Hodge Theory

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1 Overview of the Field

One way of defining Hodge theory is as the study of the integrals of algebraic differential forms and their generalizations. In this form, the subject is as old as calculus itself. Many basic questions in mathematics from differential equations to number theory are deeply intertwined with Hodge theory. Indeed, it was the study of such algebraic integrals which lead to classical theory Riemann surfaces.

The basic idea of Hodge theory is as follows: The cohomology of a topological space M is defined via a combinatorial gadget such as simplicial cochains. The addition of a differentiable structure allows one to replace simplicial cochains by differential forms. If M is compact and orientable a choice of a Riemannian metric allows us to continue this refinement: Each cohomology class has a unique harmonic representative.

If M is a complex manifold which admits a metric compatible with the complex structure of M, we get the further structure of a $\mathbf{C}^*$ action on the space of harmonic differential forms. Via this action, the cohomology groups of a smooth complex projective variety inherit pure Hodge structures $H^k(X, \mathbf{C}) = \oplus_{p+q=k} H^{p,q}(X)$. In the 1970's, Deligne generalized this construction by introducing the notion of a mixed Hodge structure and showing that the cohomology of every quasi-projective complex variety carries such a structure.

Many of the applications of Hodge theory to algebraic geometry relate to algebraic cycles. This is a classical subject which dates back all the way to the work of Abel and Jacobi. In the 1920's Lefschetz took up the problem of determining when a 2-cycle C on a smooth projective surface S arises from an algebraic cycle on S. Using Poincare's theory of normal functions, Lefschetz proved that C is algebraic if and only if the integral of every holomorphic 2-form on S vanishes on C [6]. Based upon this work, Hodge then formulated his famous conjecture that an integral Hodge class $w \in H^{p,p}X$ on a smooth complex projective variety X is the fundamental class of a codimension p algebraic cycle on X.

This conjecture has been open (with small alterations) since 1950. Since that time much of work in the field has been motivated at least in part by the desire to prove it, its various generalizations or its consequences. A vast theoretical arsenal has been developed to this end, and it has been relentlessly applied to solve problems in algebraic cycles.

A prime example of this is the study of variations of Hodge structure introduced by P. Griffiths. One of the goals of Griffiths' program was to prove the Hodge conjecture by studying the variations of pure Hodge structure arising from the cohomology groups of the fibers of a Lefschetz pencil. The program ran into a very serious obstacle: the intractability of the group of cycles modulo algebraic equivalence. (The proof that this group, the Griffiths group, can be non-trivial is certainly one of the early successes of Hodge theory.) However, the pursuit of variations of pure and then mixed Hodge structure built up an immense theory which includes Schmid's results on asymptotics of pure Hodge structure, the Steenbrink-Zucker-Kashiwara theory of admissible variations of mixed Hodge structure and M. Saito's theory of mixed Hodge modules.

The results of this program continue to be applied to the study of algebraic cycles. Even a list of highlights of these developments would be quite long. However, here are a three examples which come to mind: (1) Cattani, Kaplan and Deligne's proof of the algebraicity of the locus in a family where a cohomology class remains Hodge; (2) Nori's connectivity theorem; and (3) the proof of Deligne's conjecture on 1-motives by L. Barbieri-Viale, A. Rosenschon, and M. Saito.

Another example of the study of periods of algebraic differential forms leading to a new branch of mathematics is the theory of Abelian varieties. By the work of Abel and Jacobi, the period lattice Λ of a compact Riemann surface C defines an associated complex torus $\mathbf{C}/\Lambda$ called the Jacobian of C. Furthermore (i) this Jacobian is a complex projective variety; (ii) by Torelli's theorem, C can be recovered from its Jacobian. On the other hand, a generic complex torus $\mathbf{C}^r/L$ which embeds into complex projective space is not a Jacobian. This new class of objects are called Abelian varieties. The classical Schottky problem asks what relations the period lattice L of an Abelian variety must satisfy in order to arise from a compact Riemann surface.

In higher dimensions, the analogous questions are:

(a) The Torelli problem: Recovering the geometry of a variety from its from its period matrix (Hodge structure);

(b) Grothendieck theory motives.

