Functional and Metric Analysis and their Interactions

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1 Overview

Broadly speaking, the main theme of the conference was the interactions between geometry and analysis. More precisely, the workshop focused on the relation between nonlinear geometry and objects coming from functional analysis as well as its interactions with other related areas. In the past couple of decades, there has been an upsurge of interest from different areas of mathematics in problems of this sort. For instance, researchers are interested in determining when given metric spaces are coarsely embeddable into another metric space (or into a class of metric spaces such as Hilbert spaces, reflexive spaces, uniformly convex spaces, etc.) and in characterizing linear properties in terms of purely metric ones (e.g., Ribe's program). Besides metric geometrists and functional analysts, those problems are of particular interest to theoretical computer scientists working with data-mining and the travelling salesman problem ([1, 5]), to C^*-algebraists working in connection with the Novikov conjecture ([7]), to mathematical physicists who use this as a framework to study the classification of topological phases ([6, 2, 4]), and to economists working with transportation theory in the study of optimal transportation and allocation of resources (see [3]).

Despite the attention problems of this sort have been receiving lately, we believe the community is still too divided into specialized research subgroups in a way that is detrimental to the further development of the area. The conference aimed to try to remedy this situation. Our list of speakers and attendees included experts interested in a vast range of the aforementioned topics. Moreover, students and junior researchers working on related topics were a key part of our workshop; which is fundamental for the renewal of the field and for creating the necessary environment for the next experts to rise.

2 Description of the talks

Many of the talks concerned very recent developments in their field, including some important unpublished results that could only be learned about in a setting such as this workshop. The talks fit into roughly four different research areas, although there was significant overlap between these areas. The talks have been summarized below, grouped according to their main area. Following the summary, the talks are cross-listed according to their secondary or tertiary areas.

Analysis on Metric Spaces

1. **Estibalitz Durand-Cartagena**'s talk was concerned with the notion of "rough angles" in metric spaces, which have to do with three-point inequalities interpolating between the ultrametric inequality and the triangle inequality. The interplay between this property and other natural geometric properties like embeddings into Euclidean space was explored. Having small rough angles implies pure 1-unrectifiability, a frequently recurring property throughout the workshop.

2. **Sylvester Eriksson-Bique** spoke about coarse tangent fields in doubling subsets of Hilbert space, based on joint work with David and Schul. This construction was motivated on the one hand by the notion of tangent field by Alberti-Csrnyei-Preiss, and on the other by Peter Jones' β-numbers used in the analyst's traveling salesman theorem. The extension of Alberti-Csrnyei-Preiss' result to higher dimensions is a major problem in the field, and the presented work moves toward that direction.

3. **Pietro Wald** presented joint work with Bate on differentiability of Banach space valued Lipschitz functions on metric measure spaces. This line of research began with Cheeger-Kleiner who proved results of this type for Lipschitz functions taking values in Banach spaces with the Radon-Nikodym property. The results of Wald showed that it is possible to have a metric measure space for which the differentiability behavior of Lipschitz maps into ℓ^p depend on the value of p, which sharpens and simplifies a construction of Schioppia.

4. **Pedro Tradacete** spoke on doubling measures on metric spaces minimizing the doubling constant, in particular about some conditions under which the measure achieving the minimal constant is unique. The results were obtained in joint work with Conde-Alonso and Benito.

5. **Jeremy Tyson**'s talk was on notions of dimension in metric spaces interpolating between Assouad dimension and box-counting dimension and between box-counting dimension and Hausdorff dimension. These new notions of dimension can be used to distinguish spaces where Hausdorff dimension, box-counting dimension, and Assouad dimension cannot. Jeremy's work was joint with Fraser and Chrontsios Garitsis.

6. **Lisa Naples** presented research on the asymptotic structure of maximum distance minimizers, based on joint work with Alvarado, Catalano, and Merchán. These sets arise as solutions to optimization problems that can model infrastructure such as water supply to houses, and it also closely related to the analyst's traveling salesman problem, another recurring theme in the workshop.

7. **Olga Maleva**'s talk on joint work with Dymond concerned the study of points of nondifferentiability of Lipschitz functions between subsets of Banach spaces. This topic is intimately related to the work on tangent fields originating from Alberti-Csrnyei-Preiss, and also plays a classic role in nonlinear embedding theory of Banach spaces.

