Vortex Dynamics: the Crossroads of Mathematics, Physics and Applications

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1 Overview of the Field

Vortex dynamics is at the heart of many unresolved problems in fluid mechanics and is also central to numerous applications in science and engineering. The goal of the meeting was to survey the recent progress in this field focusing on applied mathematics aspects of both classical and emerging problems. We brought together about 48 participants (23 in person and 25 on-line) at different career stages and working on various aspects of vortex dynamics, spanning applied mathematical analysis, scientific computing, physical modelling as well as scientific and engineering applications. A highlight of the workshop were sessions designed to foster cross-fertilization between applied mathematics and different application areas.

Vorticity was introduced by Helmholtz in 1858 and was a major step forward in overcoming the limitations of classical irrotational fluid mechanics. It has been of critical importance in fluid mechanics ever since. Kelvin's work on vortex atoms sought to establish a fundamental theory of matter based on vorticity. Prandtl's boundary-layer theory, which resolved the 19th-century paradoxes of fluid mechanics, depends crucially on the dynamics of vorticity. In the 21st century, vortex dynamics brings together a vibrant community centered around applied mathematics and involving scholars working in an ever-growing range of application areas such as biomechanics, astrophysics, oceanography, atmospheric sciences, aeronautics, acoustics, condensed matter, and computational physics. The workshop provided plentiful interaction opportunities to experts in some of these areas who do not often get to attend the same specialized conferences.

2 Recent Developments and Objectives

The main goal of the workshop was to provide a forum for an exchange of ideas concerning recent developments and future directions in the study of vortical flows, involving both their theoretical and applied aspects. Some outstanding open problems in this field where significant progress is currently being made include

- *hydrodynamic turbulence* which remains one of the last unsolved problems in classical physics; there is a consensus that vorticity plays a key role in understating turbulent flows and ever-increasing computational power has led to high-resolution simulations probing the structures of vortices and their interactions, especially in the context of the phenomenon of vortex reconnection [6, 15, 12],

- the possibility *singularity formation* in the 3D Navier-Stokes and Euler equations is one of the biggest unsolved questions in mathematical fluid mechanics (recognized by the Clay Mathematics Institute one of its seven "Millennium Problems" [7]); as demonstrated by both mathematical analysis and innovative computational studies, concentrated vorticity plays an essential role in all possible scenarios of singularity formation [19, 14],
- *exact solutions* of the Euler equations describing vortices have always been at the center of our study of fluid flow; the discovery of new families of equilibrium solutions and understating their stability have been the highlights of the recent progress in this area which has been fuel ed by new developments in complex variable theory [11, 4, 21],
- *geometric and topological* aspects of vortex motion where the application of modern methods of differential geometry and topology makes it possible to extract unifying features of diverse flow phenomena arising in fields such as geophysics, biofluids, and superfluids, as illustrated in the recent collection [3],
- *laboratory techniques* such as particle-image velocimetry (PIV) and laser Doppler velocimetry (LDV) have revealed the internal structures of vortices in a variety of experimental flow configurations; in particular, for the first time they made it possible to directly measure space- and time-resolved vorticity fields in complex flow allowing for a critical verification of theoretical models [17, 23],
- *quantum turbulence* is a new frontier in physics where vortices, understood as lattice defects, have played a key role in understanding the equilibria and dynamics in Bose-Einstein condensates; these results have been obtained using a combination of mathematical analysis and computational studies [1, 5],
- *vortex models* allow us to simplify the mathematical description of many flow phenomena dominated by vorticity; as such, they are instrumental in understating complex flows arising in numerous scientific and engineering applications, including animal locomotion (flying and swimming), pattern emergence in collective behavior (fish schooling [8]), aerodynamics of cars, planes and wind turbines [20], and atmospheric dynamics (extreme weather events).

Vortex dynamics has been the topic of meetings that had regularly taken place in the past, usually under the auspices of the International Union for Theoretical and Applied Mechanics (IUTAM). These meetings included the symposia in Copenhagen (2008), Cambridge (2012), Fukuoka (2013), Venice (2016), Carry-le-Rouet (2017) and San Diego (2019). Vortex dynamics was also a recurrent theme during the semester-long thematic program "Topological dynamics in the physical and biological sciences" (Isaac Newton Institute in Cambridge, UK, Fall 2012), "Multiscale Scientific Computing" (Fields Institute, January–April 2016), "Applied and Computational Complex Analysis" (Isaac Newton Institute in Cambridge, UK, Fall 2019), and "Mathematical Hydrodynamics" (Fields Institute, Fall 2020).

3 Workshops highlights

One of the goals of the workshop was to foster interdisciplinary collaborations, especially, among mathematicians and researchers working in "application" domains. With this goal in mind, presentations during the workshop were arranged in several thematic streams reflecting some of the areas listed in Section 2. Most of these streams featured at least one presentation given by a senior researcher and intended to give an overview of the topic and highlight some open questions.

