08ss045 The stable trace formula, automorphic forms, and Galois representations

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1 Overview of the Field

The heart of Langlands' program reinterpreting much of number theory in terms of automorphic forms is his *Functoriality Conjecture*. This is a conjecture associating the automorphic representations on a pair of connected reductive groups over a number field F, say G and H, whenever there is a homomorphism of the appropriate type between the corresponding L-groups

$$h : {}^L H \to {}^L G.$$

The structure of functoriality is largely captured in terms of parametrization of automorphic representations of a given group G by homomorphisms

$$\phi : \mathcal{L}_F \to {}^L G$$

where $\mathcal{L}_F$ is the hypothetical Langlands group.

In the analogous conjectured parametrization of irreducible representations of a reductive group G over a local field K, the Langlands group is replaced by the Weil-Deligne group W_K; in this setting one can state precise conjectures in terms of known objects, and these conjectures were proved when $G = GL(n)$ during the 1990s. This represents the first step to the extension of local class field theory to the non-commutative context, whereas the Langlands functoriality conjectures for automorphic forms are to be understood as generalizations of global class field theory. In particular, the Artin conjecture on holomorphy of Artin L-functions follows from Langlands' Functoriality Conjecture applied when H is the trivial group.

The most comprehensive technique for proving the Functoriality Conjecture in the cases in which it has been established is the Arthur-Selberg trace formula. This was used by Lafforgue to prove the global Langlands conjecture for $GL(n)$ over a global field of positive characteristic, and has had a variety of striking applications to number fields, including the Jacquet-Langlands correspondence and its generalizations in higher dimensions and the Arthur-Clozel theory of cyclic base change for $GL(n)$. Each special case of functoriality has been of enormous importance in number theory. Until recently, applications of the trace formula were limited to the cases in which it could be stabilized. With the proof of the fundamental lemma for unitary groups by Laumon and Ngô, followed by its extension to all groups by Ngô – together with the proof that the fundamental lemma depends only on the residue field, first by Waldspurger, more recently by Cluckers, Hales, and Loeser – Arthur's stabilization of the trace formula is close to being complete, and the simple trace formula can be stabilized in a number of situations. This makes it possible to carry out the applications of the trace formula anticipated more than two decades ago by Langlands, Kottwitz, and Arthur.

2 Objectives of the Summer School

The primary goal of the summer school was to contribute to creating a situation where number theorists will be able to make use of the most recent developments in the theory of automorphic forms on higher-dimensional groups with no less ease than they have hitherto done with the $GL(2)$-theory. This necessarily involves coming to terms with the stable trace formula. Applications of the stable trace formula of special interest to number theorists include

(i) the determination of multiplicities of automorphic representations;

(ii) the proof of functoriality in the special case when $G = GL(n)$ and H is a classical group (symplectic, orthogonal, or unitary), with the hypothetical Langlands parameter ϕ introduced above replaced by a discrete automorphic representation of $GL(n)$;

(iii) the parametrization of irreducible representations of classical groups over local fields in terms of the local Langlands parametrization for $GL(n)$; and

(iv) the calculation of the zeta functions of Shimura varieties, and the analysis of the corresponding Galois representations.

The first three topics are treated in the book on functoriality and the twisted trace formula that James Arthur is in the process of completing. The fourth topic, together with special cases of functoriality when H is a unitary group, is developed in the series of books in preparation by participants in the Paris automorphic forms seminar. A concrete objective of the summer school, then, was to help prepare participants to read these books. This applies as well to virtual participants who will at some future date be able to watch video recordings of the proceedings online.

3 Presentation Highlights

The seven-day program was organized as a series of daily themes, beginning with basic facts about automorphic representations of $GL(n)$ and classical groups and leading up to applications of the stable trace formula to functoriality and the cohomology of Shimura varieties. The daily themes are presented in order:

3.1 Day 1: Framework of reciprocity and functoriality

(a) Classical modular forms and associated Galois representations in the light of automorphic representations of GL(2) (C. Skinner)

(b) Automorphic representations of GL(n) and classical groups (J. Cogdell)

(c) Introduction to Langlands reciprocity for Galois representations (M. Harris)

(d) Introduction to Langlands functoriality for classical groups (T. Gee)

3.2 Day 2: Basic representation theory of GL(n) over local fields and classical groups

(a) Introduction to representation theory of p-adic classical groups (A. Minguez)

(b) Introduction to harmonic analysis on p-adic groups (J. Arthur)

(c) (Tempered) cohomological representations of GL(n) and unitary groups (D. Shelstad)

(d) d. Local Langlands correspondence for GL(n) over p-adic fields (M. Harris)

3.3 Day 3: Introduction to the simple trace formula

(a) The trace formula for cocompact groups (J. Bellaïche)

(b) Introduction to the simple trace formula (J.-P. Labesse)

(c) Applications of the simple trace formula (E. Lapid)

3.4 Day 4: Introduction to endoscopy

(a) Introduction to stable conjugacy (T. Hales)

(b) The stable trace formula, part I (S. W. Shin)

(c) The stable trace formula, part II (J.-P. Labesse)

(d) Endoscopic transfer of unramified representations (J. Bellaïche)

(e) Endoscopy for real groups (D. Shelstad)

3.5 Day 5: Functoriality and the stable trace formula

(a) Introduction to functoriality for classical groups (J. Cogdell)

(b) Functorial transfer for classical groups, statements (J. Arthur)

(c) Functorial transfer for classical groups, sketch of proofs (J. Arthur)

(d) Simple stable base change and descent for $U(n)$ (M. Harris)

3.6 Day 6: Shimura varieties

(a) Introduction to Shimura varieties (L. Fargues)

(b) Integral models of PEL Shimura varieties (L. Fargues, in two parts)

(c) Points on special fibers of PEL Shimura varieties, following Kottwitz (S. Morel, in two parts)

3.7 Day 7: Shimura varieties

(a) Newton stratification of special fibers of PEL Shimura varieties (E. Mantovan)

(b) Points on special fibers of PEL Shimura varieties and Igusa varieties (S. W. Shin)

4 Outcome of the Meeting

Registered to attend the meeting was the maximum number of 42 participants, all but one of whom did in fact attend. Among the participants were seven graduate students, ten recent post-docs, and twelve confirmed researchers who are not specialists in the topics of the summer school, primarily number theorists. The remaining participants were speakers either at the summer school or, in a few cases, in the subsequent week's program.

Seven number theorists were invited to organize and moderate evening sessions to review each day's lectures, to answer questions that could not be treated in detail during the formal presentations, and to provide opportunities for the less experienced students and post-docs to familiarize themselves with foundational material. The seven moderators (Darmon, Nekovar, Kisin, Skinner, Iovita, Böckle, and Prasanna) succeeded in convincing most of the day's lecturers to attend the lively evening sessions, which lasted 2-3 hours and represented an important addition to the formal program.

With a view to making the week's lectures available to a broader audience, all lectures were videotaped, with the generous support of the French Agence Nationale de la Recherche as well as NSERC.