Banff International Research Station Proceedings 2016

BIRS

Contents

Five-day Workshop Reports

Chapter 1

Creative Writing in Mathematics and Science (16w5093)

January 10 - 15, 2016

Organizer(s): Florin Diacu (University of Victoria), Marjorie Senechal (Smith College)

Overview

Our workshop series, Creative Writing in Mathematics and Science, aims to create a critical yet supportive community of writers who share the goal of reaching the general public through journalism, popularizations, fiction, biographies, plays, poems, and other forms of literary writing such as, increasingly, op-eds and blogs. Like its five predecessors, this workshop brought together 20 mathematicians, scientists, and journalists to discuss their work-in-progress.[1] The community of writers that is emerging from the series is important to all of us. Some are well-established, others are novices; some are mathematicians (and scientists), others professional writers and, increasingly, some are both. We stay in touch, reading each other's drafts, attending public readings (if we can get to them), writing letters of recommendation for one another and blurbs for books; we read each other's manuscripts for publishers. The participant list varies from year to year, with some overlap (which is beneficial for the workshop itself). This year in our workshop announcement we encouraged inquiries, and six participants attended for the first time. We are grateful to BIRS for its continued encouragement and support.

How The Workshop Was Run

As in the past, some of our participants identify themselves primarily as writers, others as mathematicians or scientists who also write about their field. Our common goal was to exchange ideas, help improve each others writing, and find new and ingenious ways to reach the general public. All six workshops in this series have been guided by three axioms:

1. Every participant must bring work-in-progress to be critiqued by the group.

2. We all of us are the teachers and we all of us are students. We have much to teach, and learn from, each other.

3. All genres are welcome. Novelist can learn from poets, journalists from playwrights, biographers from popular-science writers, and so on in all permutations.

[1] To everyone's regret and the workshop's loss, my co-organizer, Florin Diacu, was unable to attend the workshop, so there were actually only 19 of us.

This year we tried a new daily schedule (the reports of past workshops explain how we did it before). Each day focussed on particular genre: nonfiction, biography, poetry, and drama/fiction. We (the organizers) selected participants to lead a morning discussion on the day's theme (not to teach the subject). This left time, in the afternoons, for critiquing individual work.
These discussions were very different in format. Don O'Shea, who led the non-fiction discussion, chose an article (by a former workshop participant, not present at this one) for us all to read and critique. Siobhan Roberts, who led the discussion of biography, began by recounting some of the challenges she faced in writing biographies of mathematicians who are still alive. Ellen Maddow and Paul Zimet, the founders of the New York theater company The Talking Band, gave us exercises that raised central questions in writing drama and fiction. Emily Grosholz led the poetry discussion; see her report below for details. On Friday morning we discussed the workshop itself and planned for the future. On Monday and Tuesday evenings, we gathered after dinner to discuss topics of common interest. The public reading of our work, mentioned above, was held on Wednesday evening. Since we didn't give ourselves an afternoon off, many of us went for a hike on Thursday evening. .

Responses of the Participants

We (the organizers) asked the participants to describe what they gained from the workshop. Since several of them work in more than one genre, we have not organized their responses by genre. We list them in alphabetical order instead.

- At the BIRS 2016 workshop I received feedback on poems and creative nonfiction pieces. We spoke about truth and precision in creative writing which borrow from the scientific method. In addition, I participated actively in discussions around other projects. Topics ranged from math/science communication, the concept of mathematical proof, and permutation in theatrical performance. Overall, the experience was extremely engaging and fruitful. I have revised several poems in direct response to the experience. I have also expanded my network of colleagues working at the interface of arts and sciences. **Madhur Anand**

- My thanks for a week with a wonderful group of writers, poets, playwrights, biographers, mathematicians, and scientists; our discussions suggested multiple ways of approaching our writing tasks, the work we heard was at a very high level, and my own poetry manuscript benefitted greatly from comments both general and specific. Each of the genres offered opportunities for new insight. in explanatory writing, how to think of the audience and one's purpose (e.g. "Intrigue the novice but don't embarrass the expert"), how to explore narrative structure and voice in the service of teaching texts; how to incorporate scientific thinking into novel writing; improvisational techniques with mathematical underpinnings for rapidly generating theatre plots; mathematical equations and geometrical objects as a) metaphorical devices for poets b) playful devices for coloring books and constructions. Biographical writers variously explored the lives of their mathematical and scientific protagonists and the cultural consequences of interactions between their disciplines and other arts and sciences; op-ed writers took up the serious task of explaining to newspaper readers relative risk; blog posters of mathematical poetry looked for ways to enrich online access to archives. Environmental ecologists took their own scientific vocabulary, shaken up with life stories, to create poems. So many ideas arose! A big thank you to the organizers and participants, and to BIRS for its sponsorship. **Robin Chapman**

- As one who does significant amounts of fiction and nonfiction writing, as well as poetry, I found the wide focus of this workshop extremely helpful. I had several very helpful discussions about the Fibonacci-inspired verse form that I've been working on. I also found the discussion of theater, and the improvisation session, fascinating, though it isn't an area that I've worked in. Overall, this series of workshops has been formative both to my work as editor at the CMS Notes and to my sideline of other creative writing. While it's too early to say what specific fruit this workshop may bear, a talk at the previous workshop in the series was the inspiration for a science fiction story that has appeared on a website, Issues in Earth Science, intended as enrichment material for grade school children. **Robert Dawson**

- This workshop was especially helpful because it spanned different genres. As it happens, Im trying to write about mathematics in the genre of poetry, but also in the context of two books on philosophy of mathematics, where I must carry out quite a bit of exposition of mathematical ideas. So, for example, on the first day, we agreed that diagrams can be useful additions to expository prose and mathematical notation, but we noted that correct placement is key, and also that too many side-car boxes with related information can distract from the main thrust of the argument, and weaken the exposition. And the author must think long and hard about what terms need to be defined formally. If you have to define too many terms, you may be leading the reader into waters that are too deep; yet the languages of mathematics especially in the twentieth century involve lots of special concepts and terms. The last day was devoted to poetry about mathematics and the mathematical sciences, which I led. We talked about poems with structure inspired by mathematics, and traditional structure that can be illuminated by mathematics, as well as poems about mathematical results and concepts, and poems that appropriate mathematical vocabulary and use it in odd and surprising ways. In what ways is a poem like a theorem, and in what ways is it different? Each of us (that is, the poets present) read a sample poem, which also often led to thinking about the temporal and spatial structure of poems, and their symmetries, or the poignant relation between the evanescent writers and their persistent poems (and theorems). **Emily Grosholz**

- I have, for more than 30 years, been a collector of poetry related to mathematics. In 2010 I began to share these collected poems on the Internet using a blog, Intersections Poetry with Mathematics at http://poetrywithmathematics.blogspo The challenge that I posed to my colleagues at BIRS was to evaluate the blog and to suggest not only ways to improve it and but also ways to make information in this online resource more easily accessible to interested visitors. Advice included the introduction of a SEARCH box, development of links to favorite posts, and use of social media to attract visitors to relevant posts. I am gradually implementing these strategies and continually on the lookout for others. **JoAnne Growney**

- I learned a great deal at this workshop, but my most significant take-away was the feedback I received on my own writing (including advice on what pieces of feedback I should use to change my writing and what pieces I should examine, then feel free to discard). What is important to me, personally, is that this has motivated me to return to the writing that I had neglected for so many years. The kind and thoughtful encouragement I received is key to the newfound motivation. **Helen Grundman**

- The longer I am part of the mathematical community the more I find its culture both wonderful and lacking. On the one hand I can sit down with mathematicians from Japan to Iran and immediately have common experiences and ideas, and a pathway to learning more about worlds, both internal and external. On the other hand the culture can be judgemental, exclude people for reasons that do not live up to scrutiny or seemingly deliberately hide its beauty behind elegant but hard to master languages. Yet what I feel most is the lack of introspection about the culture, making the failings harder to deal with. As a firm believer in the importance of mathematics and abstract thinking in an age of increasingly abstract information I struggle with this. It was a delight, therefore, to meet with a group of people from mathematics and beyond who embraced discussion of mathematical culture and were engaged in placing mathematics in a broader setting. Providing bridges across disciplines and engaging a wider audience of people. This conference on a practical level provided support and criticism of creative projects at a high level. Yet more importantly it gave me, and I feel many of the other participants, a chance to recharge our fundamental love for the subject and return to the challenge of how we can try to make it more accessible and welcoming to all who can benefit on a technical, numerical or aesthetic level. **Edmund Harriss**

