Complex Arrangements: Algebra, Geometry, Topology

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1 Overview of the Field

A hyperplane arrangement A is a finite collection of hyperplanes in some fixed (typically real or complex) vector space V. For simplicity, in this overview we work over the complex numbers $\mathbb{C}$. There is a host of beautiful mathematics associated to the complement $X = V\ A$. Perhaps the first interesting result in the area was Arnol'd's computation [2] of the cohomology ring of the complement of the pure braid arrangement; this was shortly followed by work of Brieskorn [3] computing the cohomology ring $H^*(X)$ of X in terms of differential forms. Subsequently, Orlik and Solomon [21] gave a presentation of the ring $H^*(X)$ as the quotient of an exterior algebra by an ideal determined by the intersection lattice $L(A)$ of the arrangement; the rank one elements of $L(A)$ are the hyperplanes, and a rank i element is a set of hyperplanes meeting in codimension i.

A far more delicate invariant of X is the fundamental group; unlike the cohomology ring, $\pi_1(X)$ is not determined by $L(A)$. In [15], Hirzebruch wrote "The topology of the complement of an arrangement of lines in P^2 is very interesting, the investigation of the fundamental group very difficult." For any group G, the lower central series is a chain of normal subgroups defined by $G_1 = G$, $G_2 = [G, G]$, and $G_{k+1} = [G, G_k]$). Because X is formal (that is, its rational homotopy type is determined by the rational cohomology ring), the ranks of the successive quotients $\phi_k = G_k/G_{k+1}$ (the LCS ranks) are combinatorially determined for $\pi_1(X)$; so some information about the fundamental group does depend on the combinatorics of $L(A)$. Combining results of Priddy, Sullivan, Kohno, and the Poincaré-Birkhoff-Witt theorem, one can show that the LCS ranks are determined by the graded pieces of the diagonal Yoneda Ext-algebra $Ext^(H^*(X))_i(\mathbb{C}, \mathbb{C})_i$; this leads to beautiful connections to Koszul algebras and duality, first observed by Shelton-Yuzvinsky [28].

2 Recent Developments and Open Problems

A good deal of the current work in the field centers around some very striking relationships relating the LCS quotients of $\pi_1(X)$ to a projective variety associated to $H^*(X)$. This variety, known as the resonance variety $R^1(A)$, is defined as follows: For each element $a \in H^*_1$, the Orlik-Solomon algebra can be turned into a cochain complex (H^*, a). The i^{th} term of this complex is simply the degree i graded piece of H^*, and the differential is given by multiplication by a:

$$(H^*, a)\colon \quad 0 \longrightarrow H^*_0 \xrightarrow{\cdot a} H^*_1 \xrightarrow{\cdot a} H^*_2 \xrightarrow{\cdot a} \cdots \xrightarrow{\cdot a} H^*_\ell \longrightarrow 0. \tag{1}$$

This complex arose in the work of Aomoto [1] on hypergeometric functions, and in the work of Esnault, Schechtman and Viehweg [10] on cohomology with coefficients in local systems. In [11], Falk gave necessary combinatorial conditions that the components of $R^1(A)$ satisfy, and conjectured that the components are linear projective spaces; this was proved simultaneously by Cohen-Suciu [7] and Libgober-Yuzvinsky [17].

In [29], Suciu conjectured a relationship between $R^1(A)$ and the LCS ranks; in particular, under certain conditions the LCS ranks are solely by the dimensions of the components of $R^1(A)$. There are several means of attacking this conjecture, which has been proved in a number of cases (Falk–Randell [13], Papadima–Suciu [24], Jambu–Papadima [16], and Lima-Filho–Schenck[18]). Nevertheless, the conjecture remains open in general. Connections between higher homotopy and resonance provide a wealth of new and exciting research avenues that are just beginning to be explored; in short, this is a field with a wonderful array of open problems which connect algebra, geometry, topology and combinatorics.