8. **Assaf Naor** spoke about the Euclidean distortion of n-point metric spaces quasisymmetrically embedding into Hilbert space. Their new result, joint with Chang and Ren, is that every such metric space has Euclidean distortion $O(\sqrt{\log(n)})$. This is a sharp upper bound and is new even when the metric space is assumed to be a subset of ℓ_1. Besides the intrinsic interest to metric geometry, there are applications to the theory of algorithms, e.g., the Sparsest Cut Problem.

Banach Space Geometry

9. **Ramón Aliaga**'s talk about properties of the Lipschitz-free spaces over complete purely 1-unrectifibale metric spaces can be placed at the frontier of the functional and metric analysis. The first part was devoted to a remarkable result due to R. Aliaga, C. Gartland, C. Petitjean, and A. Procházka which says that a free space has the Radon-Nikodým property if and only if it has the Schur property, and that this happens precisely when the underlying metric space is purely 1-unrectifiable. If the metric space is compact, all of the above is moreover equivalent to the free space being a dual Banach space. Although this is a Banach-space theoretic statement, the argument comes from metric geometry and relies on D. Bate's approximation result with locally flat functions. Another presented property was

that every element of the free space over a complete purely 1-unrectifiable space is a convex integral of molecules by R. Aliaga, E. Pernecká and R. Smith.

10. **Alexandros Eskenazis** talked about some recent advances in the Ribe program, based on joint works with Mendel and Naor. In particular, he presented new bi-Lipschitz invariants for metric spaces which characterize classical local properties of norms. Important examples are new metric characterizations of the existence of an equivalent p-uniformly smooth norm or of Rademacher type p. More broadly, Eskenazis explained a general method for obtaining a metric characterization of a given linear local property of Banach spaces.

11. **András Zsák** presented in his talk joint work with Florent Baudier and Thomas Schlumprecht, on Kaltons problem, whether every reflexive Banach space coarsely embeds into a stable metric space. He showed that every reflexive space is upper stable, and that, moreover, upper stability and Kaltons Property Q are equivalent.

12. **Triinu Veeorg**'s talk, based on joint work with T. A. Abrahamsen, R. Aliaga, V. Lima, A. Martiny, Y. Perreau, and A. Procházka, was concerned with some isometric aspects of Lipschitz-free spaces over complete subsets of $\mathbb{R}$-trees. She showed that the Daugavet points and Δ-points in the unit sphere of such free spaces coincide. Moreover, if they can be represented as convex series of elementary molecules, they also agree with "super" variants of the Daugavet/Δ-points (where slices are replaced with relatively weakly open sets), and a clear metric characterization of these properties exists.

13. **Yoël Perreau** gave a talk about Delta and super Delta points in Lipschitz-free spaces. He showed that these notions agree for generating elements of the unit sphere called molecules, and provided their metric characterization in terms of the connectability of the two defining points in the underlying metric space. From this, he derived that every infinite-dimensional Banach space can be renormed with a super Delta point.

14. **Alicia Quero de la Rosa** talked about the Pełczyński's property (V^*) in Lipschitz-free spaces, motivated by an open problem to decide whether for free spaces (V^*) is equivalent to not containing c_0 isomorphically. Quero showed that (V^*) is locally determined (towards the compact determination of non-containment of c_0) and gave several sufficient metric conditions for (V^*) to hold. The new examples of free spaces enjoying (V^*) presented here included those over locally compact purely 1-unrectifiable spaces, Hilbert spaces, or Carnot groups. The talk was based on joint work with R. Aliaga and E. Pernecká.

15. **Andrés Quilis** talked about curve-flat Lipschitz functions on a metric space M. The main result is that a Lipschitz map between metric spaces is curve-flat if and only if its linearization between the corresponding Lipschitz-free spaces is a Dunford Pettis operator, if and only if it is Radon-Nikodým, if and only if it does not fix a copy of L_1. A key tool is a result by D. Bate on the approximation of curve-flat functions by locally almost flat functions. This is a nice example of the interactions between the analysis on metric spaces and the linear properties of Lipschitz free spaces. This result provides an operator version of a theorem by Aliaga, Gartland, Petitjean and Procházka and is a joint work with Flores, Jung, Lancien, Petitjean and Procházka.

