Recursion Theory and its Applications

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15-20 Oct, 2023

1 Overview of the Field

Recursion theory, or computability theory, is a branch of mathematical logic. Recently, there have been deep applications of recursion theory to analysis, number theory, set theory, and other areas. For example, Lutz and Lutz (ACM Transactions on Computation Theory, 2018) gave a striking description of Hausdorff dimension via recursion theoretic notions and used it to give a new proof of the Kakeya conjecture in dimension 2; Day and Marks solved the decomposability problem via a transfinite priority argument; and Yinhe Peng and Liang Yu (Advances in Mathematics, 2021) proved that Turing determinacy ($\mathbf{TD}$) implies the axiom of countable choice for sets of reals, answering a long-standing question in set theory.

Other applications are being developed. For example, very little is known regarding the interplay between combinatorial, recursion-theoretic, and number-theoretic properties of the expansions of real numbers. A research line started by Slaman studies similarities between the describability of a real number in terms of Diophantine Approximation and in terms of effective Hausdorff dimension. For instance, if a real has irrationality exponent δ, then it has effective Hausdorff dimension at most $2/\delta$.

How do these applications of recursion theory arise? To a recursion theorist, a continuous function is a recursive function relative to an oracle; a null set is a set covered by a Martin-Löf test (or a Schnorr test) relative to an oracle; and a set with Hausdorff dimension zero is a set only containing reals that can be significantly compressed by a single oracle. In this way, recursion theory offers a fine-grained way to analyze some of the most central notions in mathematics; this explains why recursion theory is applicable to problems that make no reference to computation. The focus of this workshop is on understanding and extending these applications.

2 Recent Developments and Open Problems

- *Geometric measure theory.* The *point-to-set principle* relates the Hausdorff dimension of a set in $\mathbb{R}^n$ with the (relativized) Kolmogorov complexity of approximations to its members. It opens a door between recursion theory (in the guise of algorithmic randomness) and geometric measure theory. Recently Lutz and Lutz have used this principle to obtain new results about fractal geometry using tools from recursion theory. Many open questions remain related to their work.

In fractal geometry, the Besicovitch–Davies theorem is known to hold only for analytic sets (continuous images of closed sets). It is a natural question whether this can be generalized. Recently, Slaman found that the Besicovitch–Davies theorem is unprovable for co-analytic sets, while Crone–Fishman–Jackson and Peng–Wu–Yu independently proved that the theorem holds for arbitrary sets under certain large cardinal axiom assumptions. Peng–Wu–Yu's proof is quite short and uses very recent results from algorithmic randomness. What is still open is the exact consistency strength of the Besicovitch–Davies theorem for arbitrary sets. It is expected that recursion theory is going to play a central role in answering this question.

- *Set theory.* It is a long-standing open question whether the axiom of determinacy implies dependent choice. Up to now, all efforts to settle this question used exclusively set-theoretic techniques. Recent work of Peng and Yu strongly suggests that recursion theory may help solve the problem. Indeed, using some very recent results from algorithmic randomness, they proved that Turing determinacy implies some weak forms of dependent choice.

  Martin's conjecture is one of the central open problems in recursion theory. It essentially says that there are no natural unsolvable problems other than (iterations of) the halting problem. The conjecture is related to both descriptive set theory and recursion theory. Recently, Kihara, Montalban, and Patrick Lutz made some new progress on Martin's conjecture. We hope to make some further progress on this conjecture.

  Another recent fruitful interaction between recursion theory and descriptive set theory is the application of Montalban's "true stages" technique of iterated priority arguments to better understand the Borel hierarchy. Inspired by work of Day, Downey, and Westrick concerning reducibility on functions of a real variable, Day and Marks recently used true stages to settle the long-standing decomposability conjecture. At the same time, Day, Greenberg, Harrison-Trainor, and Turetsky used true stages to give a new proof of the Louveau–Saint Raymond separation theorem, with applications to reverse mathematics. It is expected that several other problems in descriptive set theory may be solved using variations on this technique, for example, perhaps it will allow a layer-by-layer analysis of the G_0-dichotomy, a project started by Lecomte and Zeleny.

- *Algorithmic randomness theory.* In recursion theory, randomness and (Cohen) genericity are fairly orthogonal concepts. For example, a result of Nies, Stephan, and Terwijn states that if X is 2-random and Y is 2-generic then X and Y form a minimal pair with respect to Turing reducibility. However, for weaker notions of randomness and genericity, the situation is more delicate. A result of Kautz states that every 2-random X computes some 1-generic Y. This was subsequently strengthened to any X *Tuing below* some 2-random (Barmpalias, Day, Lewis–Pye) and X Demuth random (Bienvenu and Porter). However, the reason why this holds remains unclear. All the above results seem to obtain the ability for randoms to compute generics via the ability to compute fast-growing functions (in the sense of hyperimmunity), which immediately raises the question of whether it simply is the case that for every 1-random X, X computes a 1-generic if and only if X has hyperimmune degree. This question has been open for a long time now. It would also be interesting to better understand the relationship between randomness and genericity for forcing notions other than Cohen's (for example Mathias genericity).

