Geometric Mechanics: Continuous and discrete, finite and infinite

George Patrick (Saskatchewan)
Jerrold Marsden (Caltech)
Juan-Pablo Ortega (CNRS, Universite de Franche-Comte)
Mark Roberts (Surrey, UK)
Jedrzej Sniatycki (Calgary)
Cristina Stoica (Wilfrid Laurier)

August 12 – August 17, 2007

1 Overview of the Field

Geometry, as learned by students of grade school, is the business of lines, angles, circles and triangles. It's useful because, locally, the concepts of geometry are similar to objects of our world. Using geometry, we can compute areas, heights, and angles. Almost all of us know some geometry. Many of us need it, from time to time.

It's mostly about lines. A line is the shortest distance between two points. On a manifold M, with Riemannian metric g, the length of a curve $m(t)$ from $t = a$ to $t = b$ is computed from

$$\text{length}^2 \equiv \int_a^b g\big(m'(t), m'(t)\big)\, dt.$$

To find the "straight line" between m_1 and m_2 is to minimize this length functional among curves $m(t)$ with $m(a) = m_1$ and $m(b) = m_2$.

What has that got to do with mechanics? With $F = ma$? To begin with, another notion of "straight line" is the path of a unforced particle. That is Newton's first law. At a far deeper level, Hamilton's principle [1, 43] of mechanics states that frictionless motion in a configuration space Q in general occurs on curves $q(t)$ which extremalize the functional

$$S \equiv \int_a^b L\big(q'(t)\big)\, dt, \tag{1}$$

where L (the Lagrangian) is typically the difference of the kinetic and potential energies. From L one can construct a two form ω on TQ, and an energy function E on TQ, such that the solutions of the variational principle (1) are exactly the integral curves of the Hamiltonian vector field X_E i.e. integral curves the unique vector field that satisfies

$$i_{X_E}\omega = dE. \tag{2}$$

Morphisms that preserve ω preserve these equations, which shows that conservative classical mechanics fundamentally occurs in the context of the category of symplectic manifolds.

Generally, we have a pair (M, ω, E), where ω is a closed nondegenerate two form on M, and E is a smooth function on M, and the evolution of the system is again via equations (2). If a Lie group acts of M in such a way that ω and E are invariant, then the solutions are preserved. These are the *Hamiltonian systems with symmetry*, and they are nongeneric in the set of all Hamiltonian systems. It is useful to separate the part of the dynamics that occurs due to symmetry from the generic part. By results collectively referred to as Noether's theorems, for every symmetry, there is a conserved quantity for the dynamics; and the conserved quantities in turn generate the symmetries. These conserved quantities, counting one for every independent symmetry, are typically packaged into a single *momentum map* from the phase space to the dual of the Lie algebra of the symmetry group. The quotients in the symplectic category are complicated because, besides reducing orbits to points, they also have to account for the momentum map, or else one does not get to a generic context. This is *reduction*, and the foundational work is [45]. Sequential partial reductions—reduction by stages—are an important part of the theory, because they allow for finer grained elimination of symmetry [39, 40]. And there are also ways to reduce from the basic Lagrangian variational principles [16, 17, 31, 42, 44]

Smooth quotients are only expected when the symmetry group acts freely, and singularities are expected where there is isotropy. In Hamiltonian systems, there are two sources for singularities in the quotients: from isotropy of the symmetry and from singularity in the momentum map. Because the symmetries and the momentum are so intimately related, these singularities occur at exactly the same places in phase space [2]. The singular quotients in symplectic geometry are Whitney stratified spaces, where the stratifications are by isotropy type and momentum [5, 54, 57, 67].

Equilibria are of course fundamental organizing solutions for dynamical systems. In the presence of symmetry one can seek equilibria in the quotients; these are the *relative equilibria* and they are solutions that are the actions of one parameter groups of the symmetry. Relative equilibria, such as occur in the restricted three body problem of celestial mechanics, are of great physical interest, and have been studied since the outset of mechanics. One of the important tools of Geometric Mechanics is special coordinates near relative equilibria, derived from symplectic geometry's various linear splittings together with the (equivariant) Darboux theorem [56, 58, 59, 65, 71]. The coordinates are helpful to determine the structure of the set of relative equilibria, as in [53, 58, 61, 64, 66, 74]. Extensions to relative periodic orbits occur in [55, 74, 75], and numerical continuation is considered in [76].