16. **Tommaso Russo** answered in his joint work with Carlo Alberto De Bernardi and Jacopo Somaglia a question of Fonf and Lindenstrauss, and constructed a $\sqrt{2}+$-separated and 1-dense subgroup of $\ell_2(\Gamma)$, Γ being a cardinality satisfying $\Gamma^\omega = \Gamma$, for which there exists a symmetric, bounded, convex body, whose translates via that subgroup form a tiling of $\ell_2(\Gamma)$.

17. **Richard Smith** presented his recently developed Choquet theory for Lipschitz-free spaces. The classical De Leeuw's representation of elements of free spaces by Radon measures is universal, but not unique. Smith introduced a quasi-order on the representing measures that allows the selection of the "best" representations with notably agreeable properties that find applications in the isometric theory of free spaces. In particular, they provide a complete characterization of the extreme points of the closed unit ball of any free space. The latter part of the talk was based on joint work with R. Aliaga and E. Pernecká.

18. **Damian Sobota** provided in his talk, new isometric descriptions of separable $C(K)$-spaces in terms of real-valued functions on infinite trees. This result led to proving that certain isometric copies of separable $C(L)$-spaces in a $C(K)$ space are actually complemented. This work was achieved jointly with Jakub Rondos.

19. **Florent Baudier** presented ultraprobabilities as a bridge between the Ribe program and the Kalton program. The goal of the Ribe program is to reformulate in purely metric terms local properties of Banach spaces. The Kalton program has the same goal for asymptotic properties. In the reflexive setting, ultraprobabilities and weak limits over ultrafilters are the asymptotic counterparts of usual conditional expectations for local properties. The range of potential applications is very wide. Baudier gave examples based on joint works with Aceves or Fovelle.

20. **Alejandro Chávez-Domínguez** studies in this joint work with Bruno M. Braga the small scale geometry of operator spaces. While completely coarse maps between operator spaces are linear, as previous work by the same authors proved, it is shown, that the small scale variant is much richer.

21. **Michal Doucha** talked about recent progress concerning the existence of strictly convex renormings of Banach spaces that are invariant with respect to an action of a group by linear isometries. He presented some new results showing that the solution depends not only on the Banach space in question, but also on the acting group and the type of action. This work is motivated by a question by M. de la Salle and extends results by Lancien, Ferenczi and Rosendal. It is a nice example of interactions between questions about group actions and classical renorming problems from Banach space geometry.

22. **Audrey Fovelle** talked about p-concentration properties of Lipschitz maps from countably branching Hamming graphs to Banach spaces, for $p \in (1, \infty]$. This tool was introduced by Kalton and Randrianarivony in 2008 for maps with values in asymptotically uniformly smooth (AUS) Banach spaces. In a previous work Fovelle properly formalized these properties. By a precise study of the stability of these properties under ℓ_p-sums, she provides in this talk the first example of a non AUS Banach space with this p-concentration property. In fact, she even shows that such a space can be found with arbitrarily high Szlenk index. This question was open since the Kalton-Randrianarivony paper.

23. **Miguel Martin** presented in his talk several costructions of Banach spaces for which the set of norm attaining functionals do not contain non trivial cones.

24. **Rubén Medina** presented the solution to a problem by Dilworth, Kutzarova and Ovstrovskii, and found the growth rate of the Banach-Mazur distance between the Lipschitz free space over Laakso graphs and ℓ_1^N, of the corresponding dimension.

25. **Estelle Basset** presented new joint results with G. Lancien and A. Procházka on the isomorphic distinction of Lipschitz-free spaces, obtained by studying their dentability index. Basset gave a construction of countable complete discrete metric spaces such that the dentability indices of their free spaces are arbitrarily large countable ordinals. This leads to an uncountable family of mutually non-isomorphic free spaces over countable complete discrete metric spaces. Moreover, it provides the first known examples of free spaces over countable complete discrete metric spaces that do not isomorphically embed into any free space over a uniformly discrete or compact purely 1-unrectifiable metric space, since the dentability indices of the latter two are bounded by a countable ordinal.

26. **Valentin Ferenczi** talked about extremes of interpolation scales of Banach spaces and uniform homeomorphisms between spheres of Banach spaces. In 1995, M. Daher gave conditions so that the spheres of the spaces in the interior of a complex interpolation scale are uniformly homeomorphic. Ferenczi gave sufficient conditions for the validity of this result and related ones on the extremes of the scale, with applications to uniform homeomorphism between spheres of a uniformly convex Banach space with a 1-FDD and the sphere of ℓ_2. This is a joint work with W. Corrła, R. Gesing, and P. Tradacete.

