with log canonical thresholds.

Victor Batyrev: On topological invariants of Calabi–Yau 3-folds constructed by toric geometry

Victor Batyrev gave an accessible introductory talk applications on toric geometry to mirror symmetry. Using planar polygons as a working example, he explained how one can view in two different ways the combinatorial data of the polygon Δ equipped in addition with a triangulation and a piecewise-linear convex function ϕ on Δ supporting this triangulation. These two ways are "dual" in the sense that one can view the integer lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ as parametrizing either the characters or the cocharacters of the torus $(\mathbf{C}^*)^2$. Both lead to explicit geometric constructions: one to a 3-dimensional quasiprojective toric variety X_Σ together with an ample line bundle and the other to a family of smooth affine curves in $(\mathbf{C}^*)^2$ together with a degeneration into a union of affine lines. The example of the planar polygon with a triangulation shows the way to connect the complex structure of an affine curve in $(\mathbf{C}^*)^2$ with a Kähler class of X_Σ. This connection generalizes to higher dimensions. In particular, toric geometry allows us to construct many examples of Calabi-Yau 3-folds together with their mirrors, compute their Hodge (and Betti) numbers on both sides, and see a confirmation of predictions of mirror symmetry in these cases.

Tom Braden: Geometry and representation theory of hypertoric varieties

Tom Braden gave an introduction to the geometry of hypertoric varieties, which are a particularly concrete and computationally accessible example of (holomorphic) symplectic resolutions, and are analogues of toric varieties in this setting. He explained how hypertoric varieties share many important properties with other important holomorphic symplectic varieties such as cotangent bundles to flag varieties and Hilbert schemes of points on symplectic surfaces, but like toric varieties, they give a test case where many things can be computed "by hand" using elementary combinatorics. He then gave a brief overview of recent work joint with Licata, Proudfoot, and Webster, which computes the BGG category $\mathcal{O}$ of the hypertoric enveloping algebra, which is a deformation of its ring of functions.

Suyoung Choi: Torus actions and cohomology of complex Bott manifolds

A Bott manifold is a closed smooth manifold obtained as the total space of an iterated $\mathbb{P}^1$-bundle over a point, where each fibration is a projectivization of the Whitney sum of two complex line bundles. A Bott manifold is known to be a toric manifold, and hence, it admits a natural half dimensional compact torus action. A closed smooth manifold is called a cohomology Bott manifold if its cohomology ring is isomorphic to that of some Bott manifold. In his talk Choi showed that any cohomology ring isomorphism between two cohomology Bott manifolds which admit a half dimensional effective smooth compact torus action preserves their Pontrjagin classes. As a corollary, there is only a finite number of manifolds homotopy equivalent to given Bott manifold which admit a smooth action of a torus.

Michael Davis: Generalized moment angle complexes, graph products of groups and related constructions

In joint work with Boris Okun, Davis studies the cohomology of discrete groups G with certain non-trivial local coefficients. The groups G are Artin groups, Bestvina–Brady groups, or the fundamental groups of certain $K(G, 1)$ moment-angle complexes and their generalizations. The local coefficient systems are given by either (1) the integral group algebra $\mathbb{Z}[G]$, or (2) a certain completion of the real group algebra $\mathbb{R}[G]$ acting on $L^2(G)$ the Hilbert space of square summable functions on G.

The main results were natural decompositions of these cohomology groups.

Corrado De Concini: Infinitesimal index and some cohomology computations

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Given a G-manifold M with a G-equivariant one form ω we consider the zeroes M^0 of the corresponding moment map and define a map, called infinitesimal index, of $S[\mathfrak{g}^*]^G$-modules from the equivariant cohomology of M^0 with compact support to the space of invariant distributions on $\mathfrak{g}^*$.

In the case in which G is a torus, N is a linear complex representation of G, $M = T^*N$ with tautological one form we explained how this is used to compute the equivariant cohomology of M^0 with compact support using certain spaces of polynomial which appear in approximation theory.

The talk was based on joint work with C. Procesi and M. Vergne.