3 Scientific Progress Made

In Fall of 2004 MSRI held a semester-long program on arrangements. The semester was wonderfully stimulating and served both to advance existing projects and foster new collaborations. Interaction among participants, postdocs, and graduate students at MSRI made clear the need for a good, up-to-date central reference for the field. In 1990 Orlik and Terao published ”Arrangements of Hyperplanes” [22], but the field has advanced dramatically since that time.

The aim of this workshop was twofold; the first objective was to work on a ”state of the art” book on hyperplane arrangements, and the second objective was to work on various joint research projects. Both of these aims were achieved; indeed, the meeting turned out to be crucial to the development of the book. We ended up restructuring the first third of the chapters, and changing the flow of the book quite dramatically. It is impossible to emphasize enough how important it was for us to all be in the same place, debating and challenging each other; we departed with a clear concensus on the selection of topics, flow, and notation. With seven authors involved, this is crucial. We plan to finish the first draft and hand it over to the publisher by the end of the summer. In addition to work on the book, the meeting also served to further a number of ongoing research collaborations:

- Cohen and Suciu made significant progress in their ongoing study of the boundary manifolds of hyperplane arrangements. Shortly after the Banff meeting, they completed a 29-page paper “The boundary manifold of a complex line arrangement”, available at `http://arxiv.org/math.GT/0607274`. This paper has now been submitted to Geometry & Topology Monographs.

- Denham and Suciu discussed their on-going work on moment-angle complexes, monomial ideals, and coordinate subspace arrangements. Shortly after the Banff meeting, they produced a substantial, expanded revision of their first paper on the subject, following referee’s comments. The paper, available at `math.AT/0512497`, was submitted to the Pure and Applied Mathematics Quarterly, for the Bob MacPherson Festschrift. They also discussed several follow-up projects, to be pursued in-depth this Fall at MSRI and Oberwolfach.

- Discussions between Cohen, Denham, and Falk facilitated progress on a project on resonance and critical points of arrangements. For a H of A, let α_H be a linear form with kernel H. A collection of weights $(\dots\lambda_H\dots)$ gives rise to a “master function” $\Phi_\lambda = \prod_{H\in A}\alpha_H^{\lambda_H}$, and a corresponding element $a = d\log\Phi_\lambda$ in H_1^*. Recent work on “discriminantal” arrangements suggests a relationship between resonance, $a \in R^k$, and the critical set of the master function, $\text{crit}(\Phi_\lambda) = \{x \in V \mid a(x) = 0\}$. In the case where A is a free arrangement, the above participants have shown that resonance informs on the codimension of the critical set: if $a \in R^k$, then $\operatorname{codim}\text{crit}(\Phi_\lambda) \leq k$. Discussions at the workshop focused on the potential reverse implication: does knowledge of the (codimension of the) critical set of Φ_λ inform on the resonance of $a = d\log\Phi_\lambda$?

- Falk and Yuzvinsky worked to put the finishing touches on their paper ‘Cohomology of Orlik-Solomon algebras, multinets, and pencils of curves’ and how to incorporate the very interesting comments they received after the first draft of the paper had appeared on the arXiv.

- Denham and Schenck made large strides on understanding the connection of resonance varieties, the support varieties of various Ext modules, and Fitting ideals. In particular, they Recent work shows that for arrangements, the first resonance variety $R^1(A)$ is equal to $V(annExt^{\ell-1}(F(A),S))$, where S is a symmetric algebra and $F(A)$ is a finitely generated, graded S–module depending only on the cohomology ring of $X(\mathcal{A})$.

Denham and Schenck were able to generalize this result on the cohomology ring of an arrangement complement in two directions; showing that $F(A)$ may be replaced with an arbitrary S–module M, and that

$$R^k(M) = \bigcup_{k' \leq k} V(annExt^{\ell-k'}(M,S)),$$

where $R^k(M)$ is defined in terms of Koszul cohomology. By way of the Cartan-Eilenberg spectral sequence, this yields a stratification of $R^k(M)$ related to the filtration of M.

4 Outcome of the Meeting

The meeting yielded a firm outline and division of labor for the book on arrangements, as well as advances on a large number of research projects, described above. Overall, it was an extremely fruitful visit for all participants!

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