Modeling Coarse Geometry by C^* Algebras

27. **Ruy Exel** gave a talk about flows on uniform Roe algebras $C_u^*(X)$ of uniformly locally finite metric spaces X. Flows are defined as one parameter automorphism groups which are strongly continuous.

Exel's talk focused on flows for which finite propagation operators are points of differentiability, called *coarse flows*. Observing that every flow on $C_u^*(X)$ is given by a (possibly unbounded) "Hamiltonian" operator h, Exel characterized precisely which operators are allowed, provided that the metric space X satisfies Yu's property A. Another aspect of the talk was the discussion of cocycle equivalence and cocycle perturbation of flows, culminating in the result stating that, again under property A, every coarse flow is a cocycle perturbation of a coarse diagonal flow (i.e., a flow whose Hamiltonian is a diagonal operator whose diagonal entries are given by a coarse function on X). Exel's talk was based on a joint work with Bruno de Mendona Braga and Alcides Buss.

28. **Rufus Willett** gave a survey talk about expander graphs and K-theory with a special emphasis on discussing different properties of expanders, the current state of art in the field and on some important open questions in the area. In his talk, Willett explained how some important conjectures in topology and geometry can be solved if one knows enough about K-theory of certain appropriate C^*-algebras, the so-called "Roe algebras". On the other hand, Willett explained why expanders (sequences of sparse, highly connected graphs) lead to pathological behavior in these K-theory groups, with different properties of expanders leading to different types of pathology.

29. **Narutaka Ozawa** gave a talk on this recent article which showed that the quasi-local algebra of a sequence of expander graphs is strictly larger than its uniform Roe algebra. Precisely, for bounded linear operators acting on a Hilbert space over a uniformly locally finite metric space, there are two notions of "localness", finite-propagation and quasi-locality. The distinction of these two is similar to that of compactly supported functions and functions vanishing at infinity on that metric space. John Roe has asked whether quasi-local operators can be approximated in norm by finite-propagation operators. It has been proved over the time that this is the case provided that the underlying space is sufficiently nice. On the other hand, Ozawa recently found the first example of a quasi-local operator that is not approximately finite-propagation. Ozawa's proof in based on an analysis of the embeddability of matrix algebras into these two C^*-algebras.

30. **Diego Martnez** gave a talk about his recent solution for the rigidity problem for Roe algebras. Martnez explained how a given a metric space gives rise to a notion of bounded operators with "bounded displacement" and how this leads to the construction of the so called Roe algebras, C^*-algebras first considered by Roe in the 1990's and that inherits much of the large-scale geometric structure of the space. Matnez talk, which was based on a joint work with Federico Vigolo, discussed why these algebras are always a coarse invariant, that is, why their isomorphism class does not depend on the local geometric aspects of the starting space.

31. **Piotr Nowak** discussed higher-dimensional analogs of Kazhdan projections in matrix algebras over group C^*-algebras and Roe algebras. These projections are constructed in the framework of cohomology with coefficients in unitary representations and in certain cases give rise to non-trivial K-theory classes. He also showed a connection with ℓ_2-Betti numbers of a group and with surjectivity of different Baum-Connes type assembly maps. Nowak's talk was based on a joint work with Kang Li and Sanaz Pooya.

Geometric Group Theory and Group Actions

32. **Christian Rosendal** started his talk defining asymptotically spherical topological groups. This is a notion which says that spheres of large radius with respect to any maximal length function are still spherical with respect to any other maximal length function, which is a strengthening of a related condition introduced by Sebastian Hurtado, called bounded eccentricity. Rosendal's main result was a partial characterization of which groups are asymptotically spherical. He also presented an example of a discrete, bounded eccentric group who fails to be asymptotically spherical. Rosendal's talk was based on a joint work with J. Zomback.

33. **Tessera Romain** was about cuts of finite subgraphs of hyperbolic groups, including applications. The cut of a finite graph X is the minimal size of a set E of edges such that after removing E, the size of the remaining connected components of X is at most $|X|/2$. Given an infinite graph G, one

may look for every n at the maximal cut of a subgraph of size n. This yields the separation profile, introduced by Benjamini-Schramm and Timar. This provides a powerful obstruction to the existence of Lipschitz embeddings between Cayley graphs. An important application being the following theorem: an amenable group Lipschitz embeds in a hyperbolic group if and only if it is virtually nilpotent. Alternatively, one may focus on the cut of balls in G, which imposes restriction on subsets which coarsely separate G. An application is the following theorem, recently obtained with Oussama Bensaid and Anthony Genevois: a hyperbolic group is coarsely separated by a subgraph of subexponential growth if and only it virtually splits over a 2-ended subgroup.