Given the venue of the workshop, we aimed to attract a significant number to researchers from Eastern Asia, although due to reasons beyond their control a number of participants from Japan and South Korea was eventually unable to attend in person. We were however able to bring together a diverse group of researchers based on four continents and spanning the entire career spectrum. The main thematic areas of the workshop are listed below together with the corresponding presentations.

3.1 Topology and Geometry in Classical and Quantum Fluid Mechanics

- **Dynamics and bifurcations of critical points of vorticity with applications to wake transitions**

  Morten Bröns (Technical University of Denmark)

  The wake of the fluid flow around a circular cylinder typically organizes itself into a set of discrete vortices, the periodic von Kärmän vortex street. This is denoted a 2S wake as two single vortices, one of each sign, are shed during one period. If the cylinder also oscillates, more complex wake patterns may arise. One is the P+S wake where a Pair and a Single vortex are shed. Identifying the center of a vortex by a local extremum of vorticity we develop a bifurcation theory for such points with time as the parameter, hence providing a topological description of the creation and destruction of vortices. Applying this theory to numerical simulations of the wake behind an oscillating cylinder, we will show that the transition from a P to a P+S wake as the amplitude of the cylinder oscillations increases is a complex sequence of topological and dynamical bifurcations and not just a single event as previously tacitly assumed. The transition goes through states which have a spatio-temporal structure where vortices are created and destroyed. These states cannot be described by combinations of the simple symbols S and P, and we propose an extended classification scheme to describe these intermediary wakes.

- **Quantum vortex knots and links under zero helicity condition**

  Renzo L. Ricca (Univ. Milano-Bicocca)

  By exploiting the hydrodynamic formulation of the Gross-Pitaevskii equation (GPE) for quantum fluids, we present new results on the creation and evolution of vortex defects forming knots and links in the condensate. First, we prove that for such systems circulation is quantized, and kinetic helicity remains conserved and identically zero in time [2]. Then, we consider the direct and inverse topological cascades of knots and links, showing that these can be produced by the decay of complex knots, or by the interaction of simple, unknotted, unlinked loops [25]. In relation to the Seifert surfaces given by the isophase minimal surfaces spanning knots and links, we show that these are critical markers for energy. Finally, because of the zero helicity condition, we show that there is a continuous exchange of writhe and twist helicity, with the possible production of new defects as a manifestation of the Aharonov-Bohm effect [9].

- **A topological study on cascade decay of fluid vortex knots/links**

  Xin Liu (Beijing Univ. Technology)

  Topology plays a more and more important role in the research of vortex dynamics. In this talk, we focus on cascade evolutions of vortex knots/links in classical and quantum fluids, where an evolving procedure from a high-topological complexity state to a low-complexity state contains a series of midway stages connected by reconnection events serving as topological non-conservative transitions. Placing emphasis on the mathematics behind the phenomenon, we propose a geometric method for the study [18]: using the orthogonal polynomial basis (such as Legendre, Hermitian, etc.) to span an algebraic space, such that a knot polynomial obtains a set of coordinates, and thus the corresponding knot/link is represented by a point in the space. Then an evolving pathway becomes a geodesic route in the space from the knot/link point to the origin, with the complexity degree monotonically decreasing; and an intermediate stage is able to be interpreted as a midpoint on the decaying pathway. The method has been proven useful in explaining some recent laboratory and numerical observations in classical and quantum fluids [16, 17] as well as recombinant DNA plasmids [22, 24]. Joint work with Renzo Ricca (U Milano-Biccoca, Italy) and Xinfei Li (Guangxi U Science and Technology, China).

- **Topological invariants and Nambu bracket for ideal fluid dynamics and magnetohydrodynamics**

  Yasuhide Fukumoto (Kyushu Univ.)

  For the ideal magnetohydrodynamics (MHD), the cross helicity, the total mass, the total entropy, and the magnetic helicity constitute the complete set of the Casimir invariants with respect to the Lie-Poisson bracket. These may be regarded as the topological invariants associated with the particle relabeling symmetry. We apply Noether's theorem to a variational framework, using the Lagrangian label function, a map from the Eulerian position to the Lagrangian position, and prove that the Noether charge is the cross helicity, the volume integral of the scalar product of the velocity field and a frozen-in field. We then construct the Nambu bracket for the ideal MHD, using the three Casimirs, the cross helicity, the total entropy, and the magnetic helicity. The resulting bracket has the desired properties that it has fewer redundant terms and that the coefficients are all constants. The Lie-Poisson bracket induced from the Nambu bracket brings an extension of the known one, which automatically guarantees the cross-helicity to be a Casimir invariant. With this form, the isomagnetovortical perturbations are explicitly written out in terms of the Casimirs.

- **Point vortex dynamics in background fields on surfaces**

  Yuuki Shimizu (The Univ. Tokyo)

  Point vortex dynamics are widely adopted as simplified models for two-dimensional Euler flows with localized vortex structures. Recent advancements have considered the effects of advection by background fields and the dynamics on curved surfaces. This talk will review these developments and introduce the mathematical justification of point vortex dynamics in background fields on surfaces as weak solutions to the Euler equations in the sense of de Rham currents.