- I presented a few scenes from the mathematical play I am working on as part of my senior project thesis at College of the Atlantic. My play, currently titled Problem Set, explores the narratives and art of mathematics. In the first scene, Definitions, Professor Wentworth introduces the suspension of disbelief in theater and geometry to the audience. In the second scene, Misfits, Pi and the Koch Snowflake meet in the waiting room at a psychologist's office and discuss what it means to be normal. At the public reading, I also presented a short monologue that juxtaposed the musicality and rhythms of a mathematician working through a problem and an actor warming up before a show. The feedback from the other participants in the workshop was very helpful for me in focusing the scope of my project. I had so many disparate mathematical ideas that I wanted to explore through my writing, but it was too large a scope to contain in just one play. I was really struggling with which concepts to concentrate on developing deeply for this play in particular. After hearing everyone's feedback, I've decided to focus on considering the process of learning and conceptualizing mathematics in my play, and set aside the stories of historical mathematicians like Archimedes for another play. As a senior undergraduate, it was really encouraging to meet the other participants, who are all engaging with the sorts of really exciting interdisciplinary ideas that I want to spend the rest of my life exploring. I am so thankful for the opportunity to attend this workshop. Talking with and learning from all of these amazing people in such a beautiful place was an experience that will definitely inspire me and my work in the future. **Corrie Ingall**

- I brought to Banff a few chapters of a manuscript I have been working on. This is to be a nontraditional textbook for an introductory course on analysis with a specific focus on methods of mathematics. Usually books at that stage focus exclusively on proof, and I aim in this project to expand that focus. I also intend to introduce students / readers to the culture and history of mathematics and infuse the narrative with significant amount of philosophical digressions. During the workshop I have received much valuable and constructive feedback about the book. In particular participants suggested that I separate out the various informal comments from the technical mathematical content, drop some of the preliminary text, possibly consider alternative forms of publishing. Each and every piece of comment I received on my work was constructive and, I believe, will allow this manuscript to be much more effective in the end. The workshop discussions on other participants work have also been rewarding and eye-opening for me. Once again I was reminded of the immensity of the task of the writer and the poet. Once again I observed the craft nature of writing and the sheer amount of work that goes into creating something that is of value to someone besides its creator. And once again I could see clearly that readers can help improve work at any stage. All in all, I am delighted and honored that I had the opportunity to participate in the workshop. I made new connections with people who I now see as my friends as well as my colleagues, and I worked towards m own personal writing goals, all the while being reminded of the very collaborative and community aspects of the two life pursuits I find myself dedicated to: mathematics and writing. **Gizem Karaali**

- My purpose in attending the creative writing workshop at BIRS was to get inspiration for a starting point for a new play with mathematical content and structure. My play Delicious Rivers was inspired by my participation in similar workshops organized by Marjorie Senechal. Once again it was energizing to be in a room with a group of people, from a variety of disciplines grappling with the challenges of writing about complex ideas with clarity and elegance. Books that were suggested by fellow participants that might provide ideas for my new play are <u>II Died for Beauty – Dorothy Wrinch and the Cultures of Science</u> by Marjorie Senechal, and <u>Euler's Gem: the Polyhedron Formula and the Birth of Topology</u> by David S. Richeson. **Ellen Maddow**

- I came to the workshop with three short Op-Ed articles for national (print and digital) newspapers. One was a draft of my annual Pi Day article for Slate Magazine, and the other two were extracted opinions that stemmed from my forthcoming book Fluke: The Math and Myth of Coincidence. Those articles led to many valued intelligent conversations, discussions, opinions, and constructive ideas that continued with residual discussions even days after our five day residency through the BIRS list. From the intelligence and collegiality of workshop participants, my articles were rewritten and very much improved. Everyone talked about how much they benefitted from the sessions and about how they felt a sense of accomplishment along with an awareness of work to be done. It was my third BIRS workshop in math and science writing, the third with a very capable, wise, and professional group. I came away with a comfortable list of ideas for improvements and an exhilarating sense of accomplishment, a feeling that would be hard to come by in my usual world of solitary writing. The satisfaction went far beyond the welcomed critiques of my own work. Reading and listening to the works of my colleagues, to the prose and poetry, gave me a strong sense of my own contribution to the writing cause. I felt a sense of value and benevolence at being able to give constructive criticism to others, when I could. I felt exhilarated by the rewarding opportunity of reading the works of others, because reading those works and listening to the presentations of those works gave me, the lonely writer, a sense of writing from the other side. From that other side one gets an invaluable sense of how readers read and interpret what the writer means to say, but doesnt always succeed in delivering the intended meaning. As everyone knows, getting to Banff is not an easy trip, especially for someone coming from as far away as Vermont. Yet each time I come, I leave with a feeling that the trip was well worth the weariness. I leave with a feeling of great accomplishment and satisfaction that my work has been constructively criticized by a team of knowledgeable colleagues who genuinely care about the works of others. **Joe Mazur**

- The workshop raised–in my mind, anyway–two big questions. One relates to the importance of understanding in creative writing about mathematics, both the extent to which a writer must understand the mathematics to convey something meaningful about it and the degree to which a writer can hope to impart understanding to lay readers. The other question–a perennial one–is about truth: the distinction between fiction and nonfiction, and the importance (or not) of coming down squarely in one category or the other. Even more valuable to me than the feedback from my fellow workshoppers on my own work–though I did greatly appreciate the encouragement and suggestions I received about my nascent screenplay–was the insights the workshop gave me into the process and product of other writers. I very much enjoyed seeing how a poet writes prose, for example, and I hope to bring some of Madhur's economy of language, attention to sound, and careful selection of sometimes startling details to my own work in the future. And as someone who too often waits passively for inspiration to strike, I may try to replicate on my own a theater-type exercise like the ones Paul and Ellen led. The serendipitous juxtapositions of ideas that result would, I think, jog my at times too-logical brain into a more creative, possibly more productive state. I left Banff with a list of books to read, a number of writing ideas to pursue, and confirmation that there exists at the intersection of mathematics and writing a vibrant community of which I am thrilled to be a part. **katharine merow**

- One of the more memorable discussions that I remember from the BIRS Workshop took place on the first day, when we took up the non-fiction genre. On the one hand there is a desire by some writers in the mathematics community to convey mathematical concepts more directly than the popular press has done so far. A recent example in this direction is Cedric Villani's research memoir Birth of a Theorem, an unfiltered, day-by-day account of the discoveries for which he was awarded the Fields medal. The gambit on the part of the author is that by giving up the task of explaining the mathematics in elementary terms, it will be possible to create a "fly on the wall" experience for the general reader. On the other hand there is a concern about the responsibility of mathematical non-fiction authors to their content and readers. Within the discipline, Piper Herron's has recently raised serious concerns about a lack of inclusiveness in the mathematics community. Her dissertation offers a more generous approach to communicating mathematical abstractions through writing. Villani and Herron are just two authors whose writing has very recently drawn attention to the larger question of audience. Less recent, but still influential in such debates is "On Proof and Progress in Mathematics" by Bill Thurston. **Philip Ording**

- I found the workshop extraordinarily valuable. Among the themes that had particular salience for me were the role of truth in biography. What happens when a story is retold by an individual so that he or she believes it, but other evidence suggests that it cannot be true? Another striking takeaway was the contrast between how a piece would be read, and what the piece actually said. (Im thinking in particular of how Joe Mazurs draft editorials were read, and the workshop participants sense of how they would be interpreted by the readership that Joe wanted to reach.) In other discussions, I was struck by remarks indicating that a writers assumptions about what a reader will read are very much at odds with readers actual behavior. Apart from the intrinsic interest of the discussions, I personally felt renewed and charged up by the institute. I arrived, I think, a bit weary and even a little burned put, and I came away excited and reinvigorated. **Donal O'Shea**

