Off-shell supersymmetry via graph theory and superspace

Charles F. Doran (University of Washington), Gregory D. Landweber (University of Oregon)

22 July 2006–29 July 2006

1 Introduction

The purpose of this workshop was to study physical supersymmetry (SUSY), i.e., representations of the super Poincaré group and algebra. In particular, we are working towards a classification of *off-shell* supersymmetry, representations of supersymmetry on unconstrained spaces of fields. Supersymmetry is well understood *on-shell*, in terms of fields dynamically constrained to satisfy equations of motion determined by a Lagrangian. In contrast, off-shell supersymmetry is understood only up through spacetime dimension 6, which falls short of the 10 or 11 dimensions required for string theory or M-theory. Indeed, navigating the passage from on-shell to off-shell supersymmetry is called by Gates as "The Fundamental Supersymmetry Challenge" [6].

Traditionally, off-shell fields have been constructed by considering superfields on superspace, a supersymmetric extension of spacetime. Our approach is different, instead studying the reductions of supersymmetric theories to one time dimension, in terms of mechanics. Our primary tools are Adinkras, graph-theoretic diagrams which encapsulate the combinatorial data required to classify the one-dimensional supersymmetric theories. In this workshop, we use Adinkras to classify known theories and generate new ones, in an attempt to understand the deeper meanings of Adinkras and test their limits.

2 Formal Morning Sessions

Each morning of the workshop, one of the participants led a formal session lasting approximately three hours. These sessions were not conventional talks, but rather directed discussions, meant to advertise a particular problem in the field and provide all of the necessary background for the group to work on it during the remainder of the workshop. On the first day of the workshop, organizer Charles Doran welcomed the participants and led an introductory discussion. The other formal morning sessions were as follows:

2.1 Mike Faux: Physics issues in 4D SUSY

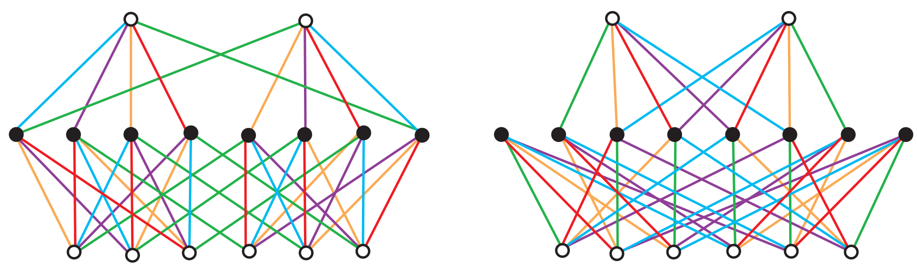
Mike Faux described the physics context and motivations for many of the mathematical problems in our research program. He gave an overview of the important known representations of supersymmetry in spacetime dimensions ranging from one to eleven dimensions, elucidating which subset of these were known off-shell and which subset were known only on-shell. He also explained some of the remarkable features which appear in quantum field theory when supersymmetry is included, among these the Green-Schwarz anomaly cancellation mechanism which appears in 10D $N = 1$ supersymmetry reflecting the magic of string theory.

Faux then explained several puzzles associated with the case of 4D $N = 2$ supersymmetry. These fell into two categories. First were puzzles associated with understanding duality relationships among the minimal multiplets in this class and second were puzzles associated with understanding the natural embedding of the 4D $N = 2$ multiplets into representations of garden algebras in 1D $N = 8$ supersymmetry when these theories are dimensionally reduced to one dimension. Resolving these puzzles is an important preliminary for understanding the yet-mysterious off-shell structure associated with 4D $N = 4$ super Yang-Mills theory.

2.2 Greg Landweber: Filtered Clifford supermodules

Greg Landweber began by reviewing our joint paper [1], which gives a one-to-one correspondence between integrally graded representations of off-shell supersymmetry and filtrations on representations of Clifford algebras. This transforms our infinite dimensional problem of classifying off-shell supersymmetry into the finite dimensional problem of classifying filtrations of Clifford algebra representations. This point of view also allows us to extract numerical invariants from our representations, such as the list dimensions of the filtration levels or associated graded degrees, or the numbers and heights of the sources and sinks in the corresponding Adinkra.