34. **John Mackay** talked about groups acting on the Banach space L_1. As Mackay explained, an important classical way of studying groups is through their possible affine isometric actions on Hilbert spaces, which lead to the development of influential concepts such as Kazhdan's Property (T) and the Haagerup property. Other Banach spaces, and other types of action, are also natural and useful to study. For example, actions on L_p spaces, or actions by uniformly Lipschitz affine maps. Mackay discussed some of the recent developments in this area, including his joint work with Cornelia Drutu where they found actions on L_1 spaces for groups with hyperbolic features.

35. **Cornelia Drutu** gave a talk about how to understand infinite groups via their actions on Banach spaces. Cornelia talked about fixed point properties and proper actions on Banach spaces, which are relevant in several areas such as for the Baum-Connes conjectures, in combinatorics, for the study of expander graphs, in ergodic theory, etc. In particular, she described two notions of spectrum allowing to measure "the strength" of property (T) for infinite groups, and what can be said about them in the case of hyperbolic groups. Cornelia also discussed properties of (acylindrically) hyperbolic groups and mapping class groups, that correspond to weak versions of a-T-menability. Cornelia's talk was based on joint works with Ashot Minasyan and Mikael de la Salle, and with John Mackay.

Cross-Listing

In addition to the groups listed above, the following talks are cross-listed in other research areas.

1. **Analysis on Metric Spaces:** Ramón Aliaga, Alicia Quero de la Rosa, Rubén Medina, Andrés Quilis.

2. **Banach Space Geometry:** Sylvester Eriksson-Bique, Pietro Wald, Olga Maleva, Assaf Naor, John Mackay, Cornelia Drutu.

3. **Modeling Coarse Geometry by C^* Algebras:** John Mackay, Cornelia Drutu.

4. **Geometric Group Theory and Group Actions:** Michal Doucha.

3 Posters

Five of our junior participants presented posters on their research. The posters complemented the talks well and the presenters discussed their work with more senior experts in their field. The presenters and their posters' descriptions follow.

1. **Jan Bíma** presented a poster about nagata dimension and Lipschitz extensions into p-Banach spaces. Given two metric spaces $N \subseteq M$ and $0 < p \leq 1$, we wish to determine the smallest constant $\mathfrak{t}_p(N, M)$ such that any Lipschitz map $f : N \to Z$ into any p-Banach space Z can be extended to a Lipschitz map $f' : M \to Z$ satisfying $\mathrm{Lip}(f') \leq \mathfrak{t}_p(N, M)\mathrm{Lip}(f)$. We establish that if N has finite Nagata dimension at most d with constant γ, then $\mathfrak{t}_p(N, M) \lesssim_p \gamma(d+1)^{1/(p-1)} \log(d+2)$ for all $0 < p \leq 1$. We show that examples of spaces with finite Nagata dimension include doubling spaces, as well as minor-excluded metric graphs. We also establish that the constant $\mathfrak{t}_p(N, M)$ generally increases as p approaches zero.

2. **Helena del Rio**'s poster introduced a new class of bounded linear operators, called Range strongly exposing operators (RSE, in short), which form a natural intermediate class: weaker than absolutely strongly exposing operators, yet stronger than both uniquely quasi norm-attaining and classical norm-attaining operators. Several foundational results on norm-attaining operators are extended to the RSE setting. In particular, del Rio improved some classical results by Uhl (1976) and Schachermayer (1983) and got some analogous to Acosta (1999). Her poster was based on joint work with Geunsu Choi, Audrey Fovelle, Mingu Jung, and Miguel Martin.

3. **Giulia Fantato**'s poster investigated the existence of a uniform homeomorphism between the sphere of Ferenczis uniformly convex hereditarily indecomposable Banach space and the sphere of ℓ_2. Her approach was based on complex interpolation of families of Banach spaces: a family of norms on $\mathbb{C}^n$ together with their corresponding antiduals were considered, and Fantato showed that the interpolation yields a Hilbert space at the center of the strip. This is currently being studied in the context of a result by Daher, which guarantees the existence of uniform homeomorphisms between the spheres of spaces that arise from the interpolation of families of Banach spaces.