The stability of relative equilibria can be delicate, as the spectrum of the linearizations in Hamiltonian system cannot all be in the negative real half plane, and asymptotic stability is impossible. Using energy and momenta for the Lyapunov functions can provide conditions for robust stability. But generically there are also regions where such Lyapunov stability fails and the spectra is purely imaginary. This situation is a an important target of one of the principle advances in Science in the last century: the KAM theory, which arose from early questions about the stability of the solar system [4], and in many cases forms of stability may be recovered even here [24, 25, 52]. KAM stability, or, in higher dimensions, stability over exponentially long times is delicate. It can be destroyed through resonance [7], or by the introduction of arbitrarily small dissipation (dissipation induced instability) [9, 22, 33, 34].

There are other ways to write equations (2). Given functions f and g, the Poisson bracket of $\{f, g\}$ of f and g is

$$\{f, g\} \equiv \omega(X_f, X_g)$$

and the solutions of the variational principle (1) are equivalent to

$$\frac{df}{dt} = \{f, E\}.$$

This leads to Poisson geometry [43, 72]. There is little advantage between symplectic and Poisson geometry at this level. But in the quotients by the actions of groups the symplectic geometry occur as the leaves of the Poisson spaces; the latter coherently assembles the former into a whole. Both are important, and they are different viewpoints.

There are extensions. If a mechanical system is constrained in its velocities, such as a penny, ball, or egg, rolling on a table, then the geometry is altered. This is nonholonomic mechanics [13], and it necessitates generalizations, from symplectic to semi-symplectic, and Poisson to almost Poisson. Different categories mean different morphisms, and different basic properties. For example, energy is conserved, but not momentum, or you couldn't turn any rolling vehicle (local nonconservation of angular momentum). Since momentum and symmetry in holonomic mechanics are related, the quotients in these generalizations take a different form. The geometry of nonholonomic systems is complicated, and its development relatively recent [6, 11, 37, 38, 60, 68].

Other important extensions occur in field theories i.e. essentially, partial differential equations. A classic example is the realization of the motion of a perfect fluid as a geodesic flow on the group of volume preserving diffeomorphisms [3, 23]. Of course differential geometry and field theories are not strangers: the theories of general relativity, and gauge field theories [50, 51], are all heavily geometric, and no summary can do justice to the developments. Field theories may be generally cast geometrically, in a similar way as geometric mechanics, as is seen in the works [27, 28, 29], and the references therein.

All these ideal conservative systems are central. Most fundamentally they arise as above from variational principles, and give rise, as already mentioned, to the category of symplectic and Poisson manifolds, which themselves are of great current interest to Mathematics. For mechanical systems, such as rigid body systems, the phase spaces are finite dimensional, while for field theories they are infinite dimensional. While physical systems are seldom exactly symmetric, or exactly conservative, there are often physical regimes for which they are nearly so, and thus for which the dominant or organizing behaviors are those of the ideal systems. For example, small dissipation might be superposed on a conservative system, or a system might be obtained by breaking the symmetry of a perfectly symmetric system. The symmetric, conservative systems are an organizing center for the subject.

Discrete analogies occur, in 1956 [70], and then rediscovered in [18], as integration algorithms for the ordinary differential equations of holonomic mechanics. It was long known that the flows of mechanics are canonical, and so preserve the volume of phase space. In the geometric setting, the flows preserve the symplectic form, and so are morphisms of the symplectic manifolds. Integration algorithms that are iterations of a single symplectic map preserve the dynamical features of classical mechanical systems, over long times, better than generic algorithms. Thus began in 1990 a cascade of work of geometric integration algorithms, as discussed for example in the books [30, 36], and extended to nonholonomic systems [19, 48]. Significant is the development of discrete mechanics, which is towards discrete models that reflect physical reality so well that they have a status with continuous models. The continuous models are fundamentally variational. The variational foundations of discrete Lagrangian mechanics, where the continuous tangent bundle phase space is replaced by pairs of configurations, and the action integral is a sum over sequences of pairs, occur in [49], and as developed as such in [46, 63, 73]. This is extended to nonholonomic systems in [21, 20], and to first order variational field theories in [41].