Graham Denham: Topological aspects of partial product spaces: a survey

The notion of a partial product space is a relatively recent unification of various combinatorial constructions in topology. This construction is variously known as the generalized moment-angle complex, or (more euphoniously) as the polyhedral product functor. Some instances of it are closely related to Davis and Januszkiewicz's quasitoric manifolds: these include the moment-angle complexes (Buchstaber and Panov) and homotopy orbit spaces for quasitoric manifolds. By making suitable choices, one also obtains classifying spaces for right-angled Artin groups and Coxeter groups, as well as certain real and complex subspace arrangements.

One advantage to this generality is that some topological information about such spaces can sometimes be expressed directly in combinatorial terms: presentations of cohomology rings; a homotopy-theoretic decomposition of the suspension of a partial product space; descriptions of rational homotopy Lie algebras and the Pontryagin algebra. I will give an introductory overview of some remarkable results along these lines.

Nickolai Erokhovets: Buchstaber invariant of simple polytopes

Surfaces and convex polytopes lie in the focus of a scientific study since antiquity. Toric topology gives a new fruitful connection between polytopes and surfaces. Namely, for a given convex simple n-dimensional polytope P with m facets there is a canonical way to build an $(m+n)$-dimensional moment-angle manifold $\mathcal{Z}_P$ with a canonical action of the torus T^m such that the topological type of $\mathcal{Z}_P$ depends only on the combinatorial type of P, and $\mathcal{Z}_P/T^m = P$. Then combinatorial properties of simple polytope P can be investigated from the point of view of the topology of $\mathcal{Z}_P$ and vice versa. For example, bigraded Betti numbers $\beta^{-i,2j}$ defined by the canonical moment-angle cell structure on $\mathcal{Z}_P$ are combinatorial invariants of P.

Definition 3.1. A Buchstaber number $s(P)$ is the maximal dimension of a torus subgroup $H \cong T^s \subset T^m$, which acts freely on $\mathcal{Z}_P$.

The problem stated by Victor M. Buchstaber in 2002 is to find a simple combinatorial description of the s-number.

From the definition it is not difficult to see that $1 \leqslant s(P) \leqslant m-n$.

A very important case is $s(P) = m-n$. Then the quotient space $M^{2n} = \mathcal{Z}_P/T^m$ is a so-called quasitoric manifold. It is known that $b_{2i}(M^{2n}) = h_i(P)$, where $\{b_{2i}\}$ are classical Betti numbers, and $(h_0, h_1, \ldots, h_n)$ is an h-vector of a polytope. However $s(P) = m-n$ is not the general case. For example, this equality is not valid for the polytope, dual to the cyclic polytope $C^n(m)$ for $m \geqslant 2^n$.

The s-number can be defined for any simplicial complex K in such a way that for a simple polytope P we have $s(P) = s(K)$, where $K = \partial P^*$ – the boundary complex of the dual simplicial polytope.

At present moment the following problems in this field are actual: to find a simple combinatorial description that gives an effective method to calculate the s-number in important special cases, to find a connection between values of $s(K)$ of different simple polytopes and complexes, to find a connection with other combinatorial invariants, to study the behavior of the s-number under factor mappings of simplicial complexes (see the definition below).

We discuss the results about $s(P)$ like the following:

(1) $s(P) = 1$ if and only if P is a simplex.
(2) For any $q \geqslant 2$ there exists a polytope P such that $m-n = q$ and $s(P) = 2$. In fact, $s(C^n(m)^*) = 2$ for $n \geqslant 13$, and
$$n + 2 \leqslant m \leqslant n + 2 + \frac{n-13}{48}$$
(3) A surjective mapping $\pi\colon \mathrm{Vert}(K_1) \to \mathrm{Vert}(K_2)$ of the sets of vertices of two simplicial complexes K_1 and K_2 is said to be a factor mapping if $\sigma \in K_2$ if and only if $\pi^{-1}(\sigma) \in K_1$. For any factor mapping $\pi\colon K_1 \to K_2$ one can define a set of all sections – mappings $\zeta\colon \mathrm{Vert}(K_2) \to \mathrm{Vert}(K_1)$ such that $\pi \circ \zeta = \mathrm{id}$. Any section induces a simplicial mapping $K_2 \to K_1$, while the map π itself does not necessary induce a simplicial mapping $K_1 \to K_2$. Using this observation one can prove that
$$s(K_2) \leqslant s(K_1) \leqslant s(K_2) + m_1 - m_2, \quad m_i = |\,\mathrm{Vert}(K_i)|$$
(4) There are two polytopes P and Q such that their face vectors $f(P)$ and $f(Q)$, and the chromatic numbers $\gamma(P)$ and $\gamma(Q)$ are equal, but $s(P) = 2$, and $s(Q) = 3$. Therefore $s(P)$ can not be calculated using only $f(P)$ and $\gamma(P)$.

