# Moving Polymer in a Random Environment

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## 1 Overview of the Field

The behavior of polymers in a random environment has seen intense activity for a few decades, see [3] and [2], for example. A polymer is a long molecule consisting of many segments arranged from first to last, with random orientations. Such a polymer is conveniently modeled using a random walk $(S_n)_{n\geq 0}$ or a Brownian motion $(B_t)_{t\geq 0}$. For polymers, the time parameter of the random walk or Brownian motion represents the length along the polymer rather than physical time. Two kinds of random environments are commonly considered: environments that do not change with time and those that do change. Polymers in a changing random environment are often called directed polymers, while models of polymers in a stationary environment are often called trap models.

To describe these models more precisely, consider the case of a random walk in a changing environment. Here $(S_n)_{0\leq n\leq N}$ denotes a random walk of $N$ steps, taking values in $\mathbb{Z}^d$. We assume that $S_0 = 0$ and that $(S_n)$ is a nearest neighbor walk, so that $S_{n+1} - S_n = \pm 1$ with probability $\frac{1}{2}$ each. Let $P_N$ denote the probability measure for the random walk.

Next we describe the environment variables $\omega(n, z)_{n\geq 0, z\in\mathbb{Z}^d}$. These are i.i.d. random variables governed by a probability measure $\mathbb{P}_N$.

Finally, we define a probability measure $\mathbb{Q}_n$ by the following procedure, well known in statistical mechanics. For $\beta > 0$, let

$$
\begin{aligned}
H_N &= \sum_{n=0}^{n} \omega(n, S_n) \\
Z_N &= E^{\mathbb{P}_N}\left[\exp\left(-\beta H_n\right)\right] \\
d\mathbb{Q}_N &= \frac{\exp\left(-\beta H_n\right)}{Z_n} d(P_N \times \mathbb{P}_N).
\end{aligned}
$$

This framework is well known in statistical mechanics, going under the name of directed polymers. A typical question is to study the diffusive behavior of $(S_n)_{n\leq N}$ under $\mathbb{Q}_N$. Under $\mathbb{P}_N$, we know that $S_N \approx C\sqrt{N}$ (diffusive behavior). This behavior continues to hold under $\mathbb{Q}_N$, for small values of $\beta$. However, for large values of $\beta$, it is known that $|S_N|$ is typically much smaller than $\sqrt{N}$, and this behavior is called localization. Under localization, $S_N$ may even concentrate at a single point.

## 2 Scientific Progress Made

The goal of our team project is to study diffusive behavior and localization in the context of a random string.

In our first project we studied the annealed survival probability of a random string in a Poissonian trap environment. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_0)$ be a filtered probability space on which $\dot{\mathbf{W}} = \dot{\mathbf{W}}(t,x)$ is a $d$-dimensional random vector whose components are i.i.d. two-parameter white noises adapted to $\mathcal{F}_t$. We consider a *random string* $\mathbf{u}(t,x) \in \mathbb{R}^d$, which is the solution to the following stochastic heat equation (SHE)

$$\begin{aligned} \partial_t \mathbf{u}(t,x) &= \frac{1}{2}\,\partial_x^2 \mathbf{u}(t,x) + \dot{\mathbf{W}}(t,x) \\ \mathbf{u}(0,x) &= \mathbf{u}_0(x) \end{aligned} \tag{2.1}$$

on the circle $x \in [0,J]$, having endpoints identified, and $t \in [0,T]$. The initial profile $\mathbf{u}_0$ is assumed to be continuous. Note that we will use boldface letters to denote vector-valued quantities. We will be interested in the evolution of the random string in a field of obstacles centered at points coming from an independent Poisson point process. More precisely, let $(\Omega_1, \mathcal{G}, \mathbb{P}_1)$ be a second probability space on which is defined a Poisson point process $\boldsymbol{\eta}$ with intensity $\nu$ given by

$$\boldsymbol{\eta}(\omega_1) = \sum_{i \geq 1} \delta_{\boldsymbol{\xi}_i(\omega_1)}, \quad \omega_1 \in \Omega_1,$$

with points $\{\boldsymbol{\xi}_i(\omega_1)\}_{i\geq 1} \subset \mathbb{R}^d$.

