Geometric Analysis and General Relativity

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1 Overview of the Field

General Relativity is one of the fundamental theories of modern physics. Since its formulation by Einstein in 1915, it has been a cornerstone of our understanding of the universe. Mathematical research on the problems of general relativity brings together many important areas of research in partial differential equations, differential geometry, dynamical systems and analysis. The fundamental mathematical questions which arise in general relativity have stimulated the development of substantial new tools in analysis and geometry. Conversely, the introduction of modern tools from these fields have recently led to significant developments in general relativity, shedding new light on a number of deep and far reaching conjectures.

The area of interaction between the analysis of the Lorentzian Einstein equation, the field equation of general relativity, and other geometric partial differential equations is large. Important classes of hyperbolic equations such as the wave maps equation and the Yang-Mills equation are connected with the Einstein equation at a fundamental level, and harmonic analysis methods used in the study of those equations are being applied to the Einstein equation. Fluids as well as kinetic models such as the Vlasov equation appear as matter models.

The Einstein equation shares issues of convergence, collapse and stability with important geometric evolution equations such as the Ricci flow and the mean curvature flow. Asymptotic behavior at singularities as well as questions related to asymptotic geometrization are being intensely studied in the case of the Einstein equation as well as in the cases of the Ricci and the mean curvature flows. There is the potential for continued cross fertilization between these fields. Elliptic problems and techniques arise in studying the initial data sets in the context of the Cauchy problem.

The analysis of the Riemannian Einstein equation plays a fundamental role in modern Riemannian geometry and geometric analysis. Recent developments in General Relativity have brought many ideas and techniques which have been developed in the Riemannian context to bear on the Lorentzian Einstein equations. An example of this type of development is the realization that apparent horizons, which are relevant for our understanding of quasilocal aspects of black holds, have many properties in common with minimal surfaces. Many recent results in this area have been inspired in part by this connection. Other areas of geometry which have played a significant role in general relativity in recent years include the geometrization of three-manifolds; convergence and collapsing phenomena in Riemannian manifolds with bounds on their curvatures, the study of manifolds admitting metrics of positive scalar curvature, and curvature flows for hypersurfaces.

2 Recent Developments and Open Problems

One of the fundamental mathematical results in relativity is the establishment of the global nonlinear stability of Minkowski space due to Christadoulou and Klainerman. Important recent work by Lindblad and Rodnianski has provided an alternative treatment of this. One of the most outstanding problems in the field concerns the analogous question for black hole spacetimes, namely the nonlinear stability of the Kerr black holes. While this problem is unlikely to be resolved in the near future a number of recent results are relevant to it and will likely be part of the ultimate resolution. This includes the analysis of the interior of black holes for certain spherically symmetric non-vacuum spacetimes due to Dafermos; and the analysis of the decay for solutions of the wave equation and Price Law decay within black hole spacetimes due to Dafermos and Rodnianski. Stability results have also been recently established for $U(1)$-symmetric spacetimes by Choquet-Bruhat and for certain hyperbolic models by Andersson and Moncrief.

Intimately related to these questions are the Cosmic Censorship Conjectures due to Penrose, which relate to the presence of naked singularities and the fundamental issue of determinism for the physical model posed by the Einstein field equations. The recent verification of the Strong Cosmic Censorship for the Gowdy spacetimes by Ringström represents a significant positive result in this area.

Christodoulou has proved that a concentration of radiation can lead to the formation of closed trapped surfaces in vacuum, in particular assuming the standard weak cosmic censorship schenario this leads to formation of black holes arising solely from the focusing of sufficiently strong incoming gravitational waves. This result and the so-called short-pulse method it relies on can be expected to lead to important advances and there is already related work by Klainerman and Rodnianski as well as others. This work complements the earlier work of Schoen and Yau, and others on the formation of black holes due to concentration of matter. There remain important open problems in this direction.

Other important recent advances concerning the Cauchy problem for the Einstein equations have come from Klainerman-Rodnianski and Szeftel on well-posedness and extention criteria for the vacuum equations and from Planchon-Rodnianski on the question of uniqueness without a loss in regularity.

The quasi-local model for black holes alluded to above is developed in the context of the study of marginally outer trapped tubes. This notion was introduced by Hayward and subsequently developed by Ashtekar and Krishnan. Significant recent mathematical work in this area has been done by Andersson, Dafermos, Galloway, Mars, Metzger, Schoen and Simon. New PhDs such as Eichmair and Williams have recently emerged into the field having done work in this area.