In regards to (a), there has been much progress over the years, including the proof of the generic Torelli theorem for projective hypersurfaces as well as the spectacular discovery of mirror symmetry: Many questions in enumerative geometry of Calabi–Yau manifolds and related varieties can be solved by looking at the how the period map of an associated family of mirror manifolds degenerates. As for (b), it is hoped that many of the new tools being created to study degenerations of mixed Hodge structures will shed new light on the Hodge conjecture, filtrations on Chow groups and many other related topics.

2 Recent Developments and Open Problems

2.1 Hodge conjecture and normal functions

The Hodge conjecture remains the major open problem in the field. While there is no proof as yet, there are several interesting new lines of attack. One of these is an idea of P. Griffiths and M. Green which involves looking for singularities of normal functions. To see why this line of attack is reasonable it helps to recall, first, what a normal function is and then the role that normal functions played in Lefschetz proof of the (1,1) theorem: the only non-trivial case where the Hodge conjecture is known.

Let X be a smooth projective complex variety with fixed ample line bundle $\mathcal{O}_X(1)$. A Hodge class $\alpha \in H^{2n}(X, \mathbf{Z})$ is called primitive if the intersection of α with the first Chern class of $\mathcal{O}_X(1)$ vanishes. The line bundle gives an embedding of X in the projective space $P = |\mathcal{O}_X(1)|$ of sections of the line bundle. If $s \in H^0(X, \mathcal{O}_X(1))$ and $V(s) \subset X$ is smooth, then $\alpha \cap V(s)$ is null-homologous in $V(s)$. From this, one can get an element α_s of the Griffiths intermediate Jacobian $J(H^{2n-1}(V(s))$ associated the the Hodge structure $H^{2n-1}(V(s)$. This intermediate Jacobian is a complex torus which is algebraic for $n = 1$ but not in general. From a Hodge class one therefore obtains a section of the bundle $J(H^{2n-1}) \to P^{\mathrm{sm}}$ of intermediate Jacobians of the locus over the locus P^{sm} of smooth hyperplane sections. Lefschetz realized this and used it in his book *Analysis situs* to prove his 1-1 theorem. In fact, as Lefschetz realized, any primitive cohomology class gives an analytic section of the family of intermediate Jacobians over the locus P^{sm}. Moreover, Lefschetz realized that the section behaves reasonably near the boundary of P^{sm} in P if and only if the class α is Hodge. In modern language, this allowed him to deduce by GAGA that the section is algebraic. Finally, Lefschetz applied Jacobi inversion to prove the 1-1 theorem by restricting to a pencil L in P and using the algebraicity of the section over L to build a cycle on X.

There are a number of problems in extending Lefschetz' approach to proof the Hodge conjecture in general. The most glaring and difficult is the failure of Jacobi inversion. The simple fact is that, while $J(H^1(V(s)))$ is an abelian variety parametrizing null-homologous divisor classes in $V(s)$ (for $V(s)$ smooth), $J(H^{2n-1}(V(s))$ has no such description in general.

Green and Griffiths looked for another line of attack using normal functions. Their influential idea is to study the locus on $P \setminus P^{\mathrm{sm}}$ where the section of the family of intermediate Jacobians is singular. To each such point there is a associated a local invariant which the authors call the *singularity* at the point. This singularity is essentially a cohomology class in the singular hyperplane section of X corresponding to the point in $P \setminus P^{\mathrm{sm}}$.

2.2 Twistor D-modules

Another recent development in Hodge theory is the theory of *twistor D-modules* pioneered by several authors including C. Sabbah and C. Hertling based on ideas from physics whose Hodge theoretic interpretation is most influenced by the work of C. Simpson. This theory was used by Sabbah and T. Mochizuki to prove the Kashiwara conjecture on the existence of a decomposition theorem for semi-simple perverse sheaves [7].

3 Presentation Highlights

3.1 Patrick Brosnan, Singularities of admissible normal functions

This talk was on work with H. Fang, Z. Nie and G. Pearlstein on the Green-Griffiths program for studying the Hodge conjecture via singularities of admissible normal functions [1].