  What sort of uniformly distributed sequences result in notions such as ML-randomness or Schnorr randomness? Avigad considered computable sequences of distinct integers $(a_n)_{n\geq 1}$ and proved that for every Schnorr random real X, the sequence $(nX)_{n\geq 1}$ is uniformly distributed modulo 1, while there are reals that satisfy the notion but are not even Kurtz random. Recently, Becher and Grigorieff extended Avigad's notion, and characterized Martin-Löf and Schnorr randomness in terms of uniform distribution with respect to Σ^0_1 sets. Their theorem intersects with the characterization of randomness in terms of an effective version of Birkhoff's ergodic theorem (Franklin et al. and independently Bienvenu et al.) We believe that all ML-random reals X have the discrepancy of almost all real numbers; this remains to be proved. We still do not know what part of discrepancy theory holds for uniform distribution with respect to Σ^0_1 sets. It also remains to understand what real numbers satisfy Avigad's notion. We conjecture that any real Martin-Löf-random for a (computable) measure of positive Fourier dimension satisfies it (this is a problem raised during the Algorithmic Randomness workshop at AIM (August 2020) organized by Hirschfeldt, Miller, Reimann, and Slaman). Another open question regards the relationship between algorithmic randomness and notions beyond uniform distribution modulo 1, such as Poisson genericity, which was introduced by Rudnick and studied by Peres and Weiss, and the pair correlation function studied by Rudnick. We do not know whether there are computable instances. It is an open problem how to characterize these notions by initial segment complexity.

- *Recursion theory and Number theory.* While an existential definition of $\mathbb{Z}$ in $\mathbb{Q}$ would imply that Hilbert's Tenth Problem for $\mathbb{Q}$ is undecidable, a conjecture of Mazur implies that the integers are not existentially definable in the rationals. Recently Eisenträger, R. Miller, Springer and Westrick proved that in most algebraic extensions of the rationals, the ring of integers is not existentially definable. They view the set of algebraic extensions of $\mathbb{Q}$ as a topological space homeomorphic to Cantor space. They show that the set of fields that have an existentially definable ring of integers is a meager set. On the other hand, by work of Koenigsmann and Park, there is a universal definition of the ring of integers in finite extensions of the rationals, but these results do not extend to most algebraic infinite extensions: the set of algebraic extensions of $\mathbb{Q}$ in which the ring of integers is universally definable is also a meager set.

- *Topology.* There are plenty of examples of how topological notions are deeply tied to computability theoretical ones. In recent work, Kihara and Pauly introduced the notion of the point degree spectrum of a represented space. This notion connects the Medvedev degrees with the continuous and enumeration degrees, and was used by the authors to prove results in infinite-dimensional topology, in particular obtaining a refinement of Pol's solution to Alexandrov's problem. Kihara, Ng, and Pauly showed how represented second-countable T_0 spaces can be identified with (sub)collections of the enumeration degrees. Moreover, Day, Downey, and Westrick, in work mentioned above, characterized the Bourgain rank of a function of Baire class 1 with compact Polish domain using a notion of reducibility that corresponds to parallelized continuous strong (pcs) Weihrauch reducibility. Kihara then extended the result to functions with non-compact Polish domain and drew a connection between Martin's conjecture and pcs Weihrauch degrees, showing that the degrees of single-valued functions are well-ordered.

3 Presentation Highlights

Professor Slaman gave a talk about Extending Borel's Conjecture from Measure to Dimension. In the talk, he announced a recent result that in Laver's model, for any n, any gauge function g and set $A \subseteq \mathbb{R}^n$, if A has strong Hausdorff dimension f, then A is σ-finite for the Hausdorff measure H^f. In the same talk, Ted asked whether Borel conjecture imply for all A and f, if A is not σ-finite for H^f, then A does not have strong dimension f.

In the talk Products do not preserve computable type by Mathieu Hoyrup, a compact metrizable space has computable type if for every set that is homeomorphic to that space, semicomputability is equivalent to computability. J. Miller proved that spheres have computable type, and Iljazović proved that closed manifolds have computable type. Čelar and Iljazović asked whether the product of two spaces having computable type also has computable type. We give a negative answer to that question, providing a finite simplicial complex having computable type, but whose product with the circle does not. The construction heavily relies on classical results on homotopy groups of spheres and suspensions.

Professor Khoussainov gave a talk about Automatic Structures. In the talk, he gave a survey about the theory of automatic structures.

In the talk Compression of enumerations and gain by Xiaoyan Zhang, he talked about how information in a recursive enumeration can be compressed, in the sense of relative Kolmogorov complexity (rK). We introduce a strong and a weak form of compression, and we also care about the gain of compressions. It turns out that the existence of strong gainless compression is a key to provide a join operator on r.e. rK degrees, making the degree structure dense. I'll show that strong compression and weak gainless compression exist for any recursive enumeration. He also talked about a positional game that is crucial toward obtaining a strong gainless compression.