- I brought to the workshop the seed of an idea for my next biography, about the great mathematical logician Kurt Gdel, who is perhaps best known for his Incompleteness Theorems, stating: For any consistent formal system that can express facts about basic arithmetic: 1) there are true statements that are unprovable within the system; and 2) the systems consistency cannot be proven within the system. My goal is to explore how Gdels often times fantastical and paradoxical works, and their uses and abuses, have permeated the popular imagination and continue to become increasingly relevant as we grapple with the universe and our place therein. I also led the workshop day dedicated to scientific biography, which focused loosely around the notion of fact versus fiction, both specifically in the telling of biographical stories, and generally in translating for a popular audience technical content via all genres, whether non-fiction broadly speaking, fiction, poetry, or playwriting. As a starting point, I drew upon the challenges of writing my newest book, Genius At Play, The Curious Mind of John Horton Conway (previously workshopped at BIRS), published in July 2015 by Bloomsbury. As an epigram for the Conway biography, Is borrowed from the poet Emily Dickinson: Tell all the Truth but tell it slant. For the Gdel project I am employing as a preliminary guide the logician Verena Huber-Dysons observation: There is more to TRUTH than can be caught by PROOF. Mathematical and scientific truth is precise, objective, obstinate. Memory and historical truth is nebulous, subjective, malleable. During this workshop day we addressed the question of how to craft the fodder of fact in order to tell clear and engaging scientific stories that ring true. **Siobhan Roberts**

- At BIRS, I workshopped an article on women in different mathematical subfields, finished a poem, received useful feedback on multiple poems, and participated in the poetry reading. The article has since been submitted to a journal. I very much appreciated the opportunity to connect with a community of mathematical and scientific writers. I'm especially excited to know more scientific poets! **Ursula Whitcher**

- A question I came into the workshop this year was, what firm beliefs have you held that you now question or no longer believe. This question was raised for me by the much publicized study by Princeton economists Anne Case and Angus Deaton showing that middle-aged white Americans are dying in much larger numbers than they used to. Although several of the studys findings were promptly questioned by statisticians, press commentators jumped to interpret these findings according to their own belief systems. The dubious facts fit neatly into ideological narratives, both on the left and right, confirming what the commentators already believed. The participants in the workshop gave illuminating personal examples of their own previously held beliefs that they now doubted or abandoned. This discussion continued in a slightly different form during Siobhan Roberts presentation on writing scientific biography. In writing a biography of the mathematician, John Conway, who was a colorful, but unreliable storyteller, she was faced with the question about what is knowable about a person. If there are gaps in the biographers knowledge, what is permissible to surmise? This discussion led me (at Siobhans suggestion) to read Incompleteness: The Proof and Paradox of Kurt Godel by Rebecca Goldstein. In math, as in theater (and, of course, as in life) there are questions about what is knowable. Currently I am starting work on a play that looks at the same subject from multiple Points of View, (in a literal visual sense as well as a metaphorical sense) and the question arises is there one POV that is more truthful than another. So the discussions within the Creative Writing Workshop at Banff continue to churn in my mind and lead me to new readings and paths of inquiry as I work. As a writer for theater, I am grateful to have the opportunity to be with very smart people who are versed in another language (i.e. mathematics) and willing to act as generous translators for non-mathematicians like me. Of course, this is a two-way street. I believe the mathematicians and science writers have gained some new knowledge from the presentations by me and my collaborator, Ellen Maddow about how mathematical ideas can be made graspable through theatrical means. I have participated in three Creative Writing Workshops for Mathematicians at Banff, and I have come to deeply appreciate the reciprocal nature of the learning that occurs in these sessions.

Paul Zimet

Outcome of the Meeting

The meeting will, we hope, result in the completion and publication of the varied works-in-progress presented here. It will also strengthen the community of writers that this workshop series has been building. As mentioned at the beginning, each workshop (since the first) has brought together a mix of new and veteran participants, a mix that strengthens workshop discussions and helps the community grow. Participants have stayed in touch year-round, commenting on further work, celebrating publications. We also hope BIRS will let us organize another workshop on Creative Writing in Mathematics and Science. You will hear from us again.

Participants

Anand, Madhur (University of Guelph)
Chapman, Robin (University of Wisconsin)
Dawson, Robert (St. Mary's University)
Feaver, Amy (The King's University)
Grosholz, Emily (Penn State University)
Growney, JoAnne (independent)
Grundman, Helen (Bryn Mawr College)
Harriss, Edmund (University of Arkansas)
Ingall, Corrie (College of the Atlantic)
Karaali, Gizem (Pomona College)
Maddow, Ellen (independent)
Mazur, Joseph (Marlboro College)
Merow, Katharine (self)
OShea, Donal (New College of Florida)
Ording, Philip (Sarah Lawrence College)
Roberts, Siobhan (Siobhan Roberts)
Senechal, Marjorie (Smith College)
Whitcher, Ursula (American Mathematical Society)
Zimet, Paul (Talking Band)

Appendix: Examples of creative writing by this year's workshop participants

- **Madhur Anand**
  A New Index for Predicting Catastrophes (poems), McClelland and Steward, 2015.

- **Robin Chapman**
  Abundance, Cider Press, 2009. Winner of Cider Press Review Editors Book Award.
  the eelgrass meadow, Huntington Beach, CA: Tebot Bach Press, 2011. Honorable mention, Posner Poetry Award.
  One Hundred White Pelicans, Huntington Beach, CA: Tebot Bach Press, 2013.
  Six True Things, Huntington Beach, CA: Tebot Bach Press, in press.

- **Florin Diacu**
  The Lost MillenniumHistorys Timetables Under Siege (2nd. ed.), Johns Hopkins University Press, 2011. (Translations: Romanian.)
  MegadisastersThe Science of Predicting the Next Catastrophe, Oxford University Press, Oxford, U.K. and Princeton University Press, Princeton, N.J., 2009/10 (Translations: Japanese, Chinese.)

- **Emily Grosholz**
  "Time and Cosmology," Special issue of Studies in History and Philosophy of Modern Physics, 2015.
  Proportions of the Heart: Poems that Play with Mathematics (poems, with mathematical art by Robert Fathauer), Tessellations Publishing, 2014. "Leibniz, Time, and History," Special issue of Studia Leibnitiana, Band 44 / Heft 1. Franz Steiner Verlag, 2013.

- **JoAnne Growney**
  "Intersections Poetry with Mathematics blog, at http://poetrywithmathematics.blogspot.com
  http://poetrywithmathematics.blogspot.com/2012/06/can-mathematics-maximize-happiness.html
  http://poetrywithmathematics.blogspot.com/2010/03/poetry-of-logical-ideas.html

- **Edmund Harriss**
  With Alex Bellos, Snowflake Seashell Star: Colouring Adventures in Numberland, Cannongate Books, GB, 2015
  American edition: Patterns of the Universe: A Coloring Adventure and Math and Beauty, The Experiment; Act Clr Cs edition, 2015

- **Corrie Ingall**
  "Problem Set," a play. In progress.

- **Ellen Maddow**
  "Delicious Rivers", a play about Penrose tilings, developed in this workshop series. Performed at Smith College and La Mama Etc (NY) in 2007.

- **Joe Mazur**
  Fluke: The Math and Myth of Coincidence, (New York: Basic Books, 2016)
  Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers, Princeton, New Jersey: Princeton University Press, 2014)

- **Katharine Merow**
  "Math and Art Intersect in Man Ray Exhibition." MAA FOCUS, April/May 2015: pp.4-6.
  "A Toast! To Type 15!" Math Horizons, November 2015: pp.10-11.// "Math is my Femme Fatale," The Mathematical Intelligencer, 34, 1, 2012 pp42-43.

- **Philip Ording**
  Variations on a Proof: Exercises in Mathematical Style (working title), Princeton, in progress
  With R. Kossak (co-editor), Simplicity: Ideals of Practice in Mathematics and the Arts, Springer, in progress.
  "Triangle O" Bulletins of the Serving Library 9, 2015
  "A Definite Intuition," Bulletins of the Serving Library 5, 2013

- **Donal O'Shea**
  SRQDaily (srqmagazine.com), Saturday Perspectives Edition, op-eds, second Saturday of month (Serendipity and Requirements (Mar 12, 2016), The Invisible Humanities (Feb 13, 2016), Reflecting on 2015: (Jan 9, 2016), Data Science is Important (Dec 12, 2015), Hiring Faculty (Nov 14, 2015), etc..),
  The Poincare Conjecture: In Search of the Shape of the Universe, Walker, 2007

- **Siobhan Roberts**
  Michael Atiyahs Imaginative State of Mind, Quanta, 3 March 2016
  "Cogito, Ergo Summer, The New Yorker, 27 August 2015
  Genius at Play: The Curious Mind of John Horton Conway, Bloomsbury/Penguin Random House, 2015, 453 pages.
  "Conways Memento Mori, Numberphile/YouTube, 1 July 2015, documentary short film.