Global inequalities have played a large role in general relativity, going back at least as far as the proof the positive mass theorem due to Schoen and Yau. An important generalization of this was recently obtained via the proofs of the Riemannian Penrose inequality due to Huisken-Ilmanen and Bray. The long standing attempt to generalize the notion of mass in relativity to one which is locally defined has seen new ideas introduced due to the work of Shi and Tam and the subsequent results of Liu-Yau and Wang-Yau. Recognizing the need for low regularity formulations of the fundamental quantities in relativity Huisken has recently developed formulations of these global inequalities in terms of isoparametric inequalities which hold in a very rough setting.

The analysis of initial data sets and the implications of this for their spacetime developments has been the subject of a great deal of recent attention. Many mathematical developments in this area stem from the introduction of gluing techniques from geometric analysis. Corvino's construction of vacuum spacetimes which are Schwarzschild outside of a compact set, and the extension of this method by Corvino-Schoen and Chruściel-Delay have had a significant impact. The combination of this with conformal gluing methods by Chruściel, Isenberg and Pollack have led to examples of vacuum spacetimes with no constant mean curvature slices. Building on work of Holst, Nagy and Tsogtgerel, Maxwell has very recently produced the first general existence result for the vacuum constraint equations with no restrictions on the mean curvature. Nonetheless the issue of understanding the basic questions of the existence and uniqueness for solutions of the Einstein constraint equations remains largely open in general for initial data sets with arbitrary mean curvature.

3 Presentation Highlights

The talks for the meeting were solicited in two ways. The organizers asked all of the participants to send us a title and abstract if they wanted to give a talk. In addition we asked specific participants, whose recent work was known to be of widespread interest, to give talks. All daytime talks were scheduled for 50 minutes, to allow sufficient time afterward for questions and discussion. Rather than fill every available slot with talks, we made an effort to preserve a good deal of time for informal discussion and collaboration. This was very successful and groups formed spontaneously in the lecture room and lounge as well as in outdoor excursions around BIRS.

One particularly successful aspect of the meeting were the four post-dinner talks which we arranged to be given by Gerhard Huisken, Richard Schoen, Igor Rodnianski and Shing-Tung Yau. These hour long talks by well established leaders in the field were meant to introduce the background and most recent developments in primary and active areas of study within Geometric Analysis and General Relativity. This led to four wonderful talks, which were very much appreciated by all participants.

A summary of each of the talks given at the meeting follows.

Mihalis Dafermos: Superradiance, trapping, and decay for waves on Kerr spacetimes in the general subextremal case $|a| < M$

Mihalis Dafermos spoke on his joint work with Igor Rodnianski on boundedness and decay estimates for the free scalar wave equation on the exterior Kerr spacetime. A proof of boundedness and decay for the scalar wave equation on the Kerr exterior spacetime is an important test case for the black hole stability problem, a special case of which is the problem of non-linear stability in vacuum for the Kerr black hole spacetime. The black hole stability problem is one of the central open mathematical problems motivated general relativity. Recent work of Dafermos and Rodnianski, Tataru and collaborators as well as Andersson and Blue has provided such estimates for the case of a slow rotating Kerr spacetime, i.e. $|a| \ll M$. This case is characterised by the fact that superradiance can be treated as a small parameter. In his talk, Dafermos reported on his recent joint work with Rodnianski proving decay in the general subextreme case $|a| < M$. Key to this approach is the construction of distinct multiplier estimates taylored to the superradiant and trapped frequency ranges, respectively. The approach exploits in particular the insight that for the entire subextremal range, superradiant frequencies are not trapped in the high frequency limit.

Gustav Holzegel: Asymptotic behavior of spacetimes approaching a Schwarzschild solution

In his talk, Gustav Holzegel considered the problem of proving, given a vacuum spacetime which approaches a Schwarzschild solution, decay to the future for the spacetime curvature. In particular, assuming decay of appropriate norms of the Ricci rotation coefficients and their derivatives, he proved boundedness and decay for the curvature components and their derivatives. Some important difficulties arise from the fact that not all curvature components decay. An important ingredient in this work is to generalize recent work of Dafermos and Rodnianski regarding decay for the wave equation to the setting of the Bianchi equations.