The obstacles will be formed via a potential $\mathrm{V} : \mathbb{R}^d \times \Omega_1 \to [0,\infty]$

$$\mathrm{V}(\mathbf{z}, \boldsymbol{\eta}) = \sum_{i\geq 1} \mathrm{H}(\mathbf{z} - \boldsymbol{\xi}_{\boldsymbol{i}}),$$

where $\mathrm{H} : \mathbb{R}^d \to [0,\infty]$ is a non-negative, measurable function whose support of H is contained in the *closed* ball $B(\mathbf{0}, a)$ of radius $0 < a \leq 1$ centered at $\mathbf{0}$.

We will work in the product space $(\Omega \times \Omega_1, \mathcal{F} \times \mathcal{G}, \mathbb{P}_0 \times \mathbb{P}_1)$ along with the filtration $(\mathcal{F}_t \times \mathcal{G})_{t\geq 0}$. We will write $\mathbb{E}$ for the expectation with respect to $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$, and $\mathbb{E}_i$ for the expectation with respect to $\mathbb{P}_i$ for $i = 0, 1$. Our main quantity of interest is the quenched and the annealed survival probabilities given by

$$\begin{aligned} S_{T,\eta}(\omega_1) &= \mathbb{E}_0\left[\exp\left(-\int_0^T \int_0^J \mathrm{V}\Big(\mathbf{u}(s,x), \boldsymbol{\eta}(\omega_1)\Big) dx ds\right)\right], \text{ and} \\ S_T &= \mathbb{E}\left[\exp\left(-\int_0^T \int_0^J \mathrm{V}\left(\mathbf{u}(s,x), \boldsymbol{\eta}\right) dx ds\right)\right] \end{aligned}$$

respectively. Our main objective in this project was to provide asymptotics in $T, J$ on $S_T$ and $S_{T,\eta}(\omega_1)$. We have obtained upper and lower bounds on $S_T$ for large $T$ and $J$.

The second project we started working at the meeting was on the discrete stochastic heat equation. Here the range space is $\mathbb{Z}^d$ and time $t \in \mathbb{R}_+$ and distance along the string is $n \in \mathbb{Z} \cap [0,N]$

We are studying a random process $u(t,n)$. There is a collection of Poisson clocks.

1. For each $n$, there are $2d$ Poisson processes corresponding to the $2d$ coordinate directions. When one of the clocks rings, say at time $t$, then $u(t,n)$ moves one unit in the corresponding coordinate direction. Thus $u(t,x)$ stays on the lattice.

2. Let $v(t,n) = u(t,n+1) - 2u(t,n) + u(t,n-1)$ (the discrete Laplacian). Consider the coordinates $v = (v_1, \ldots, v_d)$. Consider $d$ Poisson processes, corresponding to $v_i(t,n)$ respectively. When one of the clocks rings, then $u(t,n)$ moves one unit in the corresponding $v_i$ direction.

We can set up the following analogue of the directed polymer in the context of the discrete random string. For each $z \in \mathbb{Z}^d$, we create an independent noise variable $\omega(t,z)$ and an independent Poisson process. When the clock rings, say at time $t$, we replace $\omega(t,z)$ by an independent copy of itself. Then we set up the directed polymer measure

$$dP_u^{T,\omega} = \frac{\exp\left(-\beta \sum_{z\in\mathbb{Z}^d} \int_0^T \omega(t,u(t,z))dt\right)}{Z_T(\omega)} dP_u$$

Our aim is to understand the asymptotics of $P_u^{T,\omega}$.

# 3 Outcome of the Meeting

At this meeting we finalised our first results, and made them available on the arxiv [1]. We submitted the paper for publication. We were also able to prove the asymptotics in $J$ of $S_T$ and are currently writing it up. We plan to continue to work on the discrete polymer during the coming year.

# References


[1] Athreya S, Joseph M, Mueller C. Sausage Volume of the Random String and Survival in a medium of Poisson Traps. *arXiv preprint arXiv:2212.03166.*

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