Local index theorem in noncommutative geometry

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There have been two new proofs of the Connes-Moscovici local index theorem [2] produced by some of the organisers and described in [5] and [3] [4]. The latter proof applies also to the situation where the standard spectral triple [1] consisting of a C^* algebra $\mathcal{A}$ acting on Hilbert space H and an unbounded self adjoint operator D with $[D, a]$ bounded for all a in a dense subalgebra of $\mathcal{A}$ is replaced by a semifinite spectral triple. The latter means D is affiliated to a semifinite von Neumann algebra $\mathcal{N}$ which contains $\mathcal{A}$ and the resolvent of D is in the compact operators in $\mathcal{N}$.

The two week period was divided into three parts due to the fact that Alan Carey and Nigel Higson were only able to come for one week each and only overlapped by three days. The organisers agreed that the main focus of the fortnight would be on examples and applications.

For the first three to four days the emphasis was on checking the details of a preprint by Pask and Rennie in which the semifinite local index theorem was applied to certain graph C^*-algebras. The algebras studied admit a natural action of the circle group and were constrained by the requirement that the algebra should admit a trace. There were problems with the construction of a suitable trace and so considerable effort went into understanding whether the trace was continuous in an appropriate sense. After much effort, these problems were satisfactorily resolved.

After Carey and Azamov arrived talks were organised on a preprint of Azamov, Dodds and Sukochev in which the Krein spectral shift function was constructed in the semifinite von Neumann algebra setting. The question of whether it is related to spectral flow was raised. For a certain path of unbounded operators equality of the two was verified in the case of finite von Neumann algebras. It was conjectured that in general they are not directly related but that there might be a way to use spectral flow to 'subtract' discontinuities in the spectral shift function. Azamov promised to report back on the outcome of this idea after returning to Adelaide.

Phillips contributed a number of missing results and proofs to a manuscript in preparation in which an overview of the analytic approach to spectral flow in semifinite ven Neumann algebras is given. The principle objective was to outline the analytic definition of spectral flow when one was in the situation of paths of operators in a von Neumann algebra with non-trivial center. A secondary objective was to answer some natural questions which had arisen in the 10 years since Phillips original paper on this subject. The ms also contains many examples and the details of these were discussed. The ms is now nearing completion and will be the first publication arising from the BIRS interaction.

Upon Higson's arrival there were informal lectures organised on KK theory. The object was to understand how to use KK theory to understand extensions of the Pask-Rennie preprint to other settings. Higson was able to clarify some of the constructions in KK theory that might be relevant in applications of the local index theorem to graph algebras. Lectures were given by Rennie on the Cuntz algebra and $SU_q(2)$ as graph algebras and conjectural applications of the semifinite local index theorem to them.

There were a number of small group research sessions investigating various questions related to these potential applications. There were also informal discussions of applications of the local index theorem in other settings such as subelliptic operators.

After the departure of Carey and Higson, Azamov pursued the relation of the spectral shift function to spectral flow, while Phillips and Rennie made significant progress on the Cuntz algebra example and some related problems. The progress centred around understanding the KK pairing being computed by the spectral flow formula in the Cuntz algebra example. The $SU_q(2)$ example was examined again in light of the progress on the Cuntz example, but little headway was made.

References

[1] A. Connes, *Non-commutative Geometry*, Academic Press, San Diego, 1994.

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[3] A. Carey, J. Phillips, A. Rennie, F. Sukochev The local index theorem in semifinite von Neumann algebras I. Spectral Flow *Advances in Math* to appear

[4] A. Carey, J. Phillips, A. Rennie, F. Sukochev The Local Index Theorem in semifinite von Neumann algebras II: the even case, *Advances in Math* to appear.

[5] N. Higson, The Local Index Formula in Noncommutative Geometry, *Contemporary Developments in Algebraic K-Theory, ictp Lecture Notes*, no. 15 (2003), 444–536.