3.2 Vortex models

- **Symmetry and motion of particles and swimmers in fluid flows.**

  Kenta Ishimoto (Kyoto Univ.)

  When studying the movement of fluids, our understanding of their behavior often hinges on monitoring the motion of tiny particles suspended within the fluid. This presentation will delve into the dynamics of microscopic objects, with a specific focus on those possessing the ability to propel themselves, such as living organisms. Nearly a century ago, G. B. Jeffery formulated an exact solution to the Stokes equation for a spheroidal object in a simple shear, uncovering a periodic trajectory for its orientation vector, now known as Jeffery's orbit. In this talk, by introducing the concept of hydrodynamic symmetry and expanding upon Jeffery's equation based on the equation of motion, we broaden its applicability to encompass general chiral axisymmetric objects. Additionally, we will illustrate through classical multi-scale perturbation theory how a range of intricate self-propelled movements can be elucidated using a generalized asymptotic form of Jeffery's equation [13].

- **Vortex simulation of swimming of a fish-like body**

  Sung-Ik Sohn (Gangneung-Wonju National Univ)

  We investigate the undulatory motion of a body translating through a quiescent fluid. This study is motivated by the anguilliform swimming of aquatic animals such as eels. We use an inviscid vortex shedding model to simulate the swimming motion. The body and separated vortices from the trailing edge of the body are described by vortex sheets. The model demonstrates the self-propulsion of the swimming body and the formation of a vortex street shed from the body. We discuss various kinematics and dynamics of a swimming body such as the forward and lateral velocities, as well as forces, in comparison with Navier-Stokes simulations. We also examine the wake pattern and swimming efficiency, which depends on recoil motions of lateral translation and rotation of a body.

- **How insects fly: the attached leading-edge vortex and its vorticity dynamics**

  Long Chen (Beihang Unversity)

  Understanding the fluid physics of natural flyers and achieving enhanced unmanned flights of micro air vehicles is a recurring topic in the development of modern aerodynamics. Specifically, bumblebees fly in a low Reynolds number regime around 1000, where the stall angle and the maximum lift coefficient of steady airfoils are both significantly reduced. This leads to the well-known conundrum that "bumblebees cannot fly". Over the last 30 years, the mystery of how insect wings can generate superb high lift that cannot be predicted by steady aerodynamic principles has been unfolded. The formation and sustained attachment of a leading-edge vortex (LEV) explains most of the high lift generated during the reciprocating revolving motion. Unlike the dynamic stall on fixed wings, the LEV on an insect wing is quickly formed and remains attached to the dorsal wing surface until the end of a half stroke. Several hypotheses have been proposed to explain the sustained LEV attachment but less understanding of its vorticity dynamics has been gained. Therefore, our recent works transformed the vorticity transport equation into a co-rotating frame and explained the fundamental vorticity transport inside the LEV using both computational fluid dynamics and oil tank data. Evidence for previous hypotheses has been identified and two novel vortex-tilting-based mechanisms that contribute to LEV attachment, i.e., planetary vorticity tilting and dual-stage radial-tangential vortex tilting, have been identified. Moreover, the LEV intensity is found to be saturated when the arc length at the radius of gyration falls between 2 to 4chord lengths, which exists over a large variety of wing geometries and kinematics. The impacts of the Reynolds number, Rossby number, and wing aspect ratio are also discussed. This presentation gives a summary of these achievements in LEV dynamics, stabilizing mechanisms, and high lift generation.

- **Membrane flutter in three-dimensional inviscid flow**

  Christiana Mavroyiakoumou (New York University)

  Many previous works have studied fluid-structure interactions induced by thin flexible bodies. In most of these studies, the body is nearly inextensible, with a moderate bending modulus. Here we consider softer materials – extensible membranes – that have zero bending modulus and undergo significant stretching in a fluid flow that can lead to flutter. Examples include rubber, textile fabric, and the skin of swimming and flying animals. We develop a mathematical model and numerical method to study the large-amplitude flutter of rectangular membranes that shed a trailing vortex-sheet wake in a 3D inviscid flow. This extends our previous work on membrane dynamics in a 2D flow, where the membrane is a 1D curvilinear segment that undergoes small and large deflections. Here we consider 12 distinct boundary conditions at the membrane edges and compute the stability thresholds and the subsequent large-amplitude dynamics across the three-parameter space of membrane mass ratio, pretension, and stretching rigidity. We find that 3D dynamics in the 12 cases naturally form four groups based on the conditions at the leading and trailing edges. The conditions at the side edges are generally less important but may have qualitative effects on the membrane dynamics – e.g. steady versus unsteady, periodic versus chaotic, or the variety of spanwise curvature distributions –depending on the group and the physical parameter values.