3.2 Sabin Cautis, Abelian monodromy extension property

This presentation introduced a completely new way of compactifying algebraic varieties called the *abelian monodromy extension.* Cautis says that a pair $(X, \bar{X})$ has the abelian monodromy extension property, given a Zariski open subset $U \subset S$ of a normal variety and a map $U \to X$, the map extends to a map $S \to \bar{X}$ if and only if the local monodromy (i.e., the induced morphism on local π_1) is abelian. The motivating example is the pair $(M_g, \bar{M}_g)$ consisting of the Moduli space of curves and the Deligne-Mumford compactification. The interest for Hodge theory is the canonical nature of the compactification and the use of monodromy.

3.3 Herb Clemens, Exploring the Hodge problem

This presentation of joint work with Christian Schnell proposed a template for attacking the Hodge conjecture on even-dimensional complex projective manifolds by induction on the dimension. A rough outline, given by Clemens, is as follows

1. Extend intermediate Jacobians $J(X)$ of sufficiently ample hypersurfaces X of W to somewhat larger abelian topological groups, denoted $K(X)$;
2. extend Voisin's version of the Abel-Jacobi map on Noether-Lefschetz loci to a map from topological cycles into $K(X)$;
3. define a 'generalized' normal function associated to a 'strongly primitive' Hodge class on W, also taking values in $K(X)$;
4. solve, almost everywhere, a topological version of Jacobi inversion for the generalized normal function'
5. use 4) to find a Noether-Lefschetz locus where topological Jacobi inversion breaks down.

3.4 El Zein and de Cataldo

Both El Zein and de Cataldo discussed independent work (Cataldo's was joint with L. Migliorini) on the geometric properties of the perverse filtration [4].

3.5 Phillip Griffiths, Arakelov Equalities

This fascinating lecture described joint work with Mark Green and Matt Kerr on Arakelov inequalities. Griffiths explains that classically Arakelov finiteness is the finiteness of the number of non-isotrivial families whose generic fibers are curves of genus g and with chosen singular fibers. The proof is based on bounds on the degree of the Hodge bundle and a rigidity result. (The arithmetic version of this result was proved by Faltings.) He then goes on to explain why the first part of the proof is purely Hodge theoretic depending on formulas for Chern classes of Hodge bundles. Griffiths then states a theorem giving a formula for these classes for arbitrary variations.

3.6 Claus Hertling, A generalization of Hodge structures from oscillating integrals

Abstract: The physicists Cecotti and Vafa studied already '91 a generalization of variation of Hodge structure which puts together meromorphic connections with irregular poles of order 2 and real structure. It is related to, but richer than harmonic bundles. A favorable situation where it turns up are holomorphic functions which have isolated singularities and except from that good topological behaviour (tame). There the oscillating integrals play a key role. One can formulate (partially still a conjecture) a correspondence between nilpotent orbits of pure structures and (limit) mixed structures. In the special case of regular singularities there are classifying spaces with negative holomorphic sectional curvature, but in order to make them complete one has to glue in smaller classifying spaces and allow for singularities. Also, several recent results of Sabbah and of Mochizuki concern these generalizations directly or indirectly.

3.7 Matthew Kerr, Global Hodge theory of Calabi-Yau fibrations (joint work with P. Griffiths and M. Green)

Abstract: is talk is about 1-parameter families of elliptic curves, K3 surfaces, and CY 3-folds – objects which arise, for example, in the theory of modular forms and in mirror symmetry – with particular attention to the role played by singular fibers. Instead of looking at the geometry of the family directly, one often studies the associated VHS, and the degrees of related vector bundles on the parameter space are a tool for studying global behavior.

In his classic study of minimal elliptic fibrations, Kodaira described all possible singular fibers and their relation to the A-D-E classification (from Lie/singularity theory). We will first recall this and how one can relate fiber types to the Euler characteristic of the total space and the degree of the Hodge bundles.

What is interesting is how these relations generalize (or fail to generalize) to higher dimensions (K3, CY 3-fold), and the related nonexistence (or existence) of non-isotrivial families with no singular fibers. We will describe some results along these (global) lines, and suggest how this should fit with our earlier classification of (local) degenerations of CY 3-fold VHS's related to mirror symmetry.