Alan Rendall: Higher dimensional cosmological models

Alan Rendall presented some joint results with Arne Goedeke on solutions of the vacuum Einstein equations in dimensions greater than four. The central question discussed is whether spatially homogeneous models which are forever expanding are geodesically complete in the future. This fact is known to hold in four dimensions, but the proof does not directly generalize to the higher dimensional case. Rendall introduced a sufficient condition for completeness and showed that it is satisfied in a class of models of dimension five by means of Kaluza-Klein reduction. The talk was concluded by a discussion of the prospects for obtaining a more global understanding of this problem.

Hans Ringström: Models of the universe with arbitrary compact spatial topology

The current standard model of the universe is spatially homogeneous, isotropic and spatially flat. Furthermore, the matter content is described by two perfect fluids (dust and radiation) and there is a positive cosmological constant. Such a model can be well approximated by a solution to the Einstein- Vlasov equations with a positive cosmological constant. Motivated by this observation, Hans Ringström has studied the properties of cosmological models with Vlasov matter and a positive cosmological constants. The results described in the talk included far-reaching statements on the stability of such models, as well as a proof that the assumption that the universe is close in our past to the standard Friedmann model does not restrict the spatial topology of the universe.

Shing-Tung Yau: Quasi-local mass in general relativity

Starting with a background for the notion energy and momentum in general relativity, Yau discussed his approach to the problem of defining a quasi-local mass in general relativity, which has been developed in collaboration with Mu-Tao Wang and Po-Ning Chen, based on earlier work of Shi and Tam, as well as Liu and Yau. An important background to the definition of quasi-local mass introduced by Yau is the Brown-York analysis of the boundary terms in in the gravitational Lagrangian. The above mentioned quasi-local mass definitions all rely on an imbedding of a 2-sphere in a reference space which gives a quasi-local mass notion via the Brown-York analysis after making a gauge choice. The novel idea in the approach of Wang and Yau is to make use of an optimal imbedding of the 2-sphere in Minkowski space. This essentially reduces the gauge freedom in the definition of the quasi-local mass to a choice of a vector in the reference Minkowski space, a notion related to a choice of observer. The talk of M-T Wang gave further details of this construction.

Mu-Tao Wang: Limit of quasilocal mass and isometric embeddings into Minkowski space

In his talk based on joint work with S.-T. Yau, Mu-Tao Wang discussed how the limit of quasilocal mass on a family of surfaces in spacetime can be evaluated in terms of the mean curvature vectors and showed that this gives a uniform description of ADM mass and Bond mass for asymptotically flat and asymptotically null spaces, respectively. He then explained how the related variational problem for quasilocal mass anchors a ground state as a hypersurface in Minkowski space.

Marcus Khuri: The Static Metric Extension Conjecture

There are several competing definitions of quasi-local mass in General Relativity. A very promising and natural candidate, proposed by R. Bartnik, seeks to localize the total or ADM mass. Fundamental to understanding Bartniks construction is the question of existence and uniqueness for a canonical geometric boundary value problem associated with the static vacuum Einstein equations. In his talk, Marcus Khuri discussed joint work with Michael Anderson, which confirms that existence holds (under a nondegeneracy condition) but also shows that uniqueness fails. He concluded by discussing the possible implications of this result.

Marc Mars: The Bray and Khuri approach to the general Penrose inequality in two particularly simple cases

Bray and Khuri have put forward an interesting new approach to address the Penrose inequality for arbitrary initial data sets. The two simplest possible situations where one can think of applying these results involve spherically symmetric initial data and static initial data. Restricting to the spherical case, Marc Mars was able to extend previous results by Bray and Khuri when the outermost horizon is strictly stable and non-minimal to the general case. For the static case, he showed that there are slices of the Kruskal spacetime with mass m for which the outermost generalized apparent horizon has area strictly larger than $16\pi m^2$ and presented a Penrose inequality for general static initial data sets satisfying the dominant energy condition, provided the mean curvature of the degenerate components satisfies an integral inequality.

Frans Pretorius: The instability of 5-dimensional black strings

5 dimensional black strings were shown to be unstable to long-wavelength perturbations by Gregory and Laflamme. Entropy considerations imply the prefered end-state of the unstable spacetime is a sequence of black holes with spherical topology. For this to happen, the black string event horizon would have to bifurcate, accompanied by a naked singularity. This would be an example of generic violation of cosmic censorship in 5 dimensional Einstein gravity. Frans Pretorius presented recent numerical results, joint with L. Lehner, which aim at illucidating the end-state of the Gregory-Laflamme instability.

