Recent Advances in Algebraic and Enumerative Combinatorics

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1 A short overview of the subject area

Algebraic and enumerative combinatorics is concerned with objects that have both a combinatorial and an algebraic interpretation. It is a highly active area of the mathematical sciences, with many connections and applications to other areas, including algebraic geometry, representation theory, topology, mathematical physics and statistical mechanics. Enumerative questions arise in the mathematical sciences for a variety of reasons. For example, non-combinatorial structures often have a discrete substructure when they are studied up to topological equivalence. Moreover, some non-combinatorial structures can be discretized, and are therefore susceptible to combinatorial techniques, and the original question is then recovered by the appropriate limit taking (the dimer problem in statistical mechanics is an instance of this).

Algebraic and enumerative combinatorics is a highly active area of the mathematical sciences, and was the subject of a special year at MSRI in 1996/97, organized by Billera, Bjorner, Greene, Stanley and Simion. The one week workshop at BIRS focussed on three topics, discussed in detail below, in which substantial progress has been made in the last five years. The topics are interrelated, as is essential for a workshop of this length. The workshop was successful in bringing together a very strong international collection of active researchers in algebraic and enumerative combinatorics, as well as other areas of the mathematical sciences in which substantial enumerative questions with a strong algebraic or analytic foundation have arisen. The connections that have been made between algebraic combinatorics and other areas in the last few years have contributed to rich and significant lines of research. Such research requires a familiarity of several research disciplines, and the workshop facilitated this contact between algebraic and enumerative combinatorics and other fields in an effective way.

1.1 SYMMETRIC FUNCTIONS AND GROUP REPRESENTATIONS

Symmetric functions and group representations have played a central role in algebraic combinatorics. This was initiated by the work of Philip Hall [17], which recast much of the classical theory of symmetric functions in terms of linear algebra. Central to this theory is the Schur function, and its generalizations. These include zonal polynomials, Jack symmetric functions, Macdonald symmetric functions, shifted and super Schur functions, and Schubert polynomials. These functions have many equivalent definitions, which bring together different areas of mathematics. For example, on the algebraic side, they are generating functions (via a change of basis) for characters of group representations: for the Schur functions themselves, these are the symmetric group and the complex general linear group; for the super Schur functions, these are Lie superalgebras; for the Schubert polynomials, the group of complex upper triangular matrices. On the combinatorial side, they are generating functions in the monomial basis for two dimensional arrays, called Young tableaux in the case of Schur functions.

The Littlewood-Richardson coefficients are the connection coefficients for expanding the product of two Schur functions into a linear combination of Schur functions. These must be non-negative integers because of their algebraic interpretation as the multiplicity of the irreducible representations of the symmetric group in the tensor product of two irreducible representations. The Littlewood-Richardson rule is a classical combinatorial rule for determining this coefficient, as the number of sequences with certain properties. A new rule for these coefficients, called the honeycomb model, has been used recently by Knutson and Tao [23] to prove the Saturation Conjecture. This result concerns the places where the Littlewood-Richardson coefficient is 0, but relates to a number of other topics, including Horn's Conjecture for the eigenvalues of a sum of Hermitian matrices (shown by Klyachko [19]). A recent paper of Fulton [11] gives a detailed discussion of various situations in which Littlewood-Richardson coefficients play an important role.

Recently, Lapointe and Vinet [25] proved that the coefficients of the Jack symmetric functions are polynomials in the Jack parameter. Knop and Sahi [22] improved this result by showing that the coefficients of the polynomial are nonnegative (conjectured by Macdonald and Stanley).

Goulden, Harer and Jackson [15] have shown, through the use of matrix integrals, that the Jack parameter may be associated with a topological invariant of embeddings of graphs in surfaces and with the Euler characteristic of the moduli space of curves. There is also a connection with finite reflection groups, through the work of Beerends and Opdam [5], and Beerends [3], and with Hilbert schemes, through the work of Nakajima [26]. (The link, from de Concini and Procesi, is that the moduli space can be viewed as a complex vector space with certain hyperplanes removed.) These bring us closer to a combinatorial interpretation of the Jack parameter. Moreover, Knop and Sahi [21] have recently given a tableau interpretation for the Jack function.

Schubert polynomials originally arose in the work of Demazure and Bernstein-Gel'fand-Gel'fand describing the cohomology of flag varieties. The combinatorial theory of in the case of the symmetric group was developed by Lascoux and Schutzenberger and extended by Macdonald, Billey and Haiman, Fomin and Kirillov, Fulton, Pragacz and Ratajski, and others. Many combinatorial problems remain, including finding a generalization of the Littlewood-Richardson rule, for multiplying Schubert polynomials. Also desirable are better formulas for the quantum cohomology, equivariant cohomology and K-theory analogs of Schubert polynomials.