- **Marjorie Senechal**
  "The Simplicity Postulate", in R. Kossak and P. Ording, eds, Simplicity: Ideals of Practice in Mathematics and the Arts, Springer, to appear.
  I Died for Beauty: Dorothy Wrinch and the Cultures of Science, Oxford, 2013
  "Figures, Fingers, and Rings", The Huffington Post, May 28, 2013
  "The Map She Carried," OUP Blog, January 11, 2013

- **Paul Zimet**
  "Star Messengers," a play about Galileo at the millennium, performed at Smith College, 2000, and at La Mama Etc in NY, 2001.

The visible achievements of previous BIRS workshops include, in addition to many publications of individual participants:

- a well-attended public reading in Max Bell Hall in June 2006,
- playwright Ellen Maddows math-laced music comedy Delicious Rivers, written in collaboration with Marjorie Senechal, which was performed at La Mama Cafe in New York and at Smith College in 2006,
- the work of 20 past workshop participants showcased in The Shape of Content, an anthology of creative writing in mathematics edited by the three co-organizers of the third workshop. This book generated rave reviews. We are still looking for a publisher for a second anthology (the publisher of the first, Klaus Peters of A. K. Peters, Ltd, retired and, to our sorrow, has died .) Should we find one, we will invite all the participants in all six workshops to contribute material.
- the panel discussion Breaking Barriers: Writers, Scientists, and Mathematicians in Conversation, a joint event of our group and the writers in residence at the Banff Centre, a highlight of the workshop we held in 2010,
- a public reading of our current work at 2013 workshop, which people from the Banff Centre, including some of the writers in residence, attended. This series was videotaped and posted on BIRSs website.
- a more informal (and unrecorded) public reading at this workshop, attended by people from the Banff Centre, including many participants in the concurrent workshop.

Chapter 2

Exploiting New Advances in Mathematics to Improve Calculations in Quantum Molecular Dynamics (16w5006)

January 24 - 29, 2016

Organizer(s): Tucker Carrington (Queen's University), George A. Hagedorn (Virginia Polytechnic Institute and State University)

Introduction and Overview of the Field

The main goal of this workshop was to foster communication between chemists developing mathematical and computational tools for studying the motion of atoms in polyatomic molecules and mathematicians interested in numerical methods for high-dimensional problems and in semi-classical mechanics. Scientists in these two groups benefit from talking with one another, and they have traditionally not done communicated well. Often, the two groups have used different terminology for the same thing; sometimes they have developed very similar ideas in parallel. Development of new methods and ideas is facilitated by exchange and interaction among the groups. Although there is a lot of overlap, scientists in the two groups sometimes lose track of the priorities and interests of members of the other group. For this reason also, interaction is important.

Mathematicians studying quantum molecular problems have often concentrated on diatomic molecules and semi-classical methods, while chemists have been interested in larger molecules and new computational tools. To compute properties of molecules with more than three atoms it is necessary to deal with the "curse of dimensionality" because $3N$ coordinates are required to describe the configuration of N particles. Even if one is just studying the motion of the nuclei, a molecule with 5 atoms requires computations in 15 dimensions. Effective computational techniques exist for studying partial differential equations in up to three dimensions, but computations in higher dimensions are harder and have only recently been the focus of attention from mathematicians. This is perhaps due to the fact that most engineering problems are 3D because we live in a 3D world. It is always possible to eliminate, from the 3N coordinates, three coordinates associated with the motion of the center of mass. When one is interested in total angular momentum equal to zero ($J = 0$) solutions, it also possible to remove three more coordinates reducing the dimensionality to $3N - 6$. One can easily write down the time–dependent and time–independent Schrödinger equations one would like to solve, but they are very hard to solve because of the typically high dimensionality.

At the workshop, mathematicians and chemists presented new methods for solving Schrödinger equations. There is a lot of overlap. Both mathematicians and chemists are developing generalized surface hopping methods for non-adiabatic dynamics; [31, 11] both mathematicians and chemists are developing sparse grid and sparse basis methods for high-dimensional problems; [28, 1, 2, 3, 26, 35, 36, 17] both mathematicians and chemists are developing methods that exploit the advantages of low-rank tensor representations also for high-dimensional problems; [5, 27] both mathematicians and chemists are developing semi-classical methods [18, 19, 20, 38, 25] that exploit the relatively large masses of nuclei (compared to those of electrons). In Banff there was an exchange about developments that have been achieved so far and some cross fertilization. It helps mathematicians to be told what problems are important in chemistry. It helps chemists to be given rigorous results about what is possible. Chemists have a tendency to explore and test ideas without analyzing them in detail. Mathematicians are often able to learn from and then render more precise ideas that chemists propose. Algorithms developed by mathematicians are almost always more general than those developed by chemists. At the Banff meeting both sides searched for ways in which ideas of mathematicians can be applied to chemistry.

In many prior conferences and workshops that brought together mathematicians and chemistry working on molecular dynamics, there have been significant difficulties getting chemists and mathematicians to talk to one another in a meaningful way. Differences in nomenclature, aims, and priorities have often been barriers. As stated above, the main goal of this workshop was to facilitate the interaction between the two groups, and in this regard, the workshop was very successful.

Recent Developments and Open Problems

Techniques for High–Dimensional Problems

Methods based on sparse-grid ideas

Both chemists and mathematicians presented talks about recent developments in sparse-grid ideas. The most obvious way to solve a D dimensional differential equation is to use a method that chemists call variational and mathematicians call Galerkin and a product basis. A product (tensor product) basis is one for which each function is a product of functions of a single variable. Such ideas work well in 3 dimensions (*e.g.*, for a molecule with 3 atoms) but fail in 12 dimensions (*e.g.*, for a molecule with 6 atoms). They fail in 12 dimensions because the size of the product basis scales as n^D where n is a representative number of basis functions per coordinate and D is the number of coordinates. Because n is often about 10, n^D is about 10^{12} when D=12. On present day computers, it is not possible to even store in memory a vector representing the solution in such a basis. In the last 5 to 10 years chemists and mathematicians have begun to use ideas originally suggested by Smolyak to mitigate problems associated with product basis sets. [37, 16, 15, 10, 23, 32, 4, 14, 17, 40, 39] Many of the prominent people in this field attended the workshop. An open problem in this field is coping with the complicated kinetic energy operators that occur when curvilinear coordinates are used. Another is the difficulty of exploiting molecular symmetry with sparse-grid type methods.

Methods based on using low rank tensor

This is a field in which chemists were far ahead of mathematicians, but mathematicians have caught up quickly. Several of the key people working in this field were also at the workshop. The coefficients that represent a D dimensional wavefunction in a product basis form a Dth order tensor. In many cases that tensor can be re-written in terms of tensors of lower order. If, in addition, one can find a way to directly compute the lower order tensors from which one can build the Dth order tensor then one has a computational method that enables one to beat the curse of dimensionality. An open problem in this area is linking iterative eigensolvers and propagation methods to tensor methods in such a way that high precision can be achieved. Another open problem is the re-representation of the potential energy surface (PES). [30, 34, 8, 11, 9, 24, 29] Tensor methods cannot be used with general PESs. Many tensor-type methods work only if the PES is a sum-of-products, what mathematicians call CP format. All methods for generating SOP PESs have important deficiencies.

More Theoretical Issues

Non–adiabatic problems

There are many problems in chemistry for which interactions between different Born-Oppenheimer PESs are important. [18, 25, 38, 21, 6, 7] Both chemists and mathematicians are developing and applying methods to treat such problems. The usual starting point is to assume the nuclear motion on a single PES is determined by classical mechanics and electronic transitions between PESs are determined from the rules of quantum mechanics. It is simply too complicated to use quantum mechanics for the entire problem. The open problem here is improving the approximations that are necessarily introduced in order to patch together quantum and classical mechanics.

Mathematics of Molecular Schrödinger Operators

Molecules have both large mass particles (nuclei) and small mass particles electrons). The usual Born–Oppenheimer approximation employs a semi-classical approximation for the nuclei and an adiabatic approximation for the electrons. Although the paper of Born and Oppenheimer was published in 1927, mathematicians did not begin analyzing the situation until roughly 1980. There has now been substantial mathematical analysis that involves one PES, and there are several papers that deal with several PES's. [18, 25, 38, 22] Many of the authors of those papers attended this workshop.

The simplest non-adiabatic behavior occurs at level crossings, where two PES's intersect. Generically, the most interesting level crossings are "conical intersections," which require PES's that depend on at least two nuclear coordinates. [18, 12, 38] They do not occur in diatomic molecules. Another closely related situation is an "avoided crossing," in which two PES's approach very close to one another, but do not actually intersect. There are now a few mathematical papers that deal with that situation. [19, 20, 21, 6, 7]

The technique of "surface hopping" is frequently used in chemistry to study these situations, but there are no mathematically rigorous papers justifying these approximations in general. [33, 31, 11] That is an area which is begging for some mathematical analysis to be done. One would hope that workshops like this one would stimulate mathematical work in such a direction.