Piotr Chruściel: 5-dimensional black holes

Piotr Chruściel started by discussing the current status of uniqueness results for 3+1 stationary electrovac black hole spacetimes. He then gave a review of some higher dimensional black hole spacetimes and our knowledge of their global structures, including those of Myers-Perry (generalization of Kerr), Emparan-Reall and Pomeransky-Senkov (black ring) and Elvang-Figueras (Black Saturn). He then reported on recent work, joint with Cortier and Garcia-Parrado, and with Eckstein and Szybka on the global properties of higher dimensional black hole spacetimes. In particular, this work shows that the Pomeransky-Senkov and Black Saturn spacetimes are singularity-free in the exterior of the horizon, and gives an analytical extension which is a candidate for a maximal extension.

Gerhard Huisken: Foliations, flows and rigidity in asymptotically flat 3-manifolds

The talk of Gerhard Huisken was centered around the inverse mean curvature flow, the mean curvature flow and their role in proving geometric inequalities involving such notions as the isoperimetric ratio, quasi-local mass (eg. Geroch or Hawking mass) and the ADM mass defined at infinity. The just mentioned geometric flows define canonical foliations in 3-manifolds with interesting monotonicity properties which play a central role in the proofs of geometric inequalities. A prime example of such an application is the proof by Huisken and Illmanen of the Riemannian Penrose inequality.

David Maxwell: The Conformal Method and Concrete Examples

Recent advances by Holst, et. al and Maxwell concerning the conformal method and non-CMC initial data might have led an optimist to conjecture that the conformal method could be just as effective for constructing non-CMC data as it is in the CMC case. In his talk, David Maxwell discussed some concrete examples that indicate that this is not true. He showed that for certain reasonable conformal data violating a near-CMC condition there cannot be a unique solution: there are either no solutions or more than one. For some examples, he was able to establish that there exist multiple solutions. These concrete examples are independently interesting as they exhibit a number of new phenomena for the conformal method, including existence of certain solutions under a very weak near-CMC hypotheses, explicit dependence on the choice of conformal class representative, and extreme sensitivity of the solution theory with respect to a coupling constant in the Einstein constraints.

Hubert Bray: On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity

Hubert Bray introduced geometric axioms for the metric and the connection of a spacetime where the gravitational influence of the connection may be interpreted as dark matter. In particular he showed how these axioms lead to the Einstein-Klein-Gordon equations with a cosmological constant, where the scalar field of the Klein-Gordon equation represents the deviation of the connection from the standard Levi-Civita connection on the tangent bundle and is interpreted as dark matter. This form of dark matter is compatible with the CDM cosmological model of the universe. In addition, unlike the WIMP model of dark matter, this dark matter is automatically cold (as is observed) in a homogeneous, isotropic universe. He concluded by showing how this scalar field dark matter, which naturally forms dark matter density waves due to its wave nature, may cause the observed barred spiral pattern density waves in many disk galaxies and triaxial shapes with plausible brightness profiles in many elliptical galaxies. If correct, this would provide a unified explanation for spirals and bars in spiral galaxies and for the brightness profiles of elliptical galaxies. The results of preliminary computer simulations were shown and compared with photos of actual galaxies.

Richard Schoen: Singularities in positive mass arguments

Richard Schoen discussed the obstructions to proving positive mass theorems in the presence of singularities. There have been a few instances when singularities have been shown to be allowable, but there is no comprehensive characterization of them.

Spyridon Alexakis: On the black hole uniqueness problem

The classical uniqueness theorem for stationary black hole spacetimes due to Carter and Robinson relies on the existence of a second, rotational, Killing field. A construction by Hawking yields such a Killing field under the assumption that the spacetime is analytic. However, until recently the existence of the Hawking Killing field for stationary black hole spacetimes containing a bifurcate Killing horizon has been open. In recent joint work with Ionescu and Klainerman, Alexakis has established the existence of a Hawking Killing field without assuming analyticity. The current version of the black hole uniqueness theorem requires a smallness assumption and is therefore valid only for spacetimes close, in a certain sense, to the Kerr spacetime. One of the main ideas in the proof is the use of an inequality of Carleman type to prove a uniqueness result for a wave equation on the exterior of the horizon, given data on the horizon.

