Topological methods for aperiodic tilings

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1 Overview of the Field

The study of aperiodic tilings began with the work of Hao Wang in the 1950's. It was energized by examples given by Raphael Robinson in the 1960's and, more famously, Roger Penrose in the 1970's. See [Se, GS] for overviews of the subject. Penrose's example was striking because it admitted rotational symmetries which are impossible in periodic tilings. Perhaps the most significant development was in the early 1980's, when physical materials, now called quasi-crystals, were discovered which possessed the same rotational symmetries forbidden in periodic structures and yet displayed a high degree or regularity [SBGC]. The field since then has been characterized by an interesting mix of a wide variety of mathematical and physical subjects. Among the former are discrete geometry, harmonic analysis, ergodic theory, operator algebras and topology. This workshop was designed to highlight recent progress in the areas of topology and ergodic theory [Sa1].

Ergodic theory and dynamical systems are natural tools for the study of aperiodic patterns. Rather than study a single aperiodic pattern, dynamical systems prefers to study a collection of such objects which are globally invariant under translations (and perhaps other rigid motions of the underlying space as well). In fact, many of the examples are exactly that: a class of objects with very similar properties. At the same time, it is very natural to construct a class from a single pattern by looking at all other patterns which have exactly the same collection of local data. More specifically, given an aperiodic pattern P, look at all P' such that the intersection of P' with any bounded region appears somewhere in P. Finally, this approach of looking at a collection of patterns grew naturally in physical models for quasi-crystals.

The key step in this approach is establishing some kind of finiteness condition of the collection. Looking at such systems from a probabilistic point of view, this means finding a finite measure which is invariant under the translation action. In most examples, this measure not only exists, but is unique.

From a topological point of view, the aim is to find some natural topology in which the action is continuous and the collection is compact. Compactness is the analyst's analogue of finite. Such topologies were easily developed - they are natural extensions of topologies on shift spaces which are standard in symbolic dynamics.

To elaborate a little, suppose one considers point sets in a Euclidean space which are uniformly dense (for some fixed positive radius R, the R-balls around all points cover the space) and uniformly discrete (for some fixed radius r, the r-balls around the points are pairwise disjoint). Such a set is called a Delone set, but here we require the same two constants to be good for all elements of the collection. A neighbourhood base for the topology is as follows. For a fixed point set P, open set U in the underlying space and positive constant ϵ, look at all other points sets P' such that $P' \cap U$ is within ϵ of the restriction of $P \cap U$. (This is not quite correct because of what is happening near the boundary of U, but it will suffice for the moment.)

This topology first appeared as a tool, since it makes the collections of interest compact and hence many standard results of dynamics may apply. Later, it was observed by a number of people that the local structure of the space could be described quite explicitly, as follows. A collection of tilings is said to have *finite local complexity* if, for fixed $R > 0$, the number of different patterns which occur inside an R-ball as one considers all patterns in the collection, modulo translation, is finite. (This can be generalized by replacing the group of translations by other subgroups of the isometry group of the underlying space.) This is often satisfied and it implies that, locally, the space is the cartesian product of a Euclidean space and a totally disconnected space. This result was improved to a global version by Sadun and Williams [SW]: the space is a fibre bundle over the a torus (with the same dimension as the space being tiled) with totally disconnected fibres.

Anderson and Putnam [AP] considered tilings obtained by the so-called substitution method. These possess a type of self-similarity present in both the Robinson and Penrose examples. They observed that a good deal could also be said about the global structure of the space of tilings. This was done by presenting the space as an inverse limit of simpler spaces which, in most examples, are branched manifolds. In particular, it meant that the cohomology of the space was a computable invariant. This idea was extended by Bellissard, Benedetti and Gambaudo [BBG] and also by Gähler and Sadun [Sa2] to more general tiling spaces. In another direction, Forrest, Hunton and Kellendonk [FHK] took a different approach to provide an effective method for computing the cohomology of those tilings which arise from the so-called projection method.

The computation of the cohomology of tiling spaces was motivated by another important factor. Bellissard gave a construction of a C^*-algebra from a tiling [Be1, KeP1, BHZ]. This C^*-algebra is a model for electron motion in a quasicrystal. In this framework, the K-theory of this C^*-algebra carries important information about the physics. Due to a result of Connes, this K-theory is the same as the K-theory of the tiling space itself. Cohomology and K-theory are closely related, especially for low-dimensional spaces.

2 Recent Developments and Open Problems

There have been a number of interesting developments over the past few years. The over-riding theme has been to understand exactly what the cohomology of the tiling space is measuring. There seems to be a sense that it is a quantitative measure of aperiodicity, but this has not been formulated in a precise way.

