CONVERGENCE OF LOOP-ERASED RANDOM WALK TO SLE(2) IN THE NATURAL PARAMETRIZATION

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1 Overview of the Field

The area of our research project is probability and statistical mechanics. More specifically, we study the loop-erased random walk (LERW) – a prominent discrete model in statistical mechanics – and its continuous scaling limit, Schramm-Loewner evolution (SLE) with parameter 2. The LERW was invented by Greg Lawler [3] in order to study the self-avoiding walk (SAW). While that approach did not turn out to be fruitful for analyzing the SAW, the LERW is extensively studied today, most notably for its intimate connection to the uniform spanning tree through Wilson's algorithm.

SLE was invented by Oded Schramm [7] as a candidate for the scaling limit of LERW. Under the assumptions of conformal invariance of the scaling limit (which was conjectured to be true for the LERW) and a "domain Markov property" (which is easy to show for the LERW), Schramm proved that the only possible scaling limit was SLE with parameter κ: a random curve satisfying the Loewner equation with a Brownian motion of variance κ as its driving function. In his original paper, Schramm determined that if LERW had a conformally invariant scaling limit, one would have to have $\kappa = 2$. Then, in a later paper [4], Lawler, Schramm and Werner verified that LERW does indeed converge to SLE(2). In addition, many other prominent models from statistical physics such as the uniform spanning tree, the self-avoiding walk, the Ising model at criticality, the Gaussian free field, and critical percolation contain discrete curves that scale or are conjectured to scale to SLE(κ) for various values of κ. Establishing that SLE is the scaling limit of these models rigorously confirms many of the predictions that had previously been made using conformal field theory.

2 Goal of the Research Project

In the paper [4], Lawler, Schramm and Werner proved the weak convergence of LERW to SLE(2) with respect to the supremum norm on curves modulo reparametrization. The goal of our project is to prove weak convergence in the stronger topology that takes into account the parametrization of the LERW. Namely, if we let X^n be the LERW from the origin to the unit circle on the lattice $(1/n)\mathbb{Z}^2$ and M_n be the number of steps of X^n, then one expects that $Y_n(t) = X^n(\mathbf{E}[M_n]t)$ should converge weakly (as n tends to infinity) in the supremum norm to a suitably parametrized version of SLE(2). Although other models have been shown to scale to SLE, none of them have been proved to converge as parametrized curves.

3 Recent Developments

There are two recent developments that make this problem appear tractable. The first is the identification of what the suitable parametrization for the SLE(2) curve should be. In the original definition of SLE by Schramm, the SLE curves were parametrized so that their capacity (a measure of how big the curves look in the unit disc when viewed from the origin) grew linearly. This was the best way to analyze the curves by way of the Loewner equation but is not natural when one considers the SLE curves as scaling limits of discrete models. Indeed, Beffara showed that the Hausdorff dimension of SLE(κ) is $d = 1 + \kappa/8$ almost surely ($\kappa \leq 8$). This suggests that for a discrete model to converge to SLE(κ) as a parametrized curve, the parametrization on the SLE(κ) curve should be such that scaling the curve by a factor of r in space is equivalent to scaling by a factor of r^d in time. This "natural parametrization" of SLE has recently been shown to exist by Lawler and Sheffield [5]. It is SLE(2) in this parametrization that one expects the LERW Y_n defined above to scale to. Note that $d = 5/4$ for SLE(2), and the fact that $\mathbf{E}[M_n]$ grows like $n^{5/4}$ was established by Kenyon.

The second result that will be useful for this problem is a tail bound on M_n. Recent work by Barlow and Masson [1] gives both upper and lower exponential tail bounds on M_n. As we describe below, this allows us to establish a tightness result that gives subsequential weak limits of LERW in the topology induced by the supremum norm.

4 Scientific Progress Made

The natural parametrization for SLE(2) defined by Lawler and Sheffield [5] is more easily described in terms of a random Borel measure μ on $\mathbb{D}$. The measure μ is supported on the SLE(2) curve and in essence gives the amount of time that the curve (in the natural parametrization) spends in each subset of $\mathbb{D}$. By the conformal invariance and the domain Markov property of SLE(2) one expects that it should satisfy the following.