We also investigate the properties of simple polytopes with $m = n+3$ vertices. In this case we calculate the bigraded cohomology ring of the moment-angle manifold $\mathcal{Z}_P$, find the value of $s(P)$, and prove that it can be expressed in terms of the bigraded Betti numbers.

Jelena Grbić: Higher Whitehead products and polyhedral product functors

In her lecture, Grbić looked at the unstable homotopy type of moment-angle complexes related to shifted complexes. In addition, she described how higher Whitehead products arise in the homotopy theory of moment-angle complexes and their relations with the loop homology of DavisJanuszkiewicz spaces.

Kiumars Kaveh: Convex bodies associated to actions of reductive groups

Kiumars Kaveh gave an introductory talk on his recent work with Khovanskii on Newton-Okounkov bodies. In toric geometry one of the fundamental correspondences is that between projective toric varieties and integral convex polytopes. The work of Kaveh-Khovanskii is a far-reaching generalization which associated to any variety a polytope Δ, obtained via semigroups of integral points. He gave many examples of connections and applications to combinatorics and algebraic geometry, such as the Hilbert function of graded algebras, the Berenstein-Kushnirenko theorem, and asymptotic theory of linear systems. In the presence of an action of a reductive group G on the variety X and a G-line bundle $L \to X$, Kaveh explained how the Newton-Okounkov polytope Δ lifts the moment polytope from symplectic geometry and is thus related to the Duistermaat-Heckman measure.

Carl Lee: Sweeping the cd-index and the toric h-vector

By sweeping a hyperplane across a simple convex d-polytope P , the h-vector, $h(P^*) = (h_0, ..., h_d)$, of its dual can be computedthe edges in P are oriented in the direction of the sweep and h_i equals the number of vertices of outdegree i. Moreover, the nonempty faces of P can be partitioned to explicitly reflect the formula for the h-vector. For a general convex polytope, in place of the h-vector, one often considers the flag f-vector and flag h-vector as well their encoding into the cd-index, and also the toric h-vector (which does not contain the full information of the flag h-vector, but provides the middle per- versity intersection homology Betti numbers of the associated toric variety when P is rational). Given a convex polytope P, we describe formulas for the cd-index of P and for the toric h-vector of P^* from a sweeping of P. These arise from analyzing Stanleys S-shelling of P^*. We describe a partition of the faces of the complete truncation of P to provide an interpretation of what the components of the cd-index are counting. One corollary is a quick way to compute the toric h-vector directly from the cd-index.

Santiago Lopez de Medrano: Intersections of quadrics and the polyhedral product functor

Santiago Lopez de Medrano (LdM) reported on earlier work of his starting in 1984 together with current developments. This early work gave beautiful connections to dynamical systems, intersections of two quadrics; these intersections are now known to be given by moment-angle complexes.

The starting point is a holomorphic flow specified by the differential equations

$$dz_i/dT = \lambda_i z_i$$

for $1 \leq i \leq n$ arising in work of Camacho, Kuiper, and Palis (CKP). LdM then introduces the complex variety determined by the intersection of two quadrics specified by the two equations

$$\sum_i \lambda_i \|z_i\|^2 = 0,$$

and

$$\sum_i \|z_i\|^2 = 1.$$

The motivation for studying the intersection of these two quadrics is the result of CKP that this intersection gives a complete invariant for distinguishing whether two such systems are equivalent. LdM gave an almost complete classification of the smooth topological type of these intersections in many cases described next.

The equations above are said to satisfy the weak hyperbolicity condition provided the complex line segments between λ_i and λ_j do not meet the origin in $\mathbb{C}$ for all $i \neq j$. LdM had proven that the intersection of these two quadrics which satisfy the weak hyperbolicity condition is given by what is now known as a moment-angle complex. In addition, he had worked out their cohomology using techniques in differential topology. This direction was then developed further in basic work of Bosio and Meersseman.

The confluence of this work together with properties of the generalized moment-angle complex (polyhedral product functor) then led to a unifying picture in work of the speaker and S. Gitler. Lopez de Medrano illustrated this part of the development of the subject as well as close connections to toric topology with natural examples, a synthesis of these different directions together with the current state of the art.