In this direction, Kellendonk and Putnam [KeP2] were able to give a more geometric interpretation of cohomology by using the notion of pattern equivariant differential forms. Given a point pattern P in Euclidean space (or a tiling), a function on the space is called P-equivariant if there is $R > 0$ such that the value of f at x only depends on the pattern P within a ball of radius R around x. More precisely, if $(P - x) \cap B(0, R) = (P - y) \cap B(0, R)$ then $f(x) = f(y)$. The collection of P-equivariant differential forms with the usual exterior differential is a chain complex and its cohomology is isomorphic to the Cech cohomology of the hull with real coefficients. On the other hand, the differential forms have more geometric content than the Cech cocycles. In particular, it is possible to average these over the space in a meaningful way and this produces a map from the cohomology to the exterior algebra of Euclidean space. The complete data contained in this combination of cohomology and this map is not fully understood. In particular, the first cohomology group seemed to play a distinguished role which is also not clear.

Continuing further in this line, Clark and Sadun [CS] gave a description of how elements of the first cohomology of the tiling space could be seen to parameterize deformations of the tilings. This provides a nice framework for dealing with various subtle issues in comparing the various ways in which two tiling spaces may be regarded as equivalent. It also raises a number of questions, both about deformations and also about various variants of the cohomology.

Barge and Diamond [BD] have given a complete classification of tiling spaces in one dimension. Much of this analysis is devoted to understanding ‘asymptotic composants’: pairs of path connected subspaces (which must be homeomorphic to the real line) which are asymptotic in some precise sense. It is too much to hope that such a treatment will extend completely to higher dimensions, but it is natural to ask what aspects will or if there is a certain class of tiling spaces for which a parallel development will work.

There has been a great deal of progress in recent years on tilings which fail to have finite local complexity with respect to translations. The first type of tiling where this condition fails is in tilings that have tiles appearing in infinitely many orientations. The most famous example is the pinwheel tiling. As mentioned above, this can be regarded as having finite local complexity by replacing the translation group of the Euclidean space by the full isometry group. However, there are a number of other examples of substitution tilings where the tiles, which are polygons, fail to meet vertex to vertex after substitution. This produces the possibility of continuous 'sheers'. Although the condition of FLC fails, Frank and Sadun [FS] have given a concrete description of the tiling space as an inverse limit of CW complexes where the sheer is represented by added dimensions in the cells. The construction works very well on specific examples, but a general theory of all such tilings is still elusive.

3 Presentation Highlights

Simultaneous with our half workshop was another on self-similar groups. Since there were common themes, especially around the concept of self-similarity, there were several introductory talks given to both groups simultaneously. Volodia Nekrashevych explained self-similar groups to the tilings workshop, and Lorenzo Sadun explained non-periodic tilings to the group theorists. These provided an interesting insight into some different fields for both groups. Some of these connections are still developing. For example, Putnam is continuing discussions in an informal way with Nekrashevych.

Our goal for the workshop was *not* to have everybody present a technical talk on his or her latest result, but rather to have an opportunity for everybody to learn about the big ideas that are driving the field. We polled the participants about what they wanted to learn, and then drafted speakers to talk about these subjects. We also allotted big time slots for the most popular subjects, so that questions could be asked and answered in depth.

Pierre Arnoux and his collaborators have been studying aperiodic substitution tilings for a number of years. Their examples are quite concrete and are strongly motivated by problems in number theory and symbolic dynamics. Put briefly, the usual continued fraction expansion can best be viewed as starting from a number (usually irrational) and 1. Here, the starting is an N-tuple instead of a pair and the goal is to develop a multi-continued fraction algorithm and to understand its geometric nature. This produces tilings in a very natural way, but unlike most from the tilings community, the tiles are fractals (such as the Rauzy fractal), rather than polygons. Arnoux gave an excellent introduction to the subject which highlighted the motivation and that ran for three hours (with a short break).

On the other hand, Arnoux and his collaborators were not at all familiar with the idea of using careful analysis of the topological structures of the space of tilings and invariants like cohomology in their study. In response, Lorenzo Sadun gave a tutorial on tiling cohomology.

Marcy Barge gave another long lecture on the state of the art in one-dimension tilings, and also discussed ideas (and partial results) for extending these results to higher dimensions. In particular, the lecture explained possible extensions of the idea of asymptotic composants to higher dimensions and showed some important motivating examples.

One area which the organizers developed quite deliberately was the use of tools from non-commutative geometry. A great deal of impetus in the subject, including the use of cohomology as an invariant, came from the work first of Bellissard and later Kellendonk on the C^*-algebras associated to aperiodic structures. Connes' program of non-commutative geometry is advancing quite rapidly and providing new tools and techniques to study such algebras, adapted from notions of classical geometry. In particular, John Pearson, a recent Ph.D. student of Bellissard, gave a long lecture on his thesis, in which he constructs spectral triples (a non-commutative analogue of a manifold together with a Dirac operator) from a Cantor set with an ultrametric. This is only the first step toward providing something along these lines for tiling spaces, but already the analysis is subtle and a surprising amount of structure may be detected. Since the meeting, there has been considerable progress in this direction by Bellissard, Julien and Savinien.