“Ballistic” refers to the free flight of a projectile. But if the projectile is to arrive at a specific target, then controlled motion is far more effective. Automatic control is an important area in engineering; devices of all sorts have to be guided and stabilized to particular states or modes of motion. Control theory is one of the main application areas of geometric mechanics, as witnessed by the recent books of Bloch [8] and Bullo and Lewis [15].

In the most primitive form, one has a number of control inputs represented by coefficients of a sum of vector fields. The question is whether there is a curve in parameter space which causes a transit between any two given points of configuration or phase space. This is the issue of controllability, and it is an early application of differential geometry: the Lie bracket of two control vector fields may be generated by alternately running one, the other, and then the reverse, so the number of control vector fields required may be reduced if those vector fields do not commute [69]. Feedback control refers to the problem of designing a way of setting the control parameters, bases on the evolving state itself, so that a particular trajectory is achieved, and stable. Recent work of this in the geometric setting occurs for example in [10, 12]. Optimal control refers to the additional requirement that some quantity, such as fuel, energy, or time, be minimized. The constraints of optimal control lead to fundamentally different variational principles than the constraints of nonholonomic mechanics. Discrete mechanics is playing a key role in control as well, with the development of applications of discrete mechanics to optimal control [32]. And there is an interesting recent application to image restoration [47].

Differential geometry and mechanics are fundamentally related, as are differential geometry and physics. Geometric Mechanics enhances the traditional approach to mechanics by the inclusion of ideas from differential geometry, nicely balanced with analytical methods. While this idea has its roots going back to the founders of mechanics, such as Jacobi, there has been a resurgence of these ideas in the past few decades, with the infusion of many new ideas and links. For instance, it is a basic fact that the standard Hopf fibration of S^3 to S^2, usually thought of as belonging to pure topology (or bundle theory) already occurs in rigid body mechanics (going back to Euler and Lagrange). The geometric approach to mechanics flourishes today: it has its own internal beauty and research (such as stability theory and singular reduction theory), as well as substantial contributions to neighboring areas, such as molecular systems, classical field theories (fluids, solids, electromagnetism, gravity, etc.), to control theory, and to computational mechanics. In Geometric Mechanics today, we use concepts and compute properties that could not be easily discerned without the differential-geometric context, some of which did not exist even a decade ago.

2 Presentation Highlights

2.1 Special Lectures

We had three special lectures, of one hour duration, in the evenings.

Jerry Marsden (Caltech), spoke about variational integrators and optimal control, and Lagrangian coherent structures (LCS). While the free dynamics of conservative systems is always ideal and therefore of special application, the application of control is wide open. Jerry explained how the established work towards discrete analogies in mechanics can help solve to solve problems in optimal control. LCS can help understand mixing, transport and barriers in fluid flows (e.g., ocean and atmosphere) and other dynamical systems. It can also be used to decide drifter deployment, and understand pollution dispersion, oil spills.

Tudor Ratiu (Ecole Polytechnique Federale de Lausanne) considered the Lagrangian and Hamiltonian structures for an ideal gauge-charged fluids [26]. These are geometrically complex infinite dimensional examples. The discussion include a Kelvin-Noether theorem non-canonical Poisson bracket associated to these systems.

Jedrzej Sniatycki (Calgary) considered the commutativity of quantization and reduction. The important new aspect is the lack of assumptions on the group action, so that the Theorem below addresses possible singularities in the classical system:

Theorem 3 *Assumptions:*

1. *Let P be a Kahler manifold, ω be the Kahler form on P and F be given by antiholomorphic directions.*
2. *Suppose that an action of a connected Lie group G on (P, ω) has an Ad_G^* equivariant momentum map $J\colon P \to \mathfrak{g}^*$ and preserves F.*
3. *Let O be a quantizable co-adjoint orbit admitting a Kahler polarization F_O such that quantization of (O, ω_O) in terms of the polarization F gives rise to an irreducible unitary representation U_O of G.*
4. *Suppose that there exists a Lagrangian subspace of (P, ω) contained in $J^{-1}(O)$.*

Then the space of square integrable wave functions, obtained by quantization of algebraic reduction at O, defines on H a projection operator Π^O such that $\Pi^O(H)$ is the closed subspace of H on which the quantization representation U is equivalent to U^O. Here H denotes the space of square integrable holomorphic sections of $L \to P$.