In general, the workshop was effective at bringing mathematicians and chemists into discussion. The long coffee breaks were key. One small illustration of the nature of the problem. Two of the participants, Caroline Lasser (mathematician) and Christoph Scheurer (chemist), had never met. This would not be unusual except for the fact that they are from the same university! At the conference they had time to learn about each other's (related) research. In another case a mathematician, George Hagedorn directed a chemist, Daniel Paláez–Ruiz, to a mathematician, Stephan de Bièvre, at Paláez–Ruiz's university.

Presentation Highlights

Several of the talks concentrated on "tensor methods" for trying to deal with the "curse of dimensionality." Those speakers included Arnaud Leclerc, Ove Christiansen, Mike Espig, Uwe Manthe, Ivan Oseledets, Reinhold Schneider, Phillip Thomas, and Edward Valeev. Other talks had a different approach that relied on using sparse grids for computations. Those included Gustavo Avila, David Lauvergnat, and Christoph Scheurer. Yet another approach using wavelets was described by Helmut Harbrecht. Other new methods for solving the time-independent Schrdinger equation were presented by James Brown, Attila Czaszar, Peter Felker, and George Hagedorn. Uwe Manthe talked about the impossibility of factorizing wavefunctions for CH_5^+.

Several speakers talked about various methods for solving time-dependent Schrödinger problems. Those included Volker Betz, Victor Batista, Christian Lubich, Vasile Gradinaru, Gabriel Hanna, Ray Kapral, David Sattlegger, Stefan Teufel, and Robert Wodraszka. Other talks dealt with closely related topics involving nuclear motion.

There were presentations on many other topics. For example, Michael Griebel and Daniel Palaez–Ruiz talked about efficient representations of electron energy surfaces. Alain Joye and Stephanie Troppmann made presentations related to open quantum systems in which a small system (such as a molecule) interacts with some very large system that is regarded as a reservoir. The talks of Ramond Kapral and Gabriel Hanna also dealt with small systems coupled to large systems.

Tomoki Ohsawa gave a presentation that concentrated on a geometric approach to quantum mechanics. Bill Poirier described a new approach to quantum mechanics. Raymond Kapral talked about surface hopping techniques that were also mentioned in several other talks.

Tucker Carrington, Caroline Lasser, and George Hagedorn talked about specific molecules and used examples to illustrate general techniques.

Johannes Keller talked about "spectrograms," which are new objects for the study of semi-classical quantum mechanics in phase space. David Tannor and James Brown also talked about phase space methods, and Jiri Vanicek talked about molecules in external electromagnetic fields. Pierre-Nicholas Roy presented a talk about path integrals.

Scientific Progress Made

As indicated in the introduction, the main goal of this workshop was the sharing of information between mathematicians and chemists. Many participants commented that this meeting was more successful than earlier such meetings that brought the two groups of researchers together. The two groups often have closely related goals, but they publish their results in different journals and often one group has no idea of progress made by the other group. The speakers from each group made an effort to address the other group, which often does not happen in such meetings. Also, some speakers welcomed the opportunity to present their work to audiences they had never addressed before.

Both groups gained insight into activities and scientific difficulties encountered by the other group. Quite clearly, the "curse of dimensionality" is of primary concern to people doing calculations, while semi-classical methods are of primary concern to people doing theoretical analysis. Semi-classical methods are appropriate for the dynamics of nuclei because of their large masses (compared to electrons). They can also be useful for large molecules where a full quantum treatment may not be feasible. Also, in problems involving non-adiabatic behavior, one often would like to deal with the nuclear motion easily, since the main emphasis is on understanding what the electrons are doing.

Fundamental Open Questions

Mathematical semi-classical analysis is based on either the assumption that $\hbar$ is small or that the ratio of masses is small. In the former case semi-classical approximations work well, and in the latter cases adiabatic approximations are appropriate. Born–Oppenheimer type approximations use both of these approximations, and most of the theory of molecular quantum mechanics is built on the ability to sequentially solve electronic and nuclear Schroedigner equations. Mathematicians are working on ways to improve and correct these approximations. It is important to know whether the corrections enable one to achieve the sort of accuracy desired by chemists. The alternative is to use numerical methods. We need more tests to determine whehter savings can be achieved with semi-classical and improved adiabatic approximations, without sacrifing accuracy.

The use of tensor methods in calculations will benefit from more theoretical analysis. We need better rank reduction methods. In the application of Smolyak methods to problems of chemical interest, more work is needed to develop ideas to exploit symmetry. It would also be beneficial to develop good black–box methods that can be used by non–experts.

Outcome of the Meeting

The two groups of researchers left the workshop with a better appreciations of the issues dealt with by the other group. Mathematicians are beginning to make important contributions to the development of methods for solving the high dimensional problems that are ubiquitous in chemistry.

Participants

Avila, Gustavo (Queen's University)
Batista, Victor (Yale University)
Betz, Volker (TU Darmstadt)
Brown, James (Queen's University)
Carrington, Tucker (Queen's University)
Christiansen, Ove (Aarhus University)
Csaszar, Attila (Eotvos Lorand University)
Espig, Mike (RWTH Aachen University)
Felker, Peter (UCLA)
Gamble, Stephanie (Virginia Tech)
Gradinaru, Vasile Catrinel (ETH Zurich)
Griebel, Michael (Universitaet Bonn)
Hagedorn, George (Virginia Tech)
Hanna, Gabriel (University of Alberta)
Harbrecht, Helmut (University of Basel)
Joye, Alain (Institut Fourier, Universit Grenoble 1)
Kapral, Raymond (University of Toronto)
Keller, Johannes (Technische UniversitŁt Mnchen)
Lasser, Caroline (Technische Universitat Munchen)
Lauvergnat, David (University of Paris-Sud)
Leclerc, Arnaud (Universit de Lorraine)

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Manthe, Uwe (Bielefeld University)
Ohsawa, Tomoki (University of Texas at Dallas)
Oseledets, Ivan (Skolkovo Institute of Science and Technology)
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Poirier, Bill (Texas Tech University)
Roy, Pierre-Nicholas (University of Waterloo)
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Scheurer, Christoph (Technische UniversitŁt Mnchen)
Schneider, Reinhold (Technische Universitat Berlin (Germany))
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Wodraszka, Robert (Queen's University)

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Chapter 3

Mathematical and Statistical Challenges in Neuroimaging Data Analysis (16w5036)

January 31 - February 5, 2016

Organizer(s): Brian Caffo (Johns Hopkins University), Linglong Kong (University of Alberta), Farouk Nathoo (University of Victoria), Todd Ogden (Columbia University), Hongtu Zhu (University of North Carolina at Chapel Hill)

Overview of the Field

Nowadays there is great need for mathematical and statistical theory and methods to analyze high dimensional, correlated, and complex neuroimaging data and clinical and genetic data obtained from various cross-sectional and clustered neuroimaging studies. However, the development of such methods for analyzing imaging data itself and integrating imaging data with genetic and clinical data has fallen seriously behind the technological advances on genomics and neuroimaging. To meet this critical and important need and challenges, the main objectives of the proposed workshop are to serve as a platform for bringing the leading figures from different disciplines including statistics, mathematics, computer science, biomedical engineering, and neuroscience, among other related sciences, exchanging new research ideas, and training the next-generation mathematicians and statisticians in the field of neuroimaging data analysis (NDA).

Recent Developments and Open Problems

Image reconstruction, segmentation and registration

Image reconstruction is to use certain iterative algorithms to reconstruct 2D and 3D images in certain imaging techniques. This is a common issue to all structural, neurochemical, and functional images. Mathematical and statistical methods have been widely used to address various technical issues arising from imaging reconstruction. Such methods include functional analysis, sparse methods, inverse problem, Fourier analysis, nonparametric methods, time series, bootstrap, and regression analysis, among many others.

Image segmentation is the process of assigning a label to every location in an image such that locations with the same label share certain visual characteristics. Image segmentation is typically used to locate important features including objects and boundaries (lines, curves, etc.) in images, which leads to a simple representation that is more meaningful and easier to analyze. Advanced segmentation methods are primarily based on mathematical and statistical methods, such as clustering methods, geometry, partial differential equation-based methods, and pattern theory, among others.