- **The N-vortex problem in doubly-periodic domains with background vorticity**

  Vikas Krishnamurthy (IIT Hyderabard)

  We study the N-vortex problem in a doubly periodic rectangular domain in the presence of a constant background vorticity field. Using a conformal mapping approach, we derive an explicit formula for the hydrodynamic Green's function. We show that the point vortices form a Hamiltonian system and that the two-vortex problem is integrable. Several fixed lattice configurations are obtained for general N, some of which consist of vortices with inhomogeneous strengths and lattice defects.

- **Blade tip vortex dynamics in the wake of asymmetric rotors**

  Thomas Leweke (CNRS / Aix-Marseille Université)

  The wakes behind rotors such as wind turbines, propellers, and helicopters are characterized by helical vortices shed from the blade tips, which can have detrimental effects on downstream structures, such as increased structural loading. Helical vortices are subject to displacement instabilities that lead to pairing between adjacent vortex loops and eventually cause the vortices to break down. Certain modes of these instabilities can be triggered by an asymmetry in the rotor generating the vortices. In three-vortex systems, like those formed by many industrial rotors, the nonlinear vortex interactions are highly complex, introducing the need for a simple model to predict their dynamics. We here present a model for helical vortex systems based on an infinite strip of periodically repeating point vortices, whose motion can be computed using a single equation. This highly simplified model is shown to accurately reproduce the helical vortex dynamics predicted by a more sophisticated filament model and observed in water channel experiments on model rotors. The model is then used to investigate different types of vortex perturbations. Perturbation direction is found to have an important effect on the evolution of the instability, and displacements are observed to induce vortex pairing more quickly than circulation changes. The effectiveness of different types of rotor asymmetries for triggering the pairing instability was also investigated experimentally, by testing various rotor configurations in a water channel. The findings of the current study can be used to design rotors that passively accelerate wake recovery and mitigate detrimental effects on downstream structures. Preliminary numerical results concerning the effect of rotor asymmetry on wind turbine wake interactions will be shown.

- **Flow analysis and control of vortex shedding using phase reduction theory**

  Makoto Iima (Hiroshima University)

  Vortex shedding, characterized by oscillating flows around non-streamlined objects at certain velocities, results from the periodic separation of vortices from the object. The Kármán's vortex street is a classic example. Understanding and controlling vortex shedding is critical to solving engineering challenges and theoretical questions, including finding optimal locations and forms of external forces. This study focuses on the analysis and control of the phase of vortex shedding. We use phase reduction theory, which has been applied to problems characterized by the limit cycle in various fields, such as engineering (mechanical vibrations), chemistry (Belousov-Zhabotinsky reactions), physiology (circadian rhythms), etc. We give an overview of phase reduction theory and established methods for calculating phase shifts due to external perturbations, before introducing a new method proposed by the author to compute the phase sensitivity function (PSF). Detailed phase shift analysis is applied to a variety of the Kármán's vortex streets. Furthermore, we study the optimization problems of an external force applied to an inclined plate: achieving entrainment with minimal energy and maximizing the locking range for a given energy. We also consider the optimization problem for a uniform, time-periodic external force within a rectangular region. The optimal location of the force distribution is found to be away from the plate, although the strongest response occurs on the plate.

- **Nonlinear dynamics of helical vortex disturbed by long-wave instability**

  Yuji Hattori (Tohoku Univ.)

  Helical vortices form around wind rotors, rotating blades of helicopter rotors, ship propellers, and many other engineering applications. The dynamical properties of the helical vortices should be understood in detail since they affect the flow and the performance of the devices. The helical vortices are subjected to long-wave instability as well as to short-wave instabilities (the elliptic instability and the curvature instability). We study the nonlinear dynamics of helical vortices destabilized by the long-wave instabilities. How the helical vortices become turbulent is of particular interest. We consider a vortex in which the centerline is a helix of constant curvature and torsion. The initial vorticity distribution on a plane perpendicular to the helix is Gaussian. Direct numerical simulation is performed to investigate the time evolution of the helical vortex disturbed by a long-wave instability mode. The 3D Navier-Stokes equations for an incompressible flow are solved using highly accurate numerical techniques assuming that the helical vortex extends periodically. Two values of the pitch are considered: $L/R = 0.2$ and 0.3. The wavenumber of the long-wave instability mode is set to $k = 1/2$. The evolution and the topology of the resulting vortices depend crucially on the pitch L/R at the nonlinear stage. In both cases, the helical vortex deforms significantly so that vortex reconnection occurs. When $L/R = 0.3$, a vortex ring is detached from the helical vortex after the vortex reconnection. As a result, the pitch of the helical vortex is doubled. A vortex ring is also created after the vortex reconnection when $L/R = 0.2$; however, it is linked with the remaining helical vortex after the reconnection. This linkage makes the vortex tubes interact strongly, which leads to a turbulent transition.