1. μ is measurable with respect to the trace of the SLE(2) curve.
2. μ is almost surely supported on the trace of the SLE(2) curve.
3. $\mathbf{E}[d\mu(z)] = G(z)\,dz$ where $G(z)$ is the "Green's function" for SLE(2) in $\mathbb{D}$.
4. Given any parametrization for SLE(2), $\mu(\cdot \cap \gamma[0,t])$ is $\gamma[0,t]$ measurable.
5. $\mathbf{E}[d\mu(z)|\gamma[0,t]] = |g_t'(z)|^{3/4} G(g_t(z))$ for $z \in \mathbb{D} \setminus \gamma[0,t]$, where g_t is the unique conformal map from $\mathbb{D} \setminus \gamma[0,t]$ to $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$.

The exponent $3/4$ arises from the fact that the dimension of SLE(2) is $5/4$.

Lawler and Sheffield [5] proved that a measure satisfying these properties exists. Moreover, this measure is unique. Given μ and an SLE(2) curve $\gamma(t)$ in any parametrization, one sets $\Theta(t) = \mu(\gamma[0,t])$, and then defines SLE(2) in the natural parametrization by

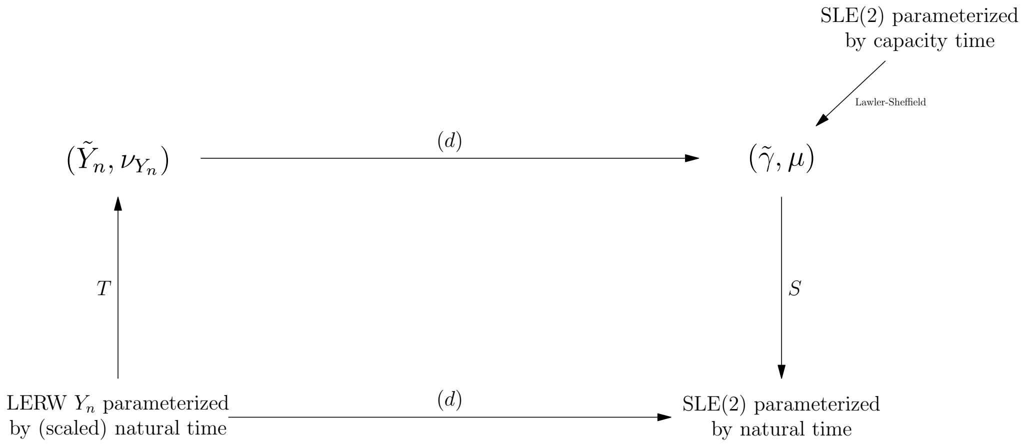
$$\gamma^*(t) = \gamma(\Theta^{-1}(t)). \tag{$*$}$$

Hence, the key step in constructing the natural parametrization is the construction of the natural measure μ.

With this in mind, we show that convergence of the natural measure for LERW to the natural measure for SLE(2) implies the convergence of the associated curves in their naturally parametrized form. Given the LERW Y_n, its natural measure is defined to be

$$\nu_{Y_n}(A) = \int_0^\infty \mathbf{1}\left\{Y_n(t) \in A\right\}\,dt. \tag{$**$}$$

Our goal is to prove that ν_{Y_n} converges weakly to μ with respect to the Prokhorov topology on measures. As summarized in the diagram below, this fact plus Lawler, Schramm and Werner's result that the LERW converges weakly to SLE(2) modulo reparametrization implies the same convergence but in the natural parametrization.



Here $\widetilde{Y}_n$ and $\widetilde{\gamma}$ are the equivalence classes of the LERW and SLE(2) curves modulo reparametrization. The mapping T takes the parametrized LERWs Y_n into the pair consisting of the equivalence class and the occupation measure $(**)$, while S maps equivalence classes of curves and occupation measures into parametrized curves via $(*)$. The map S is continuous at almost all pairs $(\widetilde{Y}, \mu)$, and therefore weak convergence on the top level of the diagram implies weak convergence on the bottom level.

Convergence on the top level follows the usual argument: we show that the pair $(\widetilde{Y}_n, \nu_{Y_n})$ is tight, and that any subsequential weak limit must have the law of $(\widetilde{\gamma}, \mu)$. Tightness of the $\widetilde{Y}_n$ follows from [4], while tightness of ν_{Y_n} is a consequence of the estimate

$$\mathbb{P}\left(\alpha^{-1} \leq \frac{M_n}{\mathbf{E}[M_n]} \leq \alpha\right) \geq 1 - Ce^{-c\alpha^{1/2}}$$

that is due to Barlow and Masson [1].