Landweber also showed explicit examples of $N = 5$ supersymmetry, developed jointly with Charles Doran. This is the simplest case which demonstrates the following two new phenomena: First, we demonstrated that supersymmetry representations are *not* completely by the dimensions of associated graded degrees, as is asserted in [7]. Rather, we constructed two distinct representations, both of dimensions$(2, 8, 6)$, as given by the following topologically different Adinkras:



(These representations can indeed be distinguished by counting their sources.) Second, we showed that there exists a one parameter family of $N = 5$ supersymmetry representations which do not admit an Adinkra at all, which Tristan Hübsch pointed out correspond to superpositions of these two distinct $N = 5$ Adinkras.

Finally, Landweber concluded with an algebro-geometric discussion of the moduli space of off-shell supersymmetric theories, or equivalently filtered Clifford supermodules, up to equivalence. While the group has considered such a moduli space for several months, this was the first time we formalized the concept, giving a precise definition of equivalence of representations. With this formal definition, Landweber conjectured that this moduli space admits a smooth stratification, with strata indexed by the possible Adinkras, and each stratum consists of precisely those representations which are smooth deformations of the given Adinkra.

2.3 Kevin Iga: Are Adinkras Enough?

Kevin Iga began by introducing the active vs. passive conceptualization of symmetries used by physicists, and motivated considering the set of field variables (formally) with their derivatives to all orders. He then introduced to the physicists the mathematical notion of a module over a ring, and discussed modules over $\mathbb{R}[\partial_\tau]$. The vector space of formal field variables with derivatives forms a module over this ring. Free modules over this ring correspond to off-shell $d = 1$ fields, and the $Q_1, \ldots, Q_N$ are linear transformations, and hence, can be represented by matrices.

Matrices also occur when describing basis changes, which are relevant for the following reason: Formally, replacing ∂_τ with 1 turns a SUSY multiplet into a Clifford algebra representation. Such representations can be described using Adinkras, as we show in our paper [2]. Taking the basis in which the Clifford algebra representation is Cliffordinkraic, we get a possible basis for the SUSY multiplet, and there is a method analogous to Gauss-Jordan elimination that determines whether or not this is a basis.

Using these ideas, Iga showed how to prove that in certain cases, we can prove Adinkraizability, and how to prove that in other cases, we do not have Adinkraizability.

2.4 Tristan Hübsch: Enough is Enough. Getting SUSY to Behave

In the process of studying the action of the supersymmetry generators on the component fields of a supermultiplet, we have discovered that the kind found in the physics literature forms a rather special subset, which we call "adinkraic" as these can be described using Adinkras [4]. Tristan Hübsch discussed the so-called "demi-adinkraic" supersymmetry actions, of which the adinkraic ones are a special case, and outlined an iterative method of decomposing such demi-adinkraic supersymmetry actions into combinations of adinkraic ones. The precise conditions which guarantee the eventual termination of this procedure are under study.

The possible need for such decompositions is in stark contrast with the comparatively simpler situation with classical Lie algebras and their representation, the computational ease and rigor of which we should like to develop for off-shell representations of supersymmetry. Hübsch briefly reviewed the main features of the "roots and weights" framework, and indicated where this approach is obstructed in supersymmetry.

In every field theory, the space of states is typically constructed as a "Fock space", and in supersymmetric theories this must begin with the so-called supersymmetric vacuum states. Hübsch described the construction of the induced supersymmetry generators, which act upon the field space as differential operators, and give rise to the system of partial differential equations that defines the supersymmetric vacuum states. This then provides a natural extension to realm of applications supersymmetric structures we have been studying.

Finally, Hübsch described an exhaustive method of constructing all the "superspace pseudo-projectors", for all N. The square of these operators is a $(\partial_\tau)^k$-multiple of themselves, with k an appropriate positive integer, and a complete set of such operators adds up to $(\partial_\tau)^k$. Besides providing an alternative way of constructing supermultiplets whose Adinkra has the topology that is a quotient of the N-cube, the pseudo-projectors are indispensable in constructing manifestly supersymmetric Lagrangians for such supermultiplets.

3 Informal Afternoon Sessions

The afternoons were reserved for informal group discussions, often continuing ideas presented that morning. This was our opportunity for experimentation and new approaches. Some of the topics we discussed include:

3.1 Complex Structures

In order to obtain a complete classification of supersymmetric theories in a four dimensional spacetime, we realized that it is necessary to consider not only the real structure of the representation, but also its various complex structures. By introducing new edges into Adinkras encoding multiplication by i, we were able to explore the consequences of imposing a complex structure on a real Adinkra. With such complex structures propagated through the Adinkra via the supersymmetry action, we were able to successfully distinguish the four dimensional chiral and anti-chiral super multiplets. Taking into account these complex structures, we found twelves distinct complex Adinkras for one-dimensional $N = 4$ supersymmetry.