- **The impact of magnetic-vortical interactions on magnetic splitting**

  Linlin Kang (Westlake University)

  This study investigates the interaction between linked magnetic and vortex tubes through direct numerical simulations of magnetohydrodynamic flow. Our findings reveal that vortex tubes have a significant impact on the morphological and dynamic evolution of magnetic tubes. In cases where there is no initial vortex tube in the magnetic system, the magnetic tube undergoes a process of splitting, resulting in the formation of several finer tubes due to three-dimensional curvature effects. However, in scenarios where an initial vortex tube overlaps with the magnetic tube, regardless of whether they share isochirality or antichirality, the vortex tube notably impedes the initial-stage splitting of the magnetic tube and the subsequent release of magnetic energy. We elucidate the mechanism of magnetic-vortical interactions on magnetic tube splitting by examining the Lorenzo force within an intrinsic curvilinear triad along the magnetic line. The Lorenzo force emerges as the primary driving force responsible for creating a vortex pair in the opposite direction, leading to magnetic tube splitting. Conversely, an additional vortex tube with the same strength as the magnetic tube can inhibit the generation of a vortex pair in the opposite direction.

- **Isomagnetovortical perturbations and wave energy of MHD flows**

  Rong Zou (Hawaii Pacific University and Kyushu University)

  Ideal magnetohydrodynamics (MHD) is a Hamiltonian system of infinite degrees of freedom and the wave energy plays an important role in determining the stability of a steady flow. According to Arnold's theorem for hydrodynamics, the steady state is an extremal point of the kinetic energy with respect to isovortical perturbations, which preserve the local circulation and the helicity, the volume integral of the inner product of velocity and vorticity. The second-order wave energy determines the instability, and it can be calculated by using the first-order Lagrange displacement, which describes the difference in the trajectories of fluid particles after and before getting disturbed. In ideal MHD, the Lorenz force destroys the conservation of helicity, but the magnetic helicity, which describes the volume integral of the inner product of the magnetic field and its vector potential, and the cross helicity, which is the integral of the inner product of the velocity and the magnetic field, are preserved. The isomagnetovortical perturbations are constructed with the magnetic helicity, cross helicity, and all the other Casimirs preserved, and then Arnold's theorem holds true for the ideal MHD flows. The formula of the energy of waves on a steady incompressible isentropic flow of the ideal MHD is established, and the evolution equations of the two Lagrangian displacement fields included in it are derived. The same idea can be extended to establish the wave energy of Hall-MHD flows.

- **Baroclinic critical layers and self-replicating vortices**

  Chen Wang (Beijing Normal University and Hongkong Baptist University)

  Recent studies have revealed a novel type of vortices that can self-replicate in stratified flows with horizontal shear. The replication is accomplished through the excitation of baroclinic critical layers, located where the wave's Doppler-shifted phase velocity matches the phase velocity of internal gravity waves. We undertake a theoretical analysis to understand this process. Using the method of matched asymptotic expansions, we investigate the evolution of a baroclinic critical layer under a steady forcing. Our analysis reveals that in the early stage, while the steady forcing establishes steady waves in the bulk of the flow, in the baroclinic critical layer, the waves' amplitudes grow algebraically with time. The growing waves force mean flows to grow, forming a strong localized jet in the critical layer. As the jet becomes stronger, it will experience a shear instability, which is referred to as the secondary instability. The instability analysis reveals that due to the unsteadiness of the mean flow, the amplitudes of unstable waves grow faster than typical normal modes. Finally, we construct a reduced model of the local vorticity to study the nonlinear evolution of the secondary instability. The model was solved numerically, which indicates that the secondary instability makes the jet roll up into vortices. At this time, the vortices act as new forces and have the ability to excite new baroclinic critical layers, and this completes a cycle of replication.

3.3 Mathematical foundations in Vortex Dynamics

- **Enstrophy variation via point-vortex collapse on inviscid flows**

  Takeshi Gotoda (Tokyo Institute of Technology)

  Enstrophy dissipation in the zero viscous limit is a remarkable property characterizing 2D turbulent flows and we expect that non-smooth solutions of the 2D Euler equations could dissipate the enstrophy. We study the enstrophy variation in terms of vortex dynamics and, in particular, focus on collisions of point vortices. Motions of point vortices on the inviscid flow are described by the point-vortex system that is formally derived from the 2D Euler equations and this system has self-similar collapsing solutions. The preceding results have shown that the triple collapse of point vortices leads to the enstrophy dissipation by introducing the filtered-Euler equations which are a regularized model of the Euler equations. In this work, we numerically show that certain four and five point-vortex solutions of the 2D filtered-Euler equations converge to self-similar collapsing orbits and dissipate the enstrophy in the zero limit of a filter scale.

- **Systematic search for singularities in 3D Euler flows**

  Xinyu Zhao (McMaster Univ.)

  The local well-posedness of smooth solutions of 3D incompressible Euler equations has been established when the initial data is in the Sobolev space H^s for $s > 5/2$. However, it is still an open question whether these solutions develop finite-time singularities. In this talk, we will present a numerical study of this question where we systematically search for initial data that may lead to potential singularity through PDE-constrained optimization. The optimization problem is solved numerically using a state-of-the-art Riemannian conjugate gradient method where the Sobolev gradients are obtained through an adjoint method. The behavior of the obtained extreme flow, which features two colliding distorted vortex rings, suggests a finite-time singularity formation. This is based on a joint work with Bartosz Protas.