Another lecture was presented by Hervé Oyono-Oyono on his solution (joint with Benameur) of the gap labeling problem of Bellissard. This problem was open for quite sometime before being solved simultaneously by three groups: Bellissard-Benedetti-Gambaudo, Benameur-Oyono and Kaminker-Putnam. All three proofs rely on Connes' index theorem for foliated spaces. But the Benameur-Oyono proof has more the flavour of non-commutative geometry by using cyclic cohomology.

There were several talks about tilings that lack finite local complexity, either because of rotational properties or because of shears. Jean Bellissard began this theme by explaining how such tilings are very natural in physics, and sketching how the K-theory and C*-algebras associated with such tilings relate to the physical properties of the tilings. Natalie Frank described a family of substitutions that have two realizations: either as a discontinuous substitution on a tiling with finite complexity, or as a continuous substitution on a tiling with shears. She posed interesting, and still unresolved, questions on how these tilings may be related. In an informal evening session, John Hunton presented the computation of the cohomology of the pinwheel tiling, something recently accomplished using higher-dimensional extensions of the techniques that Marcy Barge had discussed earlier.

Until recently, it was believed that the cohomology groups of tiling spaces were always torsion-free. A theorem to that effect had even been announced. Frank Gähler proved this wrong, and found that a number of well-known tilings have torsion in their cohomologies. (This also occurs in tilings with rotational properties, such as the pinwheel.) He explained the computational techniques that led to this discovery.

Although the meeting was built around sharing big ideas of general interest, we also wished to give junior researchers the chance to present some of their results. On Thursday we had mostly shorter talks, and split into parallel sessions in the afternoon. Gähler's and Frank's talks have already been described. Samuel Petite described progress in understanding the topology of tilings of hyperbolic space. Jean Savinien, a student of Jean Bellissard, described a new method for the computations of cohomology of tilings and its relations to K-theory based on ideas of Pimsner. Hervé Oyono-Oyono also related discussed the C^*-algebras associated to various tilings of hyperbolic space constructed by Penrose, including a description of their K-theory groups. It was interesting to contrast this work with earlier work in the Euclidean case. Antoine Julien, a student of Kellendonk's talked about his thesis work which relates cohomology with the complexity of the projection method tilings. This application of cohomology is a rather expected one, but suggests that many other possibilities exist along such lines.

Ian Putnam closed off the presentations with a description of a variant on K-theory for Smale spaces. Smale spaces are dynamical systems where the neighborhood of each point is the product of a (uniformly) contracting space and a uniformly expanding space. Substitution tilings spaces are always Smale spaces, but the concept is far more general. It is an open question how, for tiling spaces, this new invariant relates to the usual K-theory.

4 Scientific Progress Made

The biggest effects of the conference involve cross-pollination between diverse research groups and will take time to bear fruit. However, there were a number of concrete advances made at the conference.

Barge, Hunton and Sadun completed work on a large project extending ideas of Barge and Diamond from one dimension to higher dimensions. This included the computation of the pinwheel cohomology, a problem that had resisted repeated efforts for many years.

Recently, Giordano, Matui, Putnam and Skau have given a classification of minimal actions of the groups $\mathbb{Z}^d, d \geq 1$ on a Cantor set. This involves constructing a sequence of finite equivalence relations on the space whose union is an approximation to the orbit relation. But there are many technical issues involved that need to be satisfied before reaching the desired conclusion. Such dynamical systems are very closely realted to aperiodic tilings. Initially, many examples of aperiodic tilings were known to be suspensions of such systems, but Sadun and Williams actually proved that under mild hypotheses, any the hull of any tiling of d-dimensional Euclidean space is the d-fold suspension of an action of $\mathbb{Z}^d$ on a totally disconnected space. Bellissard and his collaborators had begun to consider the kind of finite approximations in the Giordano-Matui-Putnam-Skau construction in this seeting of tilings. They had a number of discussions with Giordano on this subject. Specific properties of these approximations may have implications in the setting of non-commutative geometry. The first results of these investigations have recently appeared (as a preprint). On the other hand, the lack of concrete computable examples if the Giordano-Matui-Putnam-Skau construction has been a problem. So there was a good deal of interest from these people in the kind of tilings and actions studied by Arnoux and Siegel, where the arithmetic data seems to provide quite explicit methods.

5 Outcome of the Meeting

The meeting has had the effect of generating a number of new connections between previously diverse groups: the topology of tilings people (Barge, Sadun, Putnam), arithmetic tilings (Arnoux, Siegel), dynamical systems (Giordano) and operator algebraists (Bellissard, Oyono-Oyono, Petit) and even the self-similar group theorists from the other meeting (Nekrashevych). These connections have continued to develop at other meetings since: Strobl, Austria and Leicester, U.K. in 2009 and at a large program in Luminy in 2010.

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