Sergio Dain: Linear perturbations and mass conservation for axisymmetric Einstein equations

In axial symmetry, there exists a gauge for Einstein equations such that the total mass of the spacetime can be written as a conserved, positive definite, integral on the spacelike slices. This property can be expected to play an important role in the global evolution. In this gauge the equations reduce to a coupled hyperbolic-elliptic system which is formally singular at the axis. Due to this singular behavior, the local in time existence of this system can not be analyzed by standard methods. In his talk, based on joint work with Martin Reiris, Sergio Dain studied the linear perturbation of the flat solution in this gauge (with the purpose of analyzing the principal part of the equations, which represents the main source of the difficulties), and proved existence and uniqueness of solutions of this singular linear system. The solutions are obtained in terms of integral transformations in a remarkable simple form. This representation is suitable for proving useful estimates for the non-linear case. This result is expected open the door to the study of the full Einstein equations in this gauge.

Lydia Bieri: Null Asymptotics of Solutions of the Einstein-Maxwell Equations in General Relativity and Gravitational Radiation

A major goal of mathematical general relativity and astrophysics is to precisely describe and finally observe gravitational radiation, one of the predictions of general relativity. In order to do so, one has to study the null asymptotic limits of the spacetimes for typical sources, including binary neutron stars and binary black hole mergers. In these processes typically mass and momenta are radiated away in form of gravitational waves. Demetrios Christodoulou showed that every gravitational-wave burst has a nonlinear memory. In her talk, Lydia Bieri discussed the null asymptotics for spacetimes solving the Einstein-Maxwell equations, computed the radiated energy, and derive limits at null infinity. These limits were compared with the Einstein vacuum case. The methods used were introduced in the works of Christodoulou, Klainerman, Bieri and Zipser.

Pieter Blue: Hidden symmetries and decay for the wave equation outside a Kerr black hole

The Kerr solutions to Einstein's equations describe rotating black holes. For the wave equation in flat-space and outside the non-rotating, Schwarzschild black holes, one method for proving decay is the vector-field method, which uses the energy-momentum tensor and vector-fields. Outside the Schwarzschild black hole, a key intermediate step in proving decay involved proving a Morawetz estimate using a vector-field which pointed away from the photon sphere, where null geodesics orbit the black hole. Outside the Kerr black hole, the photon orbits have a more complicated structure. Pieter Blue, in a talk based on joint work with Lars Andersson, showed that by using the hidden symmetry of Kerr, it is possible to replace the Morawetz vector-field by a fifth-order operator which, in an appropriate sense, points away from the photon orbits. This allows one to prove the necessary Morawetz estimate, also called a local energy decay estimate, which is a key step in proving pointwise decay estimates.

Igor Rodnianski: On formation of trapped surfaces

Igor Rodnianski discussed joint work with Sergiu Klainerman which extends and refines the results of Demetrios Christodoulou on the formation of trapped surfaces due to the concentration of radiation. The result of Christodoulou shows that the existence of sufficiently strong pulses of radiation, entering from a finite interval of retarded time on past null infinity, results in the formation of a trapped surface. The proof requires the incoming pulse of radiation to be highly isotropic. One of the new features of the work of Klainerman and Rodnianski is that it handles the situation where the incoming radiation is anisotropic. In particular, conditions are given where an incoming pulse supported in only a part of the cross-sections of past null infinity can be shown to form a spacetime containing a trapped surface and more generally a "scarred surface". This work may be expected to be relevant to understanding break-down phenomena for solutions of the Einstein equations with low regularity.

Michael Eichmair: Some old and some new results on MOTS and Jang's equation

In this talk, Michael Eichmair presented a synthetic overview of the geometric theory of Jang's equation, pioneered by R. Schoen and S.-T. Yau in their proof of the spacetime positive energy theorem, and its recent application to the existence theory of marginally outer trapped surfaces in initial data sets by L. Andersson, J. Metzger, and himself. He discussed recent joint work with J. Metzger on the mixed blow up behavior of Jang's equation in certain exterior domains of initial data sets. These results are akin to the classical Jenkins-Serrin-Spruck theory for minimal and constant mean curvature graphs.