- **On the triad phase coherence signature of vorticity in extreme 3D Navier-Stokes flows**

  Miguel D. Bustamante (University College Dublin)

  We report the results of our newly devised method to "extract" the most energetic and "flux-carrying" triads of Fourier modes of any 3D Navier-Stokes turbulent flow, applied to the extreme flows studied by Kang et al [14] in order to investigate the coherence signature of Fourier triad phases during strong vortex reconnection events. These extreme flows, constructed to maximise the "global" kinetic energy dissipation rate at a future time, turn out to produce transient turbulent flows characterised by strong "local" vortex reconnection events. Correspondingly, the question of the coherence of the Fourier triad phases becomes relevant. However, in the 3D Navier-Stokes case, the vast majority of the triads are not involved in the coherent structures that form, and therefore the search for the set of Fourier modes that are responsible for the interactions that lead to the observed phenomena becomes "finding a needle in a haystack". Fortunately, we managed to accomplish this impossible task via a simple post-processing procedure based on the analysis of the joint probability density function (flux vs. triad phase) over the set of triads involved in the spectral flux of energy across spatial scales. As a result, at a given time only a handful of triads (of the order of 100, corresponding usually to 0.001% of all available triads) contribute to over 60% of the total flux, with coherent triad phases, particularly during extreme vortex reconnection events. The method allows us to study important features of the distribution of the flux-carrying triads and modes, such as anisotropy, scale-by-scale budget, and the effective number of degrees of freedom of a potential minimal ODE model that could reproduce the dynamics of the full PDE.

- **Recent results on steady vortex rings of small cross-section**

  Zou Changjun (Sichuan University)

  We will talk about steady vortex rings in an ideal fluid of uniform density, which are special global solutions of the three-dimensional incompressible Euler equation. We systematically establish the existence, uniqueness, and nonlinear stability of steady vortex rings of small cross-sections for which the potential vorticity is constant throughout the core. The latter two answer a long-standing question since the pioneering work of Fraenkel and Berger [10]. Our proof is based on a combination of the Lyapunov-Schmidt reduction argument, the local Pohozaev identity technique, and the variational method, which provides a general approach for a large class of thin vortex rings.

- **Helical symmetry solutions for 3D incompressible Euler equations in an infinite cylinder**

  Cao Daomin (Chinese Academy of Sciences)

  In this talk, we are interested in solutions whose vorticities are large and concentrated uniformly near a smooth curve $Gamma(t)$ embedded in entire $\mathbb{R}^3$. This type of solution, vortex filaments, are classical objects of fluid dynamics. Under suitable assumptions, it is known to some extent that the curve evolves by its binormal flow. Two special kinds of binormal flows are traveling circles and rotating-translating helix. Solutions concentrating near a traveling circle are called vortex rings which have been studied extensively. In this talk, we will present the existence of solutions near rotating-translating helix. The general case is called vortex filament conjecture which is still a well-known open problem. This talk is based on a joint paper with Wan Jie at Beijing University of Technology.

- **Dynamics of coaxial arrays of vortex rings**

  Emad Masroor (Swarthmore College)

  A coaxial periodic array of thin-cored vortex rings is known to translate uniformly along its axis without change of shape, at a speed that depends on the thickness ratio, $\varepsilon \equiv a/R$ and the spacing $\lambda \equiv L/R$. We revisit this classical result using the modern Hamiltonian formalism of Borisov et. al (2013) and show how the topology of this flow depends on the two non-dimensional parameters ε and λ. Next, we use the Hamiltonian formalism to develop canonical equations of motion for N coaxial arrays of vortex rings and analyze the dynamics of the $N = 2$ problem. We show that this problem is amenable to quadratures, and explore its rich dynamics by means of numerical integration of the governing equations. Various regimes of inter-vortex motion are identified, and the transitions between these regimes are delineated using phase diagrams in the reduced variables.

3.4 Vortex Identifications

- **New perspectives in the fundamental theories of vortex dynamics. Part 1. Analytical integration of vorticity equation and its application in transverse field diagnosis**

  An-Kang Gao (University of Science and Technology of China)

  The Lagrangian description is a powerful method in fluid mechanics. Its potential is, however, not yet fully unleashed in the study of vorticity and vortex dynamics at high-Reynolds-number flows. In this work, a Lagrangian framework of vorticity evolution is developed and the analytical solution of the vorticity transport equation in physical space is given in the form of a Lagrangian integration. In this analytical solution, the evolution of vorticity is modulated by the history of the surface deformation tensor and the curl of fluid acceleration, with the latter being solely contributed by viscosity in a barotropic flow without nonconservative force. The surface deformation tensor, on the other hand, represents the stretching and tilting of the vorticity line. A numerical method for this Lagrangian vorticity integration is implemented in a massive-parallel open-source code, `Nektar++`, which is based on a spectral/hp element method and in which high-order modified Jacobi polynomials are used to represent the flow fields and to calculate spatial derivatives. The second-order Adams-Bashforth method is used for the time integration of the vorticity equation. With the MPI communication protocol, approximately one million Lagrangian points can be tracked simultaneously in a 3-D flow. This method is applied to the study of the secondary instability and the 3-D transition of the flow past a plunging airfoil at a high angle of attack. The Lagrangian vorticity evolution of the most unstable mode is analyzed to uncover the instability and is compared with the results from the Eulerian description. The 3-D evolution of the leading-edge vortex and the trailing-edge vortex is also analyzed to uncover the key processes that dominate the breakdown of the vortex.