2.2 Relative equilibria

James Montaldi (Manchester) presented on bifurcations of relative equilibria at zero momentum. A result of Roberts and Patrick [61], states that generically, $\mathrm{SO}(3)$ symmetric Hamiltonian systems have no equilibria. Montaldi answered why this is not the case for simple mechanical systems.

The problem of Riemann Ellipsoids has an long and important history, going back to Newton, MacLaurin, Dirichlet, Riemann and Poincare. It has its origins in the attempt to provide an explanation to the rotating figure of the Earth. Miguel Rodriguez-Olmos reviewed this problem from the point of view of Differential Geometry and discussed how the geometric perspective can give some insight into the nonlinear stability of some of its classical solutions.

Understanding the structure of the relative equilibria or periodic orbits of a system means following manifolds of them in phase space and understanding their bifurcations. It is generally impossible to do this exactly. Frank Schilder and Claudia Wulff discussed this problem, and presented the software system SYMPERCON for numerical bifurcation analysis of Hamiltonian relative periodic orbits. SYMPERCON accounts for, and takes advantage of, the nongenericity of Hamiltonian systems with symmetry in the class of all systems with symmetry.

Saari's conjecture is that every solution of the planar Newtonian N-body problem with constant moment of inertia is a relative equilibrium. Cristina Stoica (Wilfrid Laurier) presented recent work showing that, for generic rotationally-invariant vector fields in the plane, Saaris conjecture is true: the only constant-inertia solutions are the relative equilibria [66].

The stability of relative equilibria is sensitive to the topology of the orbit space of the coadjoint action of the symmetry group [62]. Claudia Wulff (Surrey) presented work proving that, in the Kirchhoff model for the motion of an axisymmetric underwater vehicle, relative equilibria that were thought to be robustly stabilized by spin are in fact only KAM stable. Furthermore, there is numerics that showing dissipation induced instability in this case.

2.3 Control

Future space missions like Terrestrial Planet Finder (NASA) and Darwin (ESA) will make use of a network of formation flying spacecraft. In these missions, the requirements on the accuracy on the relative positioning of the craft are extremely high. In addition, reconfigurations of the formation have to be performed at regular intervals with minimal energetic effort. Oliver Junge (Munich University of Technology) showed how the recently developed variational method DMOC (Discrete Mechanics and Optimal Control) for the numerical computation of optimal open-loop controls for mechanical control systems can be applied to this problem.

Suppose you have a network of sensor equipped vehicles and you want to coordinate and stabilize the motion of your fleet. Sujit Nair explained the you can couple the system to yield a multi-body Lagrangian system, and use controlled Lagrangians and matching conditions. The symmetry of these systems depends on the context: $\mathrm{SO}(3)$ for spacecraft and $\mathrm{SE}(3)$ for underwater vehicles. He also spoke about about coordinating hovercrafts for the purpose of surveillance, and showed a wonderful movie showing coordination of inverted pendula on connected rolling carts.

Andrew Lewis (Queens) spoke about energy shaping, which is a control strategy wherein one converts a given mechanical system (called the open-loop system) to another mechanical system (called the closed-loop system) with desired properties. He gave an affine differential geometric formulation of the energy shaping problem and gave a complete description of part of the problem using techniques from the formal theory of partial differential equations.

2.4 Discrete systems; numerics

Melvin Leok (Purdue University) discusses the synthesis of Lie group techniques and variational integrators to construct symplectic-momentum methods which automatically stay on Lie groups and homogeneous spaces without the need for constraints, local coordinates, or reprojection. These are integrators that simultaneously preserving the symplectic and Lie group properties.

George Patrick (Saskatchewan) presented a new development of variational discretizations, based on discrete analogues of tangent bundles, obtained by systematically extending tangent vectors to finite curve segments. He showed that existence and uniqueness of the discrete evolutions can be analyzed by blowing up the variational principles at zero time-step. These methods can automatically convert any one-step numerical method to a variational method of the same order.

Ari Stern (California Institute of Technology) presented applications of variational integrators to electromagnetism. This uses discrete versions of the exterior calculus of differential forms.

2.5 Other

In the wide group of participants, there are, inevitably, presentations that span areas or that do not seem to naturally classify with other presentations of this workshop.