Mikiya Masuda: Cohomological rigidity problem, topological toric manifolds and face numbers of simplicial cell manifolds

Three problems were discussed in the lecture. The first was the cohomological rigidity problem for real and complex toric manifolds. This asks whether cohomology rings distinguish the diffeomorphism types of the manifolds. This question builds on his earlier work in which he showed that the equivariant cohomology of a projective smooth compact toric variety, as an algebra over the cohomology ring of the torus, distinguish the manifold as variety. The lecturer described affirmative answers in the cases of certain Bott manifolds. For real Bott manifolds, he described in addition several other "rigidity" type properties.

The second problem involved new manifolds called topological toric manifolds. These were presented as candidates for the correct topological analogues of a toric manifolds. Such manifolds have local charts consisting of patches each equivariantly diffeomorphic to a sum of one-dimensional smooth representations of $(\mathbb{C}^*)^n$. For usual toric manifolds, the representations must be algebraic. Smooth compact toric varieties are a proper subset of the set of topological toric manifolds. The manifolds produced by the construction of Davis and Januszkiewicz also are a proper subset.

The third topic discussed involved the counting of face numbers of simplicial cell manifolds. Simplicial cell manifolds arise naturally in toric topology in the way that simplicial polytopes arise in toric geometry. Criteria, related to the Dehn–Sommerville equations, were presented characterizing the h-vectors of simplicial cell spheres.

Taras Panov: Complex-analytic structures on moment-angle manifolds

Moment-angle complexes are spaces acted on by a torus and parametrised by finite simplicial complexes. They are central objects in toric topology, and currently are gaining much interest in the homotopy theory. Due the their combinatorial origins, moment-angle complexes also find applications in combinatorial geometry and commutative algebra. After an introductory part describing the general properties of moment-angle complexes we shall concentrate on the complex-analytic aspects of the theory. We show that the moment-angle manifolds corresponding to complete simplicial fans admit non-Kähler complex-analytic structures. This generalises the known construction of complex-analytic structures on polytopal moment-angle manifolds, coming from identifying them as LVM-manifolds. We proceed by describing the Dolbeault cohomology and certain Hodge numbers of moment-angle manifolds by applying the Borel spectral sequence to holomorphic principal bundles over toric varieties.

The complex-analytic part of the talk was based on the joint work with Yuri Ustinovsky, arXiv:1008.4764.

Volker Puppe: Equivariant cohomology and orbit structure

Consider an action of the torus $G = (S^1)^r$ on a "nice" space X. Let X_i be the G-equivariant i-skeleton, i.e., the union of all orbits of dimension at most i. By work of Atiyah and Bredon, the equivariant cohomology $H^*_G(X)$ (with rational coefficients) is free over the polynomial ring $R = H^*(BG)$ if and only if the following "Atiyah–Bredon sequence" is exact:

$$0 \to H_G^*(X) \to H_G^*(X_0) \to H_G^{*+1}(X_1, X_0) \to \cdots \to H_G^{*+r}(X_r, X_{r-1}) \to 0.$$

The part

$$0 \to H_G^*(X) \to H_G^*(X_0) \to H_G^{*+1}(X_1, X_0)$$

is called the "Chang–Skjelbred sequence". If $H_G^*(X)$ is free over R, the latter sequence gives a powerful way to compute $H_G^*(X)$ out of data related only to the fixed points and the one-dimensional orbits.

There are T-manifolds (in fact, smooth toric varieties) such that the Atiyah–Bredon sequence is exact from the beginning up to some term $H_G^{*+j}(X_j, X_{j-1})$, but not further. The situation changes if one restricts to compact manifolds: Allday has shown that in this case for $r = 2$ torsion-freeness of $H_G^*(X)$ (i.e., exactness at the first term) implies freeness. Franz and Puppe have given counterexamples for $r \geq 3$.