It remains to show that any subsequential weak limit of $(\widetilde{Y}_n, \nu_{Y_n})$ satisfies the properties 1 through 5 that uniquely define μ. Of these, property 2 is the most straightforward to verify; it follows readily from the fact that ν_{Y_n} is supported on $\widetilde{Y}_n$. Properties 1 and 4 are nontrivial and do not follow from general facts about weak convergence. Some extra information is required and at this point we are not entirely sure what that is.

Properties 3 and 5 can be established once the following conjectures are proved.

Conjecture 4.1. *For all $z \in \mathbb{D}$ and $\epsilon > 0$ sufficiently small,*

$$\mathbf{E}\left[\nu_{Y_n}(B(z,\epsilon)) \middle| Y_n \cap B(z,\epsilon) \neq \emptyset\right] = \frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} + o(1)$$

as $n \to \infty$.

This implies property 3 by the following argument.

$$\begin{aligned} \mathbf{E}\left[\mu(B(z,\epsilon))\right] &= \lim_{n\to\infty} \mathbf{E}\left[\nu_{Y_n}(B(z,\epsilon))\right] \\ &= \lim_{n\to\infty}\left[\frac{\mathbf{E}\left[M_{\epsilon n}\right]}{\mathbf{E}\left[M_n\right]} + o(1)\right]\mathbb{P}\left(Y_n \cap B(z,\epsilon) \neq \emptyset\right) \\ &= \epsilon^{5/4}\mathbb{P}\left(\gamma \cap B(z,\epsilon) \neq \emptyset\right) \\ &\sim \epsilon^2 G(z). \end{aligned}$$

The second to last line follows from [4] while the last line is the definition of $G(z)$.

Conjecture 4.2. *Suppose that D and D' are simply connected domains in $\mathbb{C}$ containing 0 and $F : D \to D'$ is a conformal transformation such that $F(0) = 0$. Then for any $z \in D \cap \mathbb{Z}^2/n$,*

$$\mathbb{P}\left(F(z) \in X^n_{D'}\right) \sim |F'(z)|^{-3/4}\mathbb{P}\left(z \in X^n_D\right)$$

as $n \to \infty$, where X^n_D is the LERW in the domain $D \cap \mathbb{Z}^2/n$.

This implies property 5 by the following argument. First observe that

$$\int_0^\infty \mathbf{1}\left\{Y_n(s) \in A\right\} \, ds = \int_A \delta_{Y_n(s)}(z),$$

so that by an application of Fubini's Theorem

$$\int_0^\infty \mathbb{P}\left(Y_n(s) \in A\right) \, ds = \int_A \mathbb{P}\left(z \in Y_n[0,\infty]\right) \, dz.$$

Consequently for $A \subset \mathbb{D} \setminus Y_n[0,t]$,

$$\begin{aligned}
\mathbf{E}\left[\nu_{Y_n}(A) \middle| Y_n[0,t]\right] &= \int_0^\infty \mathbb{P}\left(Y_n(s) \in A \middle| Y_n[0,t]\right) \, ds \\
&= \int_A \mathbb{P}\left(z \in Y_n[t,\infty) \middle| Y_n[0,t]\right) \, dz \\
&= \int_A \mathbb{P}\left(z \in Y^n_{\mathbb{D}\setminus Y_n[0,t]}\right) \, dz \\
&\sim \int_A |(g_t^n)'(z)|^{3/4} \mathbb{P}\left(g_t^n(z) \in Y^n_{\mathbb{D}}\right) \, dz.
\end{aligned}$$

The last line is an application of Conjecture 4.2, via the map $g_t^n : \mathbb{D} \setminus Y_n[0,t] \to \mathbb{D}$ that has $g_t^n(0) = 0$ and positive derivative at zero. Since this holds for all $A \subset \mathbb{D} \setminus Y_n[0,t]$, we have

$$\mathbf{E}\left[d\nu_{Y_n}(z) \middle| Y_n[0,t]\right] \sim |(g_t^n)'(z)|^{3/4} \mathbb{P}\left(g_t^n(z) \in Y^n_{\mathbb{D}}\right) \, dz$$

as $n \to \infty$. One then uses some form of convergence of ν_{Y_n} to μ to show that the left side converges to $\mathbf{E}\left[d\mu(z) \middle| \gamma[0,t]\right]$ as $n \to \infty$, and some other form of convergence of LERW to SLE(2) to show that the right side converges to $|g_t'(z)|^{3/4} G(g_t(z)) \, dz$.

5 Outcome of the Meeting

We are currently working on proving these conjectures and intend to produce a manuscript as soon as they have been established.

References

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