- **New perspectives in the fundamental theories of vortex dynamics. Part 2. Interaction between self-spin and orbital rotation**

  Jie Yao (Peking Univ.)

  Since the foundation of vorticity and vortex dynamics by Helmholtz (1858), the vorticity field has been traditionally considered as the sole characteristic quantity for describing vortical motion. However, a fluid element possesses not only orbital rotation but also spin, which characterizes pure shear. For a long period, our understanding of these two distinct types of motion remained predominately qualitative (e.g., either axial vortices or shear layers). Meanwhile, from Helmholtz's theorems to various exact vortex model solutions of the Euler or Navier-Stokes equations, as well as various 'vortex criteria' proposed since the 1980s, the research on vortex dynamics has been mainly focused on axial vortices. However, it is essential to recognize that axial vortices represent only the strongest structures as relatively extreme events in the flow field. In contrast, shear layers, due to the presence of Kelvin-Helmholtz instability, exhibit rich diverse motion patterns, especially at high Reynolds numbers turbulence. In this talk, we present an exact and general decomposition of the vorticity vector into spin vector and orbital-rotation vector, based on the combination of normal-nilpotent and symmetric-antisymmetric decompositions of velocity gradient tensor. Then, the dynamical equations for both orbital rotation and spin are derived, which exhibit the inherent coupling of the two vector fields that implies the transition from shear layers to axial vortices by entrainment and vice versa by detrainment. Actually, every equation in terms of vorticity, such as those of enstrophy and dissipation, can be split into two coupled ones. A two-dimensional numerical example of the interaction between a pair of disturbed shear layers of opposite vorticity is shown to demonstrate this two-way coupling. We believe that this framework can provide a new platform to enhance one's quantitative and physical understanding of the formation, instability, evolution, and mutual transition of sheet-like and filament-like structures in turbulence and other complex flows.

- **Vorticity at boundaries and in wakes**

  Mark A. Stremler (Virginia Tech)

  One challenge with viewing vorticity as a fundamental flow variable is that the obvious boundary conditions are on velocity, not spatial derivatives of velocity. Historically, mathematical descriptions of vorticity boundary conditions have focused on special cases. Recent studies have generalized the formulation by considering the conservation of vorticity in a fluid volume, but these efforts are based on the assumption of the fluid being incompressible and Newtonian. Using a continuum mechanics-based approach, we present a vorticity conservation formulation that decouples fundamental kinematics and dynamics from material properties. This approach simplifies some of the previous analyses, clarifies the physical mechanisms of vorticity generation at boundaries, and facilitates the generalization of a vorticity boundary condition to a broad class of fluid mechanics problems.

  When vorticity is generated at the boundary of a solid bluff body, it is shed by the body and develops into coherent vortical structures in the wake. Historically, various related definitions have been used to define the vortex formation length, with each determining a single characteristic length over which this formation occurs at a fixed value of Reynolds number. We introduce a new definition of vortex formation length using the Proper Orthogonal Decomposition (POD) of the flow field. Using the leading modes of the POD for uniform steady flow past a circular cylinder, we identify upper and lower bounds on the vortex formation length. Having a range, rather than a single value, for the vortex formation length is consistent, both physically and quantitatively, with the hysteresis observed in the critical spacing of two in-line tandem cylinders, and we suggest that this approach may provide a better representation of the vortex formation process than a single characteristic length.

- **Applications of the vortex-surface field to flow visualization, modeling, and simulation**

  Yue Yang (Peking Univ.)

  We review the progress on the applications of the vortex-surface field (VSF). The VSF isosurface is a vortex surface consisting of vortex lines. Based on the generalized Helmholtz theorem, the VSF isosurfaces of the same threshold at different times have strong coherence. As a general flow diagnostic tool for studying vortex evolution, the numerical VSF solution is first constructed in a given flow field by solving a pseudo-transport equation driven by the instantaneous frozen vorticity, and then the VSF evolution is calculated by the two-time method. From the database of numerical simulations or experiments, the VSF can elucidate mechanisms in the flows with essential vortex dynamics, such as isotropic turbulence, wall flow transition, flow past a flapping plate, and turbulence-flame interaction. The characterization of VSFs reveals the correlation between robust statistical features and the critical quantities needed to be predicted in engineering applications, such as the friction coefficient in transition, thrust in bio-propulsion, and growth rate in interface instability. Since the VSF evolution captures the essential Lagrangian-based dynamics of vortical flows, it inspires novel numerical methods on cutting-edge hardware, e.g. graphic and quantum processors.