Image registration is the process of transforming different sets of image data into a common coordinate system, which is also called template. In NDA, image data may come from different modalities, from different times, from different viewpoints due to head motion, or from different subjects. Registration is necessary in order to be able to compare or integrate the image data obtained from these different measurements. Mathematical and statistical methods, such as functional analysis, geometry, and nonparametric methods, have been the foundation of developing various imaging registration algorithms. As results can vary somewhat depend on the specific template chosen, an important open problem is the development of methods for template choice, or methods to accommodate the uncertainty associated with this choice.

Statistical group analysis and Shape analysis Statistical group analysis is the process of analyzing a sample of images across different groups in an effort to make population level inference. For example, the groups may consist of controls and patients with a specific disorder. Imaging meta analysis, which involves combining the results from multiple group studies is a related area where new techniques for synthesizing results require development. The results of group and meta analysis can be used to inform the development of classification rules based on imaging markers for disease diagnostics and prediction. Advanced statistical methods have played a critical role in addressing various issues in statistical group analysis. Such methods include Bayesian analysis, random effects models, multiple comparison methods, meta analysis, classification methods, sparse methods, nonparametric methods, functional data analysis time series, bootstrap, and regression analysis, among many others. As the data obtained from only a single subject can itself be high-dimensional, a major challenge associated with group studies involves integrating ultra-high dimensional data with complex behavioral measures.

Shape analysis involves the representation, analysis, and processing of geometric shapes extracted from medical images across different subjects or groups. Some of the important aspects of shape analysis are to build boundary representations for a shape, to obtain a measure of distance between shapes, to estimate average shapes from a (possibly random) sample and to estimate shape variability in a sample. Shape analysis requires advanced mathematical and statistical methods including geometry, functional analysis, harmonics analysis, and parametric and nonparametric statistics.

Connectivity analysis and Multimodal analysis Connectivity analysis is to establish a pattern of anatomical links ("anatomical connectivity"), of statistical dependencies ("functional connectivity") or of causal interactions ("effective connectivity") between distinct units within a nervous system. The units correspond to individual neurons, neuronal populations, or anatomically segregated brain regions. Important mathematical and statistical methods for connectivity analysis include graph theory, social network analysis, multivariate analysis, exponential family, Markov chain Monte Carlo methods, neural network, and time series, among others.

Multimodal analysis is to develop systematic approaches for fusing image data across multiple imaging modalities, in order to find any patterns of related change that may be present. Given that any single imaging modality will only provide a partial snapshot of the true underlying neural activity, multimodal neuroimaging studies can yield a more complete picture; however, devising models that can effectively combine information from different modalities (e.g. scalp level electroencephalogram (EEG) and brain level functional Magnetic Resonance Imaging (fMRI)) is a non-trivial task. Important mathematical and statistical methods for multimodal analysis include measurement error models, multivariate analysis, neural network, and time series, among others.

Imaging genetics Imaging genetics involves the collection and analysis of a wealthy set of imaging, genetic and clinical data in order to detect susceptibility genes for complex inherited diseases including common mental disorders (e.g., schizophrenia and bipolar disorder) and neurodegenerative disorders. Understanding genetic mechanisms of inheritable mental and neurological disorders, such as autism and schizophrenia, is an important step in the development of urgently needed approaches to prevention, diagnosis, and treatment of these complex disorders. Currently, imaging provides the most effective measures of brain structure and function, and hence imaging data may serve as important endotraits that ultimately can lead to discoveries of genes for these complex disorders. Although there exist few such methods in the statistical literature, the development of tools for analyzing imaging genetics data will require advanced mathematical and statistical methods including sparse methods, multivariate analysis, regression models, and nonparametric analysis, among many others. While many tools for high-dimensional data analysis have been developed for genetic data with scalar valued phenotypes, imaging-based phenotypes are far more complex given their dimension and the inherent spatial correlation that exists in the observations comprising each image. The development of spatial models for imaging genomics is thus an open area of investigation, but one that poses significant challenges for model development, computation, and inference involving multiple testing on a massive scale.

Presentation Highlights

Day 1: February 1, 2016

The morning section I on February 1 featured two presentations about connectivity analysis delivered by Ying Guo from Emory University and Hernando Ombao from University of California, Irvine. Ying Guo's presentation was on "Exploring the brain connectivity: questions, challenges and recent findings" [1]. Brain connectivity analysis based on functional neuroimaging data has drawn significant interest in recent years. A wide range of network modelling tools have been developed for this purpose. The most commonly used methods includes full correlation, partial correlation and Bayes nets. Ying's talk presented some interesting findings in brain functional connectivity and structural connectivity using resting-state fMRI and diffusion MRI. She proposed a measure of the strength of structural connectivity underlying the functional connectivity networks estimated by independent component analysis (ICA) [2]. Hernando's presentation was on "Multi-Scale Factor Analysis of High Dimensional Time Series". Hernando introduced a multi-scale factor analysis model for EEG data [3]. By applying the algorithm he proposed to a single-subject EEG data, he found that even a small number of factors like 3 was able to capture most of the variation within each region. The connectivity between channels/voxels in the same region is generally higher than between channels/voxels from different regions, and Ying also mentioned same finding during her talk.

Jian Kang from University of Michigan and Moo K. Chung from University of Wisconsin-Madison were the presenters in the morning session II on February1. Jian's presentation was on "Posterior Mean Screening for Scalar-on-Image Regression". Neuroimaging data can be used to classify a subject's disease status or predict clinical response or behaviour. There have been many variable selection methods proposed in high-dimensional feature space, and Jian proposed a new approach called posterior mean screening by using the marginal posterior mean of regression coefficients as the screening statistic. Moo's presentation was on "Learning Large-Scale Brain Networks for Twin fMRI" [4]. In many human brain network studies, the number of voxels (p) is usually significantly larger than the number of images/participants (n). Sparse network models are usually used to fix the small n large p problem, however the computational challenge brought by optimizing L1-penalties makes it not practical to learn large-scale brain networks using sparse network models. Moo proposed a model to build sparse brain networks at the voxel level, and the minimization problem can be simply done algebraically instead of using an iterative algorithm. The computational speed gain by doing that makes it possible to use different sparse parameter instead of using a single sparse parameter, which may not be enough or optimal. The method was applied to twin fMRI data to determine the extent of heritability on functional brain networks a the voxel-level for the first time.

In the afternoon session I on February 1, Joerg Polzehl from Weiertrass Institute for Applied Analysis and Stochastics presented "Modeling high resolution MRI: Statistical issues" [5]. Most MRI data has already been through many preprocessing steps before the statistical analysis. A number of new methods have been proposed to increase spatial resolution and reduce acquisition time for MRI, such as multiple receiver coils and subsampling in K-space. However, those more complicated acquisition methods may further diminish the signal-to-noise ratio (SNR), change in the signal distribution and induce spatial correlation. By analyzing the data generating process and the resulting imaging data distribution, Joerg elaborated the effects of typical data preprocessing and the bias effects related to low SNR for the example of the diffusion tensor model in diffusion MRI. Bei Jiang from University of Alberta discussed "Modeling Placebo Response using EEG data through a Hierarchical Reduced Rank Model" [6]. There have been evidence showing that there are individual differences among depression patients on EEG, fMRI and other brain image measurements. The placebo response is a positive medical response due to placebo effect, as if there were an active medication. And it's highly prevalent among antidepressant treatment. By using EEG measurements as a matrix predictor, Bei presented a hierarchical latent class model to differentiate potential placebo responders from non-responders. Given the high dimensionality of the EEG measurements, a reduced rank regression model with a data-driven regularization was used. The application to real data of 96 placebo or drug treated depression patients showed that this model can be used to detect the placebo response and further guide the selection of effective treatment for depression patients in clinical practise.