Puppe announced the following: Let X be a compact T-manifold. If the Atiyah–Bredon sequence is exact at $H_G^*(X_j, X_j - 1)$ for all $j < n/2 - 1$, then $H_G^*(X)$ is free over R. (Exactness at $j = -1$ is to be interpreted as exactness at $H_G^*(X)$.) This is a consequence of the following result: Let $H^*(AB(X))$ be the cohomology of the Atiyah–Bredon sequence, considered as a complex of R-modules and with the term $H_G^*(X)$ omitted. Then for any compact manifold X there is an isomorphism

$$H^j(AB(X)) = \operatorname{Ext}_R^j(H_G^*(X), R[m]),$$

where $R[m]$ is the graded ring R shifted upwards by the dimension m of X.

The talk was based on joint work in progress with Chris Allday and Matthias Franz.

Nigel Ray: Applications of toric methods to cobordism theory

In this excellent survey lecture, Ray presented some basic aspects of real and complex cobordism, and explained how toric and quasitoric manifolds have enhanced our understanding of certain cobordism phenomena. The story begins in 1986, and extends to current investigations into equivariant cobordism via universal toric genera. Ray successfully made the talk accessible to all participants.

Henry Schenck: Cohomology and Chow rings of toric varieties

In this talk, Hal Schenck began with an overview of results on the cohomology and Chow rings of toric varieties. For smooth varieties, Jurkiewicz (projective) and Danilov (complete) gave the first description of these rings in the late 1970's; both with integral coefficients.

In 1990 Bifet, De Concini and Procesi showed that the integral equivariant cohomology ring $H_T^*(X_\Sigma)$ of a smooth toric variety X_Σ is isomorphic to the integral Stanley-Reisner ring A_Σ of the unimodular fan Σ, and Brion later showed that for Σ simplicial, the rational equivariant Chow ring $A_T^*(X_\Sigma)_\mathbb{Q}$ is isomorphic to the ring of rational piecewise polynomial functions $C^0(\Sigma)_\mathbb{Q}$. A result of Billera shows that for a simplicial fan, $C^0(\Sigma)_\mathbb{Q}$ is isomorphic to the rational Stanley-Reisner ring of the fan, so Brion's result is similar in spirit to that of Bifet–De Concini–Procesi. Brion and Vergne completed the picture for the simplicial case by showing that for complete, simplicial torics, $H_T^*(X_\Sigma) \simeq A_T^*(X_\Sigma)_\mathbb{Q}$.

In 2006 Payne proved that the integral, equivariant Chow ring is isomorphic to the ring of integral piecewise polynomial functions on Σ. This opens the way to analyze the Chow ring as a graded module over the polynomial ring, or as a coherent sheaf on projective space. The second part of the talk described Billera's construction of a certain chain complex to answer a question of Strang on the dimension of $C^1(\Sigma)_\mathbb{Q}$ and a famous open problem in approximation theory, which can be phrased as bounding the Castelnuovo-Mumford regularity of a certain vector bundle on the projective plane. The talk closed with a brief discussion of some recent results involving C^0 splines on polyhedral complexes (hence, the Chow ring of nonsimplicial toric varieties), using the Cartan-Eilenberg spectral sequence to obtain sufficient conditions for freeness of $C^0(\Sigma)_\mathbb{Q}$ as a module over the polynomial ring. For certain fans, there is also a surprising connection with reflection arrangements of type A and logarithmic vector fields tangent to the arrangement.

Susan Tolman: The integral cohomology of GKM spaces

Let $T = (S^1)^n$ be a torus. A GKM space is a compact Hamiltonian T-space M such that the fixed point set M^T is finite and $\dim M^K \leq 2$ for all subtori $K \subset T$ of codimension 1. Examples of GKM spaces are complex flag manifolds and toric manifolds.

Tolman recalled Kirwan's injectivity theorem and the description of the equivariant cohomology $H^*_T(M;\mathbb{R})$ by data related to M^T and the M^K as above (Chang–Skjelbred, Goresky–Kottwitz–MacPherson). The latter gives a straight-forward way to compute the real cohomology of GKM spaces in terms of a graph whose vertices are the fixed points and whose edges are given by the M^K which are spheres. The edges are labelled by weights indicating K and the speed with with the circle T/K rotates this sphere.

Unfortunately, this formula does not work for integral cohomology. Extension of the methods due to Chang–Skjelbred and GKM have been studied by Tolman–Weitsman and Franz–Puppe.