3.2 "Garden" Algebras

After discussing complex structures for representations of supersymmetry, we revisited the paper [5], with its "Garden" Algebras. The group developed a dictionary for translating between the concepts presented in this paper and various mathematical concepts related to Clifford algebras. In addition, we examined closely its treatment of real, complex, and quaternion structures on supersymmetry representations. In particular, the quaternion structure in the $N = 4$ case is essential to showing that the $N = 4$ moduli space of off-shell representations is discrete.

3.3 The Borel-Weil Theorem

Via the Borel-Weil theorem, the irreducible representations of compact Lie groups can be constructed as spaces of holomorphic sections of line bundles over homogeneous spaces. Greg Landweber is convinced that there should be an analogous correspondence between representations of supersymmetry and topological spaces, for which Adinkras describe only the 1-skeleton. Evidence for this comes from the fact that the assignment of signs to the edges of an Adinkra is the same data as is needed to determine a spin structure on a Riemann surface. This suggests that instad of considering holomorphic sections of a complex line bundle, we must consider a Dirac operator and its corresponding harmonic sections.

Landweber gave an overview of the standard Borel-Weil theorem for the group, and the group then discussed several possible approaches for extending this to our case, including: (1) Extending the supersymmetry algebra with additional generators, correpsonding to grading or dilations operators, central extensions, and multiplication operators.(2) Constructing the representation as holomorphic sections on a supermanifold. This works for complex representations and even N, but the generalization to the odd N case and more importantly the real case does is not apparent. However, it does appear closely related to the physics technique of extracting irreducible representations of symmetry by imposing superdifferential constraints on superfields. (3) Constructing Clifford algebra representations as spaces of harmonic sections for a twisted Dirac operator on spheres. While this does not give any new information about Clifford algebra representations, this approach may shed light on how to extend them to representations of supersymmetry. (4) Constructing a complex, analogous to the Dolbeault complex, for describing a free resolution of representations of supersymmetry. This imposes all superdifferential constraints simultaneously, while also allowing for irreducible representations extracted by applying gauge transformations to superfields.

4 Outcomes of the Meeting

- Hübsch and Iga made significant progress on [2], which is nearing submission to the arXiv.
- Faux and Gates worked on a paper using complex structures in the classification of 4d supersymmetry.
- Landweber and Doran have embarked on a paper to explicitly classify all representations of supersymmetry up to $N = 8$, and they also plan to explore the moduli space of supersymmetry representations.
- Based on our group discussions, Faux and Hübsch proved that under the appropriate circumstances, the supersymmetry generators must act by first order differential operators. After the workshop, Landweber refined these arguments to prove this result in greater generality. This lemma vital to showing how one-dimensional supersymmetry representations can be “oxidized” to higher dimensions, as in [3].
- In order to generate our Adinkra diagrams more efficiently, Greg Landweber wrote a computer program to manipulate these diagrams. This can also be used as a computational tool to explore Adinkras too large to be drawn by hand. Using this technology, Landweber recently discovered supersymmetric representations with $N \geq 13$ which admit (at least) two topologically distinct Adinkras.

References

[1] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, and G. Landweber, Off-shell supersymmetry and filtered Clifford supermodules, `arXiv:math-ph/0603012`.

[2] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, and G. Landweber, A classification of Adinkra topologies through representations of Clifford algebras, in preparation.

[3] C. Doran, M. Faux, S. J. Gates, Jr., T. Hübsch, K. Iga, and G. Landweber, Off-shell supersymmetric fields and mechanics (working title), in preparation.

[4] M. Faux and S. J. Gates, Jr., Adinkras: A graphical technology for supersymmetric representation theory, *Phys. Rev.* **D71** (2005), 065002, `arXiv:hep-th/0408004`.

[5] S. J. Gates, Jr., W. D. Linch, III, and J. Phillips, When superspace is not enough, `arXiv:hep-th/0211034`.

[6] S. J. Gates, Jr., W. D. Linch, III, J. Phillips, and L. Rana, The fundamental supersymmetry challenge remains, *Gravit. Cosmol.* **8** (2002), 96–100, `arXiv:hep-th/0109109`.

[7] Z. Kuznetsova, M. Rojas, and F. Toppan, Classification of irreps and invariants of the N-extended Supersymmetric Quantum Mechanics, `arXiv:hep-th/0511274`.