3.5 Hydrodynamic and Quantum Turbulence

- **Weaving turbulence with intertwined vortex tubes**

  Weiyu Shen (Peking University)

  We develop a bottom-up approach to construct turbulent flow fields consisting of intertwined vortex tubes, which sheds light on the fundamental structure of turbulence and the connection between classical and quantum turbulent flows. First, homogeneous isotropic quantum turbulence is simulated using the vortex filament method. Then, the quantum vortex filaments are transformed into spline-based parametric equations, serving as the centerlines of viscous vortex tubes. Finally, the turbulent field is constructed using intertwined vortex tubes with a quantum skeleton and finite thickness. This synthetic turbulent field satisfies a series of key statistical laws in classical turbulence, e.g., the five-thirds scaling of the energy spectrum and negative skewness of the velocity fluctuation. In this way of weaving turbulence, we can customize the turbulent fields with different Reynolds numbers and helicity components by precisely tuning the entanglement degree, thickness, and local twist of vortex tubes.

- **Dynamics of vortices in Bose-Einstein condensates**

  Tao Yang (Northwest University)

  The character of elementary excitations is very important in understanding the macroscopic quantum behaviour of a trapped Bose-Einstein condensate (BEC). The dynamics of quantized vortices are essential for understanding diverse superfluid phenomena such as critical-current densities in superconductors, quantum turbulence, and novel quantum phases in superfluids. Moreover, the transition of vortices from one topological state to another, such as tangles of vortex lines and the formation and dissipation of knotted or linked vortices, plays an important role in understanding turbulence in both classical and quantum fluids. We show the detailed dynamics of different vortex structures, such as vortex cores, lines, and rings, in 2D and 3D BECs. We propose a novel technique to generate torus vortex knots and links by entangling two simple vortex rings in a trapped BEC. The results demonstrate that helicity transfers between knots and links to helical coils can occur in both directions along different pathways. The decay of a turbulent cascade of randomly distributed vortices can induce a low circular current in an annular BEC. However, after the decay of vortices resulting from a sudden geometrical quench of the trap from annular to double concentric ring-shaped BEC, persistent current with high winding numbers can be induced from novel nonequilibrium dynamics. The circulation flows are with high stability and good uniformity free from topological excitations. We find that besides elucidating the dissipative resistance arising from the emergence of vortex dipoles, vortex dipoles can play a role as a quantum current regulator in a superfluid oscillator circuit consisting of two BEC reservoirs connected by a channel. These results are promising for new atomtronic designing, and are also helpful for quantitatively understanding quantum tunneling and interacting quantum systems driven far from equilibrium.

- **Vortex reconnection and turbulence cascade**

  Jie Yao (Beijing Institute of Technology)

  Turbulence, one of the most challenging problems left over from classical physics, has a wide range of applications in the field of engineering technology. Vortical flows are ubiquitous in nature and engineering applications, such as tornadoes, hurricanes, and trailing vortices behind an aircraft. The study of vortex dynamics, including the generation, evolution, and interaction, not only helps to understand the fundamental laws of turbulence and establish an easily understandable physical model but also provides theories for solving various technical problems. Reconnection is the process by which two approaching vortices cut and connect. As a topologically changing event, it has been a subject of considerable fundamental interest for decades - not only in (classical) viscous flows but also in quantum fluids, as well as in numerous other fields, such as plasmas, polymers, DNAs, and so on. For viscous fluid flows, reconnection is believed to play a significant role in various important phenomena, such as turbulence cascade, fine-scale mixing, and aerodynamic noise generation. In this talk, the underlying mechanism involved in vortex reconnection and its apparent role in turbulence cascade, helicity dynamics, and finite-time singularity will be delineated.

4 Outcomes of the Meeting

Since the workshop involved a significant proportion of early-career participants and participants who had not met before, its goals did not include making tangible progress on some concrete problems. On the other hand, the main objective was exchange of ideas across closely-related areas of fluid mechanics where vorticity plays a key role, such as classical and quantum turbulence, vortex dynamics, fluid-structure interactions and geophysical flows. One topic which received particular attention was application of topological methods to study different problems in vortex dynamics, especially in relation to the phenomenon of reconnection which is closely related to the issue of singularity formation in inviscid systems. Another such topic was methods for extraction and visualization of coherent vortex structures. With a uniquely interdisciplinary audience at the workshop, it was possible to discuss certain classical open problems, such a vortex reconnection or properties of various vortex equilibria, from a variety of complementary perspectives, including rigorous PDE analysis, scientific computing and physical interpretation.

With many new acquaintances struck during the workshop that may lead to collaborations in the future, there was a strong sense that it was a very successful event and plans are already being made for a follow-up event in the foreseeable future.

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