4. **Tomáš Raunig** presented a poster motivated by a question published in a paper of E. Glasner and motivated by the work of D. Kazhdan and A. Yom Din regarding the possibility to approximate functionals on a Banach space which are almost invariant with respect to an action of a discrete group by functionals that are invariant. The poster studied the case when the Banach space is a Lipschitz-free space equipped with an action induced by an action by isometries on the underlaying space. Raunig's work found a few different conditions sufficient for the answer to be positive; for example the case of free or finitely presented groups endowed with left-invariant metrics acting on themselves by translations.

5. **Henrik Wierzenius** presented a poster on his recent work on the structure of non-trivial closed subideals of the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on a Banach space X. A closed linear subspace I is called a closed J-subideal of $\mathcal{L}(X)$ if I is a closed ideal of J, and J is a closed ideal of $\mathcal{L}(X)$. The J-subideal I is non-trivial if I is not an ideal of $\mathcal{L}(X)$. The poster also included observations on closed n-subideals, a graded extension of the concept of closed subideals. The poster was based on a joint work with Hans-Olav Tylli.

4 Career Panel

We held a career panel for the purpose of educating the audience about career and funding opportunities in various countries. The panelists were Alicia Quero de la Rosa and Pedro Tradecete from Spain, Romain Tessera from France, Christian Rosendal and Chris Gartland from the United States, and Olga Maleva from the United Kingdom. The following questions were asked of each panelist.

1. Can you describe your career arc, including education and employment history?

2. How would a prospective applicant learn about a job opening at your institution or country?

3. What external funding opportunities exist for researchers working in your institution or country?

4. Are there opportunities for visiting positions at your institution or country?

We received very positive feedback from the junior participants about the usefulness of the panel.

5 Open Problem Session

We held an afternoon session where the workshop participants could propose and explain open problems. The list of problems follows.

submitted by Chris Gartland

Q1 Does $\mathcal{F}(\mathbb{R}^3)$ isomorphically embed into $\mathcal{F}(\mathbb{R}^2)$? More generally, are there $n > m \geq 2$ for which $\mathcal{F}(\mathbb{R}^n)$ isomorphically embeds into $\mathcal{F}(\mathbb{R}^m)$?

The version of the question for $n = 2, m = 1$ has a negative answer by Naor-Schechtman.

submitted by Sylvester Eriksson-Bique

A subset $A \subseteq M$ of a metric space is *r-connected* if for every $x, y \in A$, there exists $\{x_i\}_{i=0}^k \subseteq A$ such that $x_0 = x$, $x_k = y$, and $d(x_{i-1}, x_i) \leq r$ for every $i \in \{1, \dots k\}$.

We call an L-Lipschitz map $f : X \to Y$ between two metric spaces

- *L-biLipschitz* if $d(x, y) \leq Ld(f(x), f(y))$ for all $x, y \in M$.
- *L-Lipschitz-light* if diam$(A) \leq Lr$ whenever $r > 0$ and $A \subseteq M$ is r-connected with diam$(f(A)) \leq r$.
- *L-BLD* (bounded length distortion) if $\ell(\gamma) \leq L\ell(f(\gamma))$ for every curve $\gamma \subseteq M$ with finite length $\ell(\gamma) < \infty$.

Note that L-biLipschitz maps are L-Lipschitz light, and that L-Lipschitz light maps are L-BLD.

Q2 Let $M \subseteq \ell_2$ be a doubling subset of a Hilbert space. Must there exist $n \in \mathbb{N}$ and $L < \infty$ such that

(a) M admits an L-biLipschitz map into $\mathbb{R}^n$? (the Lang-Plaut problem)

(b) M admits an L-Lipschitz light map to $\mathbb{R}^n$?

(c) M admits an L-BLD map to $\mathbb{R}^n$?