The afternoon section II on February 1 featured two presentations delivered by Daniel Rowe from Marquette University and Stephen Strother from Baycrest/University of Toronto. Daniel Rowe's presentation was on "Statistical Analysis of Image Reconstructed Fully-Sampled and Sub-Sampled fMRI Data". In order to accelerate the image acquisition process, methods have been proposed by measuring less k-space data and performing image reconstruction via an estimation of missing data using other image information. Daniel's talk reviewed the measurement and reconstruction of fully-sampled and sub-sampled data in addition to their resulting statistical properties. Daniel presented that the commonly used image reconstruction sensitivity encoding (SENSE) induces long-range through-plane and in-plane correlation [7]. And by showing the potential bias and change brought by the image reconstruction, he suggested that special care needs to be taken when we obtained the preprocessed data and develop models that incorporate processing. Stephen Strother's presentation was on "Metrics for evaluating functional neuroimaging processing pipelines". The typical neuroimaging processing pipelines includes subject selection, experimental design, data acquisition, preprocessing, data analysis and pipeline processing efficacy measuring. Stephen's talk discussed the range of quantitative metrics used in the literature for evaluating the performance of functional neuroimaging processing pipelines [8]. For the preprocessing pipeline, using fixed preprocessing choices across all subjects/sessions is non-optimal and produces a conservative result with reduced SNR and detection power. Instead adapting preprocessing on a subject/session using cross-validation resampling can significantly improve pipeline performance. Also the negative effects of these common pipeline choices are likely to become worse with age and disease. For the processing pipeline, Stephen pointed out small changes within a processing pipeline may lead to large changes in the output, and the results related to human brain function may be obscured by poor or limited choices in the processing pipeline particularly as a function of age and disease.

Vikas Singh from University of Wisconsin-Madison and Jie Peng from University of California, Davis were the two speakers for the afternoon session III on February 1. Vikas Singh's talk was on "A Multi-Resolution Scheme for Analysis of Brain Connectivity Networks". Vikas presented multi-resolution analysis of shapes and connectivity networks since the multi-resolution methods are sensitive to small changes in the networks [9]. By using wavelet transform on graphs, he applied the method to cortical thickness discrimination and brain connectivity discrimination. Jie Peng's talk was on "Fiber orientation distribution function estimation by spherical needlets". Diffusion MRI (D-MRI) are widely used to reconstruct white matter fiber tracts and to provide information on structure connectivity of the brain. Fiber orientation distribution (FOD) function is a spherical probability density function (p.d.f.) that characterizes the fiber distribution at each voxel of the brain white matter. Jie discussed the estimation of FOD based on a spherical needlets representation. The proposed method leads to much better peak localization compared with existing methods based on spherical harmonics representation, particularly when the separation angles among fiber bundles are small.

Day 2: February 2, 2016

The morning session I on February 2 featured two presentations on imaging genetics using Bayesian model delivered by Farouk Nathoo from University of Victoria and Michele Guindani from University of Texas M.D. Anderson Cancer Center. Farouk Nathoo's presentation was on "A Bayesian Group-Sparse Multi-Task Regression Model for Imaging Genomics" [10]. Imaging genetics is concerned with finding associations between genetic variations and neuroimaging measures as quantitative traits. Statistically, a multivariate regression analysis can be applied by using potentially interlinked brain imaging phenotypes as response vector and the high-throughput single nucleotide polymorphism (SNP) as covariates. Farouk presented a Bayesian approach based on a continuous shrinkage prior that encourages sparsity and induces dependence in the regression coefficients corresponding to SNPs within the same gene, and cross different components of the imaging phenotypes. The proposed model allows for full posterior inference for the regression parameters using Gibbs sampling. Michele Guindani's presentation was on "Integrative Bayesian Modeling Approaches to Imaging Genetics" [11]. The data used in Michele's presentation has two subgroups, healthy controls and schizophrenic patients. By using the fMRI data and genetic covariates (SNPs implicated in schizophrenia) of all subjects, the goal is to identify brain regions with discriminating activation patterns and SNPs relevant to explain such activations in either (or both) subgroups. A hierarchical mixture model with selection of discriminating features was proposed with 2 components each describing activations in control and case groups. An alternative predictive model for disease status that takes into account direct associations between the SNPs/ROIs information and the disease status, as well as the indirect associations captured by a ROI-SNPs network was also proposed.

Bin Nan from University of Michigan and Jaroslaw Harezlak from Indiana University were the two speakers for the morning session II on February 2. Bin Nan's presentation was on ”Tuning parameter selection for voxel-wise brain connectivity estimation via low dimensional submatrices”. The tuning parameter selection accounts for the major computing cost in estimating the voxel-wise brain connectivity. Bin presented a tuning parameter selection procedure using Gap-block cross-validation via low dimensional submatrices. Jaraslaw's presentation was on ”Assessing uncertainty in dynamic functional connectivity estimation”. Traditional functional connectivity analysis typically assumes that functional networks are static in time, and dynamic functional connectivity analysis tries to analyze the functional network over time. One intuitive and straightforward method is the sliding window technique which performed by conducting analysis on a set number of scans in an fMRI session. This nonparametric approach is easy to implement, however it may also be problematic and not adequate to capture the true dynamic change of the functional network. Jaraslaw presented an algorithm based on multivariate linear process bootstrap, which allows for resample multivariate time series data. A model-free estimation of confidence intervals for the dynamically changing correlation coefficient estimate was also introduced.

Jingwen Zhang from University of North Carolina at Chapel Hill, Zhengwu Zhang from Statistical and Applied Mathematical Sciences Institute and Wei Tu from University of Alberta were the presenters in the afternoon session I on February 2. Jingwen's presentation was on ”HPRM: Hierarchical Principal Regression Model of Diffusion Tensor Bundle Statistics”. In a diffusion tensor imaging (DTI) study, diffusion properties are observed among multiple fiber bundles to understand the association between neurodevelopment and clinical variables, such as age, gender, biomarkers of subjects. Jingwen proposed Hierarchical Principal Regression Model (HPRM) on functional data to efficiently conduct joint analysis of multiple diffusion tensor tracts on both global level and individual level. The proposed model was applied to genome-wide association study on one-month-old twins to explore important genetic variants related to early human brain development. Zhengwu's presentation was on ”Robust brain structural connectivity analysis using HCP data”. One of main challenges in structural connectivity analysis is to extract precise and robust connectivity networks from the brain. Zhengwu presented a processing pipeline to reliably construct structural connectivity from the dMRI, including streamline extraction, adaptive streamline compression and robust connectivity matrix construction. Wei Tu's presentation was on ”Non-local Fuzzy C-Means Clustering with Application to Automatic Brain Hematoma Edema Segmentation using CT”. It is critical to efficiently and accurately segment the hematoma and edema region from computed tomography (CT) scans of patients with intracerebral hemorrhage. However, due to the substantial overlap between the edema and surrounding brain tissue and image artifacts, an accurate and automatic segmentation has been very challenging. Wei presented a two phase clustering algorithm by combing the fuzzy C-Means clustering and non-local smoothing. The first step applied the fuzzy clustering algorithm on the whole brain to find the hematoma tissue, which is also the region of interest (ROI). The second phase will apply the clustering algorithm on the ROI to obtain a more detailed segmentation of edema tissue.

The afternoon session II on February 2 featured three presentations delivered by Benjamin Risk from Statistical and Applied Mathematical Sciences Institute and University of North Carolina, John Muschelli from Johns Hopkins University and Chao Huang from University of North Carolina, Chapel Hill. Benjamin Risk's presentation was on ”Large covariance estimation for spatial functional data with an application to twin studies”. A structural estimation model (SEM) can be used to estimate a trait's heritability, and a mass univariate analysis can estimate an SEM at each location in the brain. Extending the model to spatial domains requires an estimation of the covariance functions. Benjamin presented a spatial function SEM using functional principal component analysis (PCA). The proposed model was applied to the imaging data of twin pairs from Human Connectome Project (HCP). John Muschelli's presentation was on ”Processing Neuroimaging Data in R: Capabilities”. R language is the most frequently used programming language by statisticians, and there have been many different packages/softwares created for neuroimaging data analysis. During the presentation, John discussed the neuroimaging processing pipeline using R, from read/write images, visualization, bias field correction, skull stripping, image registration, tissue-class segmentation and more complex modeling [12]. Chao Huang's talk was on ”FFGWAS: Fast Functional Genome Wide Association Study of Surface-based Imaging Genetic Data” [13]. More and more large-scale imaging genetic studies are being widely conducted to collect a rich set of imaging, genetic, and clinical data to detect putative genes for complexly inherited neuropsychiatric and neurodegenerative disorders. Several major big-data challenges arise from testing millions of genome-wide associations with functional signals sampled at millions of locations in the brain from thousands of subjects. Chao presented a Fast Functional Genome Wide Association Study (FFGWAS) framework to carry out whole-genome analyses of multimodal imaging data. FFG-

WAS consists of three components including (1) a multivariate varying coefficient model for modeling the relation between multiple functional imaging responses and a set of covariates (both genetic and non-genetic predictors), (2) a global sure independence screening (GSIS) procedure for reducing the dimension from a very large scale to a moderate scale, and (3) a detection procedure for detect significant cluster-locus pairs. The proposed FFG-WAS was applied to large-scale imaging genetic data analysis of ADNI data with 708 subjects, 30,000 vertices on hippocampal surface, and 501,584 SNPs.