The main theorem announced by Tolman is the following: The GKM graph determines $H^*_T(M;\mathbb{Z})$ in the following sense: If another GKM space M' gives the same GKM graph, then $H^*_T(M;\mathbb{Z}) \cong H^*_T(M';\mathbb{Z})$ as $H^*(BT;\mathbb{Z})$-algebras, and their images in $H^*(M^T \cong H^*((M')^T)$ are the same. (Note that the isomorphism $H^*(M^T) \cong H^*((M')^T)$ comes from the isomorphism of graphs.)

This leads to two questions:

- Franz–Puppe prove their result for a much broader class than GKM spaces. Does the theorem mentioned above also hold in this setting?
- What is the most effective way to compute $H^*_T(M;\mathbb{Z})$ out of the GKM graph?

This talk was based on a work in progress.

Julianna Tymoczko: Computational tools in torus-equivariant cohomology with applications to Schubert calculus

Julianna Tymoczko gave a survey talk on computational techniques in torus-equivariant cohomology with particular emphasis on applications in geometric representation theory and Schubert calculus. She introduced GKM (Goresky-Kottwitz-MacPherson) theory, which gives an explicit and combinatorial computation of torus-equivariant cohomology rings of T-spaces M under certain conditions on the orbit stratification of M. Many classical varieties occurring in geometric representation theory, Schubert calculus, and equivariant algebraic geometry are GKM spaces: examples include the flag variety $\mathcal{F}\ell ags(\mathbf{C}^n)$, Grassmannians $Gr(k, \mathbf{C}^n)$, and more generally the Kac-Moody homogeneous spaces G/B or G/P, as well as projective toric varieties. After a quick overview of the GKM description of $H^*_T(M;\mathbf{C})$ using the so-called “GKM graph”, she went on to describe several modern developments in the theory. First, she described ways to construct of H^*_T-module bases for $H^*_T(M;\mathbf{C})$ which are upper-triangular in a certain sense with respect to the GKM graph. Second, she constructed group representations on equivariant cohomology rings via graph automorphisms. Third, she briefly described how GKM theory (and good module bases) could be applied to derive Schubert calculus results.

All lectures were extremely well presented and gave clear indication to all participants that they could join the lecturers in future joint work. We are convinced that this will happen soon.

4. Developments, and mathematical scientific progress

This section is an exposition of a few of the prospects and new developments which arose during the conference.

(1) Mike Davis considered local systems together with the cohomology of certain discrete groups with coefficients in this representation arising as the fundamental group of a moment-angle complex. During the conference, the connection arose that this computation may be given by that of the cohomology of a certain

natural fibre bundle arising by "twisting" a moment-angle complex, thus giving a "parametrized version" of a moment-angle complex, extending useful constructions of many of the conference participants. The geometry and topology of the resulting bundles then give a natural setting for the associated cohomological decompositions.

(2) Mario Salvetti and Fred Cohen considered a computation of certain torsion which is an extension of a classical one carried out by Eichler and Shimura concerning modular forms in the torsion-free case. The results of these computations seem to give the torsion in the cohomology of certain fibrations in a different subject known as Anick's fibrations. This unexpected computation and connection is now an on-going project.

5. Feedback and comments from participants

We record below a sample of the very positive feedback we received from workshop participants.

- Thanks for organising such a fantastic workshop. (Jelena Grbić)
- Thanks soooooooooooooo much for a great conference. (Henry Schenck)
- thanks for the invitation and splendid organizing (Alejandro Adem)
- Thanks again for the wonderful week! It was an exceedingly interesting and pleasant conference. (Thomas)
- The Banff conference was great! (Volker Puppe)
- Once again, many thanks for organising such a wonderful conference. (Taras Panov)
- I would like to thank you for organizing the wonderful conference. (Mikiya Masuda)
- Thank you for giving me the opportunity to attend this very stimulating meeting! (Carl Lee)
- I was able to thank some of you in person, but I wanted to write to say thanks again for organizing such a fantastic workshop last week. The mix of people, topics, and perspectives made it one of the best I've ever been to. Coming from algebraic geometry, it was especially interesting to get exposure to the topologists' point of view on toric things.(Dave Anderson)
- Thank you again for the nice week in Banff. (Mario Salvetti)
- I greatly appreciate your kindness for inviting me to the wonderful conference. Thanks to you, I got a very pleasant experience. I learned many interesting mathematics and made many friends. (Yasuhiko Kamiyama)
- The conference was so nice, and it stimulated me to do my best. (Suyoung Choi)