3.8 James Lewis, Residues and algebraic cycles

This presentation recalled the counterexamples to Beilinson's formulation of the Hodge conjecture for the higher K-groups of smooth complex quasi-projective varieties. Lewis then presented an amended version of the conjecture and showed how it relates to other conjectures due to Jannsen, Voisin and Bloch-Kato.

3.9 Claude Sabbah, On Deligne's irregular Hodge Theory

In 1984 and later in 2006 Deligne gave a construction of a Hodge filtration coming from a a variation of polarized Hodge structure on a curve twisted by an exponential. (cf. Correspondence Deligne-Malgrange-Ramis, Documents mathematiques, SMF, 2007). In this talk, Sabbah recalled the construction and indicated its relationship to the Fourier-Laplace transform in the theory of polarizable twistor structures.

3.10 Morihiko Saito, Hausdorff property of the Zucker extension

If $\mathbf{H}$ is a polarizable variation of pure Hodge structure on an algebraic variety S and ν is an admissible normal function into the family of Griffiths intermediate Jacobians, then it is widely hoped that the zero locus of ν is algebraic on S. (Indeed, this is a conjecture of M. Green and M. Griffiths.) The speaker proved an important partial result in this direction in his paper Admissible normal function. More recently, Brosnan and Pearlstein used Pearlstein's $\mathbf{SL}_2$ orbit theorem for variations of mixed Hodge structure to extend Saito's initial result. In this talk, Saito extended the results of Brosnan and Pearlstein using the Neron models of Green, Griffiths and Kerr and the methods of his paper Admissible normal functions. The approach of Saito is to use analytic estimates to prove that the Zucker extension is Hausdorff as long the cohomological invariant of the Hodge class is torsion at the boundary of S in a smooth compactification $\bar{S}$ [8].

3.11 Jozef Steenbrink, Ordinary quartic double solids

The term "double solids" refers to a famous paper by Herb Clemens studying double covers $\pi : V \to \mathbf{P}^3$ with branch locus B having only ordinary double point singularities. In the talk Steenbrink dicussed the mixed Hodge structure on the third homology group of certain singular double solids. As it turns out, for certain families called *cyclide*, there is a Torelli theorem which (almost) recovers the double solid from the polarized mixed Hodge structure on $H_3(X)$.

3.12 Sampei Usui, Log Hodge structure and a geometric application

Usui sketched the construction of toroidal partial compactifications of period domains from his joint work with K. Kato and some applications [5].

3.13 Claire Voisin, Hodge loci and absolute Hodge classes

Let $f : X \to S$ be a family of smooth projective complex algebraic varieties. In the paper of Cattani, Deligne and Kaplan it is proved that the Hodge locus, i.e., the locus on S where the parallel translates of a cycle in X_s ($s \in S$) remain Hodge, is algebraic. However, in the case that f (and thus X and S) are defined over a number field, one expects that the Hodge locus is also defined over a number field. Indeed, this is a consequence of the Hodge conjecture. In the talk, Voisin gives a local criterion for the Hodge locus to be defined over a number field.

3.14 Steven Zucker

Professor Zucker sketched joint work with Joseph Ayoub on the the construction of a motive representing the **non-complex** reductive Borel-Serre compactification of a modular variety.

4 Scientific Progress Made

Most of the scientific progress made resulted from bringing together researchers in the field. This spurred collaborations that would not have happened otherwise. For example, P. Brosnan, G. Pearlstein and M. Saito started a collaboration on the Néron models of Green, Griffiths and Kerr in Hodge theory. This resulted in the paper [3]. Another collaboration was one between C. Schnell and M. Saito also on Néron models started after the conference producing [9].

M. Saito also answered in the negative the following question of P. Brosnan and G. Pearlstein which is a kind of local Hodge conjecture.

Conjecture 1. *Let H be a variation of pure Hodge structure of weight -1 over a poly-puncture disk $(\Delta^*)^r$. Let h be a Hodge class in the first local intersection cohomology group of H at 0. Then there is an admissible normal function ν on an open neighborhood of 0 such that the local invariant of ν is h.*

5 Outcome of the Meeting

The conference helped to spur several new developments. Among these is the affirmative resolution of the conjecture that the zero locus of an admissible normal function is algebraic. This was recently answered by Brosnan and Pearlstein and independently by Schnell [2, 10].

References

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