Anthony Bloch (Michigan) considered Hill's equation with random forcing terms. Andreu Lazaro (Zaragoza), Nawaf Bou-Rabee (Caltech), and Stephane Chretien discussed aspects of stochastic systems, which is an emerging area which everyone anticipates will be important [14, 35]

Katlin Grubits (Hawaii) presented "Self-assembly of particles using isotropic potentials". Eva Kanso (University of Southern California) talked about low order models of swimming. Oleg Kirillov (Moscow M.V. Lomonosov State University) considered models of rotating bodies of revolution being in frictional contact. This has applications to well-known phenomena of acoustics of friction, such as the squealing disc brakes, and singing wine glass Rouslan Krechetnikov (Carleton) considered dissipation-induced instability phenomena in both finite-dimensional mechanical systems, and an infinite-dimensional two-layer quasi-geostrophic beta-plane model, which describes the fundamental baroclinic instability in atmospheric and ocean dynamics Antonio Hernandez-Garduno (Universidad Nacional Autonoma de Mexico) discussed the averaging of Lagrangian systems, illustrating it with the example of the forced inverted pendulum.

There were presentations on general theory. Lie groupoids are a unifying thrust of geometry. Manuel de Leon (Instituto de Matematicas y Fisica Fundamental) considers geometric Hamilton–Jacobi theory on almost Lie algebroids, with applications to nonholonomic mechanical systems. Rui Loja Fernandes (Instituto Superior Tecnico) asked what happens with respect to reduction for general (non-free) proper Poisson actions. This involved defining Poisson stratified spaces and results which establish that $\bigcup_{(H)} M_{(H)}/G$ is a Poisson stratification of M/G, where there is a Poisson action of G on M. Ivan Struchiner (UniCamp–Campinas–Brasil) began by asking, up to isometries, what are all constant curvature Riemannian metrics in a neighborhood of $0 \in \mathbb{R}^2$? Similar classification problems appear in many other settings, including geometric mechanics, pdes, variational problems, etc. He presented a general approach to such classification problems using Lie algebroids and Lie groupoids. Cotangent bundles are a principle example in geometric mechanics, and indeed every regular Lagrangian system and a cotangent bundle fomulation, wherein the symplectic form is canonical. Tanya Schmah (Macquarie) presented explicitly construction of symplectic tubes, particularly for T^*Q and a group G acting by cotangent lifts. Symplectic tubes are symmetry adapted coordinates near group orbits of G, and they are one of the more important applications of the symplectic geometry in mechanics. Dan Offin (Queen's) reviewed some translations of the Maslov index, and explained how to use them to predict stability and instability for global periodic solutions determined by variational principles.

Bifurcations, even in the absence of Hamiltonian structures, are highly geometric. Luciano Buono (University of Ontario Institute of Technology) considered steady-state bifurcations in reversible-equivariant vector fields. He showed that the analysis of these bifurcations can be reduced to the study of bifurcations of an equivalent equivariant vector field with no time-reversibility, sometimes also having parameter symmetry and for which a bifurcation theory already exists.

3 Outcomes of the Meeting

This was a meeting of leading experts in applications of differential geometry to mechanics. There were a great many informal discussions, the results of which cannot be catalogued. People who work in Geometric Mechanics are scattered worldwide and there was a large benefit of having many of them in one place. The scope of the discussion was terrific, and the expansive view of the area and the activity in it was of great value. This will have affected the thinking of many of the participants. The isolation of many was diminished, and this persisted in real ways after the workshop.

It could not have been done so well elsewhere, and perhaps not at all. The participants have a very high opinion of BIRS. It is regarded as a high profile opportunity for meeting by people who do not choose to attend every conference to which they are invited. In subsequent planning, BIRS is singled out as a place to which select opportunities might be directed. The meeting directly gave rise to three other BIRS proposals. The participants want to return to BIRS.

In modern day Science, we all should organize meetings. It might not so immediate to the minds of such as the BIRS directors and high level supporters, but it is true and it should be stated, that not everyone is naturally predisposed to this activity. An important effect of BIRS is to encourage faculty in its member Universities, and elsewhere in Canada, to participate in the organization of such high level meetings. They learn, in a supportive environment, that they really can participate in the organization of meetings, at the highest level. BIRS is great help for the visibility of faculty who might not otherwise organize such these activities. Such was the case for this workshop.

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