Macdonald polynomials are a two-parameter generalization of Schur functions that also include Hall-Littlewood polynomials, Jack polynomials and zonal polynomials as special cases. Work on these initially focussed on Macdonald's 1989 conjectures about the polynomiality and non-negativity of the entries in a certain change of basis matrix, but has expanded into various areas because of the the relationship of these polynomials to, e.g., the representation theory of quantum groups, affine Hecke algebras, and the Calogero-Sutherland model in particle physics [31]. Perhaps the greatest interest in these polynomials has been raised by the manner in which they arise in the study of the diagonal action of the symmetric group on polynomials in two sets of variables. The so-called "n! Conjecture" of Garsia and Haiman has been the most notable result in this area, recently proved by Haiman [16], based on the geometry of the Hilbert scheme of n points in the plane. A number of refinements and extensions of the n! Conjecture (e.g., by F. Bergeron, Garsia, Tesler) are under active study. (The connection with Macdonald polynomials is that the coefficients in the change of basis matrix of Macdonald's 1989 conjectures are thus identified as multiplicities matrix of irreducible representations in a particular representation of the symmetric group, and thus must be non-negative.)

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1.2 RAMIFIED COVERS OF SURFACES AND HURWITZ NUMBERS

There has been a considerable amount of recent work on Hurwitz numbers, enumerating ramified covers of the sphere by surfaces of given genus. The covers are such that there is a unique point with arbitrary ramification type, the remaining branch points having elementary ramification. Work on this problem started with Hurwitz in 1895 and interest in it was reawakened largely by the work of the physicists Crescimanno and Taylor [9] in 1991 (they considered the enumeration of coverings of the sphere as a sort of string theory on the sphere). Central to the combinatorial approach to this question is Hurwitz's original construction [18] in terms of minimal ordered factorizations of permutations into transpositions such that the group generated by the transpositions acts transitively on the sheels. The construction can be obtained by considering the monodromy of the sheets around the branch points. An excellent account of the background and the connections between ramified covers of the sphere and the moduli space of curves has been given very recently by Vakil [33]. Goulden and Jackson [13] have used this approach to obtain a number of explicit expressions for Hurwitz number generating functions for low genera, and have given a polynomiality conjecture as a structure ansatz for Hurwitz numbers. Recently, Ekedahl, Lando, Shapiro and Vainshtein [12] have expressed Hurwitz numbers as Hodge integrals over moduli spaces of curves by a remarkable theorem stating that, for a genus g source and ramification type α over ∞,

$$H^g_\alpha = \frac{r!}{\#\mathrm{Aut}(\alpha)} \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,m}} \frac{1-\lambda_1+\cdots\pm\lambda_g}{\prod_i(1-\alpha_i\psi_i)}.$$

This was followed with an algebraic proof by Okounkov and Pandharipande [29] of Witten's Conjecture [35] (Kontsevich's Theorem [20]) in Quantum Field Theory. There are combinatorial aspects of this work that are of profound interest in their own right. The construction of Arnol'd Okounkov-Pandharipande ([1, 29]) is highly suggestive of a matrix model through the enumerative theory of graph embeddings.

Bousquet-Mélou and Schaeffer [7] have determined an explicit formula for the number of ramified covers of genus 0 in which ramification is arbitrary, but not specified, over every branch point.

There is considerable interest now in exploring the structure underlying the double Hurwitz numbers. [Mention Okounkov, Ionel, Pandharipande etc?] These numbers count ramified covers of the sphere in which that are two points over which there is arbitrary ramification type. It is believed that the structure underlying these should be even richer, and that the structure associated with the classical case is merely a shadow of this. There is a strongly belief that there should be an ELSV-type theorem that expresses this richer structure, but there is little information about what this theorem should be. For example, polynomiality fails on the double Hurwitz case, and this means that the theorem, if it exists, is significantly different. There were informal discussions about this case, and the prospect for using the Hurwitz encoding and the methods developed within algebraic combinatorics to elicit enumerative evidence for such a theorem. Explicit expression for the double Hurwitz numbers have been given in a number of special cases by Kuleshov and Shipiro [24].

The universal case is, of course, the triple Hurwitz problem, in which there are three points over which arbitrary ramification may be specified. This is because any target surface may be formed as a ramificiation respecting connected sum of spheres with three punctures (pairs of pants). However, there is evidence that this case, the most general case, does not have as rich a structure as the double Hurwitz case. Progress on coverings of the torus has been made by Dijkgraaf [10].