Q3 If $M \subseteq \mathbb{R}^m$ is D-doubling, can we find $n \in \mathbb{N}$ and $L < \infty$, depending only on D and not m, so that (b) or (c) above are true?

submitted by Valentin Ferenczi

Let X be a Banach space and G a bounded subgroup of the continuous linear automorphism group of X. It is known that G is discrete with respect to the strong operator topology (SOT discrete) if and only if there exist a finite set of vectors $F \subseteq X$ and a positive number $\alpha > 0$ such that for every nonidentity $g \in G \setminus \{1\}$, $\max_{x \in F} \|gx - x\| \geq \alpha$. We wish to understand the cases where the finite set F can be replaced by a singleton set. Antunes-Ferenczi-Grivaux-Rosendal (2019) give an example where $X = c_0$, the group G is SOT, and the set F has cardinality 2 but cannot be replaced by a singleton. In light of this, we are naturally lead to the following question.

Q4 If X is reflexive and G is SOT discrete, must there exist a vector $x_0 \in X$ and a positive number $\alpha > 0$ such that for every nonidentity $g \in G \setminus \{1\}$, $\|gx_0 - x_0\| \geq \alpha$?

submitted by Jesús M. F. Castillo

Q5 Let $H \subseteq \ell_1$ be a subspace. Does there exist a Banach space E such that every bounded linear operator $\varphi : H \to E$ admits a compact operator $k : H \to E$ such that $\varphi + k$ can be extended to a bounded linear operator on all of ℓ_1?

It is clear that we look for non-trivial situations in the above question: H should be uncomplemented, E should not be not separably injective, etc.

In fairly general cases, given a subspace $H \subseteq \ell_1$ as above, there is a Banach space E and a compact operator $\varphi : H \to E$ that can be extended to a bounded linear operator on all of ℓ_1, but φ admits no extension to a compact operator on ℓ_1.

Q6 Given a subspace $H \subseteq \ell_1$, is there a "natural" Banach space E (for example $E = \ell_2$) and a compact operator $\varphi : H \to E$ that can be extended to a bounded linear operator on all of ℓ_1, but φ admits no extension to a compact operator on ℓ_1?

submitted by Thomas Schlumprecht

Let M be a finite metric space. We say that M *stochastically embeds into trees* with distortion D if there exist a probability vector $\{p_i\}_{i=1}^k$ and maps $\{\phi_i : M \to T_i\}_{i=1}^k$ into graph-theoretic trees (equipped with weighted shortest-path metrics) such that $\min_i d(\phi_i(x), \phi_i(y)) \geq d(x, y)$ and $\sum_{i=1}^k d(\phi_i(x), \phi_i(y))p_i \leq Dd(x, y)$ for all $x, y \in M$.

The finite-dimensional p_λ problem asks whether there exists an increasing function $f : [1, \infty) \to [1, \infty)$ such that all n-dimensional ($n \in \mathbb{N}$), D-complemented subspaces of ℓ_1 are $f(D)$-isomorphic to ℓ_1^n. The following problem is a special case.

Q7 Does there exist an increasing function $f : [1, \infty) \to [1, \infty)$ such that, whenever M is a finite metric space that stochastically embeds into trees with distortion D, it holds that $\mathcal{F}(M)$ is $f(D)$-isomorphic to $\ell_1^{|M|-1}$?

submitted by Narutaka Ozawa

A function $f : G \to \mathbb{R}$ on a locally finite graph is *harmonic* if $f(x) = \frac{1}{|N_x|} \sum_{y \in N_x} f(y)$ for every vertex $x \in G$, where N_x is the set of vertices in G that share an edge with x. The following question was originally communicated to Taka by Yehuda Shalom.

Q8 For every finitely generated group Γ and finite symmetric generating set S, does the Cayley graph $G(\Gamma, S)$ admit an unbounded Lipschitz harmonic function.

Since all Cayley graphs with respect to finite symmetric generating sets admit a nonconstant Lipschitz harmonic function, the answer is yes for Cayley graphs for which all bounded harmonic functions are constant (called Liouville graphs).

submitted by Ramon Aliaga

For M a metric space, we denote by $\mathrm{lip}(M)$ the set of locally flat Lipschitz functions: $\mathrm{lip}(M) = \{f \in \mathrm{Lip}(M) : \forall z \in M,\ \lim_{x,y \to z} \dfrac{|f(x) - f(y)|}{d(x, y)} = 0\}$.

Q9 Let M be a purely 1-unrectifiable metric space and $N \subset M$ a compact subset. Does every $f \in \mathrm{lip}(N)$ admit an extension $F \in \mathrm{lip}(M)$ with $\mathrm{Lip}(F) = \mathrm{Lip}(f)$?