In the talk Effective presentations in effective topology and analysis by Keng Meng Ng, he gave a survey on the area.

Professor Andre Nies gave a talk about Maximal towers and ultrafilter bases in computability theory. The tower number and ultrafilter numbers are cardinal characteristics from set theory that are defined in terms of sets of natural numbers with almost inclusion. The former is the least size of a maximal tower. The latter is the least size of a collection of infinite sets with upward closure a non-principal ultrafilter. Their analogs in computability theory will be defined in terms of collections of computable sets, given as the columns of a single set. We study their complexity using Medvedev reducibility. For instance, we show that the ultrafilter number is Medvedev equivalent to the problem of finding a function that dominates all computable functions, that is, highness. In contrast, each nonlow set uniformly computes a maximal tower. Reference: Lempp, S., Miller, J. S., Nies, A., & Soskova, M. I. (2023). Maximal Towers and Ultrafilter Bases in Computability Theory. The Journal of Symbolic Logic, 88(3), 1170-1190.

In the talk Computable dualities and their applications by Alexander Melnikov, he discussed several recent results in computable topology that establish direct connections between computable structure theory and the theory of recursive Polish spaces through recently developed effective dualities. The proofs of these dualities make use of a diverse range of techniques. I will also discuss a few applications of these effective dualities that appear to be rather fundamental in the theory of effectively presented Polish spaces.

In the talk Reductions and (resolvable) combinatorial designs by Jun Le Goh, he reported on ongoing work with Belanger and Dzhafarov. In his study of the computational strength of finite pigeonhole principles (in the Weihrauch lattice), he applied combinatorial results such as graph decomposition theorems and Turán's theorem in extremal graph theory. he then discovered that certain questions he was unable to resolve were in fact equivalent to longstanding open problems in combinatorics. he presented such results, as well as results on the relationships between jumps of finite pigeonhole principles and well-studied problems in the Weihrauch lattice.

Professor Frank Stephan gave a talk about Addition Machines and the Open Problems of Floyd and Knuth. Floyd and Knuth investigated in 1990 register machines which can add, subtract and compare integers as primitive operations. They asked whether their current bound on the number of registers for multiplying and dividing fast (running in time linear in the size of the input) can be improved and whether one can output fast the powers of two summing up to a positive integer in subquadratic time. Both questions are answered positively. Furthermore, it is shown that every function computed by only one register is automatic and that the automatic functions with one input can be computed with four registers in linear time; automatic functions with a larger number of inputs can be computed with 5 registers in linear time. There is a nonautomatic function with one input which can be computed with two registers in linear time.

Professor Yue Yang gave a talk about COMPUTABILITY OVER FINITE TYPE OBJECTS. In this talk, he proposed a formalization of computability over finite type objects. Most of the talk is based on an ongoing joint work with Keng Meng Ng from Nanyang Techno- logical University, Singapore and Nazanin Tavana from Amirkabir University of Technology, Iran.

In the talk Consistency Checking for Algebraic Delay PDEs in Sequence Rings by Wei Li, she talked about the consistency checking problem for algebraic delay PDEs seeks for a general method or algorithm to determine whether an arbitrarily given system of delay PDEs has a sequence solution. She solve this problem positively by proving the effective partial differential-difference Nullstellensatz, in which she show the existence of a uniform upper bound for the number of iterated applications of the distinguished difference and derivation operators, for a reduction of this consistency-checking problem to a well-studied consistency-checking problem for polynomial equations. The main approach is our technical result about algorithms performing computations in complete decidable theories, which shows that if an algorithm performing computations restricted to definable functions is guaranteed to terminate on every input, then there is a computable upper bound for the size of the output of the algorithm in terms of the size of the input. This is joint work with A. Ovchinnikov, G. Pogudin and T. Scanlon.

Professor George Barmpalias gave a talk about Avoiding path-random trees. A tree is path-random if all of its paths are algorithmically random. Path-random trees are building blocks for models of tree-versions of Weak Konig's Lemma. Diagonalizing agains path-random trees can be challenging. He will discuss an array of such separations: positive measure trees from perfect, finite versus infinite paths and perfect from trees with one limit point. Key open questions remain, and be presented.

In the talk The computability aspect of extensions of abelian groups by Sen Yang, he talked about Aa fundamental problem in abelian group theory is to decide $\mathrm{Ext}(C, A)$, which is the group of all extensions of C by A. He investigated the reverse mathematical strength of various statements on extensions of abelian groups.

4 Scientific Progress Made

Yinhe Peng and Liang Yu discussed the countable uniformization principle under AD. They recognized that it is not clear whether countable uniformization principle over an upper cone implies the full version.

George Barmpalias and Liang Yu discussed speeding up problem. They recognized that hif over low for Ω is not implied by non-speeding up property.