1.3 RANDOM MATRICES

Recent breakthroughs in the theory of random matrices have intimate connections with combinatorics. In particular, the definitive work on the largest eigenvalue of a GUE matrix (i.e., a matrix chosen from a certain natural probability distribution on the space of $n \times n$ Hermitian matrices) due to Baik, Deift, and Johansson [2] is connected with (and in fact arose from) increasing subsequences of permutations and the Robinson-Schensted-Knuth algorithm. This work has been further developed by Borodin, Okounkov, Olshanski, Rains, Tracy, Widom, and others, though many interesting questions remain. Moreover, since the pioneering work of Voiculescu [34] it has been known that free probability theory is a fundamental tool for understanding certain aspects of random matrices. Biane [6] has shown deep connections with the "asymptotic representation theory" of the symmetric group, while Nica and Speicher [28, 27] have developed connections between free probability theory, Lagrange inversion, and non-crossing partitions, an object that has been studied extensively in combinatorics. Sniady [30] recently proved a conjecture of Biane [4, Conj. 6.4] concerning Kerov's character generator for values of irreducible characters of the symmetric group S_n on k-cycles, thereby in effect giving the second term in a certain asymptotic approximation to the values of irreducible characters of S_n.

2 Workshop speakers and talks

- Helene Barcelo: *A Discrete Homotopy Theory and its Application to Hyperplane Arrangements*
- Francois Bergeron: *Diagonal Alternants*
- Nantel Bergeron: *Combinatorial Hopf Algebras*
- Christine Bessenrodt: *Products of Characters of the Symmetric Group and Related Groups*
- Mireille Bousquet-Mélou: *Minimal Transitive Factorizations of Permutations*
- Anders Buch: *Quantum Cohomology of Partial Flag Manifolds*
- Persi Diaconis: *Set Partitions and Character Theory for Upper Triangular Matrices*
- Sergey Fomin and William Fulton: *Eigenvalues, Singular Values, and Schubert Calculus*
- Adriano Garsia: *Cohen Macauliness of the Ring of Quasi-Symmetric Functions*
- Patricia Hersh: *A Hodge Decomposition for the Complex of Injective Words*
- Michael Kleber: *Double-headed LR Coefficients and Type* E_n
- Allen Knutson: *Schubert Polynomials and Quiver Polynomials*
- Christian Krattenthaler: *Symmetric Functions Prove Asymptotic Results for Random Walks in Alcoves of Affine Weyl Groups*
- Ezra Miller: *Combinatorics of Quiver Polynomials*
- Jennifer Morse: *Tableaux Atoms and k-Schur Functions*
- Alexandru Nica: *Annular Non-crossing Permutations and Partitions, and Random Matrices*

- Eric Rains: *Vanishing Integrals of Macdonald Polynomials*
- Arun Ram: *The Radical of the Brauer Algebra*
- Victor Reiner: *Conjectures on the Cohomology of the Grassmannian*
- Michael Shapiro and Alek Vainshtein: *Cluster algebras and Poisson geometry*
- Mark Shimozono: *Buch-Fulton Factor Sequence Conjectures*
- Roland Speicher: *Free Probability and Non-Crossing Partitions*
- John Stembridge: *Graded Multiplicities in the Macdonald Kernel and a (q,t)-coinvariant Algebra*
- Craig Tracy: *A Limit Theorem for Shifted Schur Measures*
- Ravi Vakil: *A Geometric Littlewood-Richardson Rule*
- Michelle Wachs: *Posets of Graphs, Partitions and Trees*

3 Workshop participants

- Barcelo, Helene (Arizona State University)
- Bergeron, Francois (LACIM, Universite de Quebec)
- Bergeron, Nantel (York University)
- Bessenrodt, Christine (Universitdt Hannover)
- Billey, Sara (MIT)
- Bousquet-Mélou, Mireille (CNRS - Universite Bordeaux 1)
- Buch, Anders (Aarhus University)
- Diaconis, Persi (Stanford University)
- Fomin, Sergey (University of Michigan)
- Fulton, William (University of Michigan)
- Garsia, Adriano (University of California, San Diego)
- Gessel, Ira (Brandeis University)
- Goulden, Ian (Univ. Waterloo)
- Greene, Curtis (Haverford College)
- Hersh, Patricia (University of Michigan)
- Jackson, David (Univ. Waterloo)
- Kleber, Michael (Brandeis University)
- Knutson, Allen (UC Berkeley)
- Krattenthaler, Christian (Universiti Claude Bernard Lyon-I)
- Miller, Ezra (Mathematical Sciences Research Institute)
- Morse, Jennifer (University of Miami)

- Nica, Alexandru (University of Waterloo)
- Rains, Eric (ATT Shannon Labs)
- Ram, Arun (University of Wisconsin)
- Reiner, Vic (University of Minnesota)
- Shapiro, Michael (Michigan State University)
- Shimozono, Mark (Virginia Polytechnic Institute and State University)
- Sottile, Frank (University of Massachusetts)
- Speicher, Roland (Queen's University)
- Stanley, Richard (MIT)
- Stembridge, John (University of Michigan)
- Tracy, Craig (University of California, Davis)
- Vainshtein, Alek (University of Haifa)
- Vakil, Ravi (Stanford University)
- Wachs, Michelle (University of Miami)

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