Q10 Let M be a compact purely 1-unrectifiable metric space such that there exist $K < \infty$ and, for every subset $A \subseteq M$, a linear extension operator $E : \mathrm{Lip}(A) \to \mathrm{Lip}(M)$ with $\|E\| \leq K$. Must there exist $K' < \infty$ and, for every subset $A \subseteq M$, a linear extension operator $E : \mathrm{lip}(A) \to \mathrm{lip}(M)$ with $\|E\| \leq K'$?

submitted by Eva Pernecka

This question was answered in the affirmative by William B. Johnson shortly after the conclusion of the workshop.

Q11 Fix $p \in [1, \infty) \setminus \{2\}$ and $K \subset \ell_p$ compact. Does there always exist a compact superset $L \supseteq K$ and a Lipschitz retraction $\ell_p \to L$?

The answer is no if the retraction is asked to be 1-Lipschitz by results of Kopecká and Kopecká and Reich.

submitted by Etienne Matheron

Q12 Fix $p \in [1, \infty) \setminus \{2\}$ and denote by $B_1(\ell_p)$ the unit ball of the space of bounded operators on ℓ_p. When equipped with the strong operator topology, $B_1(\ell_p)$ is a Polish space. Is the set of operators with a nontrivial invariant subspace comeager?

The answer is yes for $p = 2$ because the set of operators on ℓ_2 with an eigenvalue is comeager. However, for $p \neq 2$, the set of operators on ℓ_p with an eigenvalue is meager.

submitted by Sheldon Dantas

Q13 Does every infinite-dimensional reflexive Banach space contain a strictly convex 2-dimensional subspace?

submitted by Gilles Lancien

Q14 Let X be an infinite-dimensional Banach space. Under what conditions does ℓ_2 coarsely or uniformly embed into X?

Nowak proved that ℓ_2 coarsely and uniformly embeds into ℓ_p for every $1 \leq p \leq \infty$. Ostrovskii proved that ℓ_2 coarsely and uniformly embeds into every Banach space X with an unconditional basis and nontrivial cotype. Baudier-Lancien-Schlumprecht proved that ℓ_2 does not coarsely or uniformly embed into Tsirelson space T^*.

Q15 Does ℓ_2 coarsely or uniformly embed into every infinite-dimensional superreflexive space?

Let F denote Ferenczi's superreflexive, hereditarily indecomposable space. This is a natural candidate for a counterexample. Giulia Fantato proved that S_F is uniformly homeomorphic to S_{ℓ_2} and since ℓ_2 is uniformly homeomorphic to S_{ℓ_2}, ℓ_2 uniformly embeds into F.

Q16 Does ℓ_2 coarsely embed into F?

6 Outcome of the Meeting

The conference brought researchers from different areas together, who normally would not meet in the same conference. This promoted a fruitful crosspollination between these areas and contributed to dissemination of results, techniques, and ideas.

In addition, as many graduate students and postdoctoral researchers participated, the conference served as a venue to help integrate a younger and less experienced generation of researches to the community. Besides the talks and the welcoming environment of our meeting, our conference program was specially designed to facilitate this integration by including activities as a poster session, a career panel, and a problem session.

Finally, our conference also had a virtual component which has already started to provide its benefits. As mentioned in the session about open problems above, the question proposed by Eva Perneck, **Q11**, was solved recently after the conference by William B. Johnson. Johnson was unable to attend the meeting in person, but attended it virtually.

References

[1] B. Brinkman and M. Charikar, On the impossibility of dimension reduction in l_1, *Journal of the ACM* **52** (2005), 766–788.

[2] E. Ewert and R. Meyer, Coarse Geometry and Topological Phases, *Communications in Mathematical Physics* **366** (2019), 1069–1098.

[3] A. Galichon, *Optimal transport methods in economics*, Princeton University Press, Princeton, NJ, 2016.

[4] Y. Kubota, Controlled topological phases and bulk-edge correspondence, *Communications in Mathematical Physics* **349** (2017), 493–525.

[5] A. Naor and Y. Peres, L_p compression, traveling salesmen, and stable walks, *Duke Mathematical Journal* **157** (2011), 53–108.

[6] H. Schulz-Baldes, Topological insulators from the perspective of non-commutative geometry and index theory, *Jahresbericht der Deutschen Mathematiker-Vereinigung* **118** (2016), 247–273.

[7] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Inventiones Mathematicae* **139** (2000), 201–240.