The afternoon session III on February 2 was roundtable discussion lead by John Aston from Cambridge University, Martin Lindquist from Johns Hopkins University, Hernando Ombao from University of California, Irvine, Joerg Polzehl from Weiertrass Institute for Applied Analysis and Stochastics and Hongtu Zhu from University of North Carolina at Chapel Hill. Discussion topics included the technical challenges in NDA, grant opportunities, grant review criterion, software development of proposed methodologies (Neuroconduct), the training of next generation statisticians.

Day 3: February 3, 2016

The morning session I of February 3 featured two presentations on functional data analysis. John Aston from Cambridge University presented ”Functional Data, Covariances and FPCA of Brain Data”. Functional PCA (FPCA) tries to investigate the dominant modes of variation of functional data, such as fMRI, EEG. John introduced a few different approaches to estimate network connectivity by using functional data analysis. Time changing connectivity via functional change point detection and also spatially constrained connectivity based on the used of penalized functional principal components were presented. The FPCA can be defined on the volume or on the surface, and it can be used to detect general mean shifts in image data and potentially connectivity changes. Jeffrey Morris from University of Texas M.D. Anderson Cancer Center presented ”Spatial Functional Models for Event-Related Potential Data, with Application to Smoking Cessation Study ”. Jeffrey presented a set of functional regression methods to analyze spatially correlated complex functional data such as functional imaging data [14]. Three major strategies for spatial or temporally correlated functional data are presented, 1. Functional spatial or Functional temporal processes, 2. Tensor basis functions, 3. Functional graphical models. And each method has its own benefits and drawbacks and the suitability depends on the data setting and research questions. All these three methods were applied to a smoking cessation study to assess neurological response to different types of visual stimuli.

Brain Hobbs and Jianhua Hu from University of Texas M.D. Anderson Cancer Center were the speakers for the morning session II of February 3. Brain Hobbs's presentation was on ”Recent advances in cancer imaging”. In many cancer imaging settings, radiologists often identify the presence of solid tumors over a series of a few repeated scans, and often multiple interdependent ROIs are evaluated in isolation. Independent estimation appears limiting for analysis of sparse functional data derived from dynamic imaging techniques that use physiological models to derive multiple interdependent biomarkers acquired from multiple regions of interests (ROI) within the same organ. Brain proposed statistical methods for joint estimation of sparse spatiotemporally correlated imaging-biomarkers using semi-parametric models. Joint prediction is used to identify liver metastases using perfusion characteristics from multiple ROIs acquired using dynamic computed tomography [15]. Jianhua's talk was on ”Analysis of spatially correlated functional data in tissue perfusion imaging”. Measurements from perfusion imaging modalities provide physiological correlates for neovascularization induced by tumor angiogenesis. Such measurements are often generated repeatedly over time and at multiple spatially interdependent units. To reduce model complexity and simplify the resulting inference, possible spatial correlation among neighboring units is often neglected. Jianhua presented a weighted kernel smoothing estimate of the mean function that leverages the spatial and temporal correlation, particularly, in the presence of sparse observations.

Day 4: February 4, 2016

In the morning session I on February 4, Leixi Li from University of California, Berkeley presented ”Estimation and Inference for Brain Connectivity Analysis” [16]. Previous studies have demonstrated that brain networks may degrade among Alzheimer's disease (AD) subjects compared to normal aging subjects. Amyloid beta (Aβ) is a form of protein that is toxic to neurons in the brain, and it accumulates outside neurons and forms sticky buildup called Aβ plaques. To understand how Aβ deposition are related to brain connectivity patterns in cognitively normal elder subjects, Leixi proposed two general framework to tackle the problem. First a comparison of the connectivity networks between the Aβ positive group and Aβ negative group. Second by taking the connectivity network as a predictor, the association between the connectivity network and the Aβ deposition can be modeled. Leixi introduced the symmetric tensor predictor regression model to model the association. Shuo Chen from University of Maryland presented ”Brain Connectivity Biomarkers”. Many challenges remain for group-level whole-brain connectivity network analyses because the massive connectomics connectivity metrics are correlated and the correlation structure is constrained by the extraordinarily complex, yet highly organized, topology of the underlying neural architecture. Shuo presented several novel machine learning algorithms to automatically detect topological structures, and furthermore construct network ”object” oriented statistical inference framework to identify subgraphs as network level biomarkers. Each network biomarker comprises a set of nodes (brain regions) and edges (connectivity metrics), and more importantly the network biomarker is a subgraph with organized topological structures (e.g. clique or multipartite graph).

In the morning session II on February 4, Xiao Wang from Purdue University presented ”Optimal Estimation for Quantile Regression with Functional Response”. Quantile regression is able to give a full picture of the data by estimating the $100\tau\%$ quantile of the conditional distribution of response Y given X. Quantile regression gives better estimators than mean regression when data are skewed or contain outliers since the appealing robust properties of quantiles, and it also does not require any error distribution. Quantile regression with functional response and scalar covariates has become an important statistical tool for many neuroimaging studies since the variances of errors are varying spatially within the brain. Xiao presented the optimal estimation of varying coefficient functions in the framework of reproducing kernel Hilbert space. Minimax rates of convergence under both fixed and random designs are established. An easily implementable estimator was also presented using alternating direction method of multipliers (ADMM) algorithm. Yimei Li from St. Jude Children's Research Hospital presented ”SGPP: Spatial Gaussian Predictive Process Models for Neuroimaging Data” [17]. Yimei presented a spatial Gaussian predictive process (SGPP) model to predict new neuroimaging data by using a set of covariates like age, diagnostic status and existing neuroimaging data set. The SGPP model Yimei presented uses a functional PCA model to capture global dependence, and a multivariate simultaneous autoregressive model to capture local spatial dependence as well as cross-correlations of different imaging modalities. A three-stage estimation procedure was proposed to simultaneously estimate varying regression coefficients across voxels at the global and local spatial dependence structures.

The afternoon session I of February 4 featured three presentations delivered by Marina Vannucci from Rice Universtiy, Todd Ogden from Columbia University and Anuj Srivastiava from Florida State University. Marina Vannucci's presentation was on ”A Bayesian Modeling Approach of Multiple-Subject fMRI Data”. Marina presented a Bayesian nonparametric regression model for multiple-subject fMRI data [18]. The model incorporates information on both the spatial and temporal correlation structures of the data, and allows for voxel-dependent and subject-specific parameters. It provided a joint analytical framework that allows the detection of regions of the brain that activate in response to a stimulus, while simultaneously taking into account the association, or clustering, of spatially remote voxels within and across subjects. In order to solve the computational challenge brought by the high dimensionality of the data and the large amount of parameters to be estimated, Marina presented a variational Bayes algorithm as an approximate computational technique, and its efficiency was compared to a full Monte Carlo Markov Chain (MCMC) algorithm. Todd Ogden's presentation was on ”Functional and imaging data in precision medicine” . A major goal of precision medicine is to use information gathered at the time that a patient presents for treatment to help clinicians determine, separately for each patient the particular treatment that provides the best-expected outcome. Imaging data may also be used in making patient-specific treatment decisions. Todd introduced an ongoing multi-site randomized placebo-controlled clinical trial Establishing Moderators and Biosignatures of Antidepressant Response for Clinical Care (EMBARC). The primary goals of EMBARC is selecting measurements that can be made at baseline that will help predict patient response to treatment, and therefore determine a rule, based on these measurements, that will assign the treatment that is best for each patient. Todd presented the general problem of using both scalar and functional data to guide patient- specific treatment decisions and describe some approaches that can be used to perform model fitting and variable selection [19]. Anuj Srivastiava's presentation was on ”Elastic Functional Data Analysis for Modeling Shapes of Anatomical Structures” [**?**]. A variety of anatomical structures in human brain can be represented as functions (curves or surfaces) on intervals or spheres. Morphological analysis and statistical modeling of such data faces the following challenges: the representation spaces are curved, the data is seldom registered, the classical Hilbert structure is problematic, and (nowadays) there is a tremendous amount of data to deal with. Elastic functional data analysis provides a unified framework for dealing with nonlinear geometries and simultaneous registration of function data, and leads to efficient computer algorithms. It has proven to outperform all recent methods in registering functional data. The Functional PCA, resulting from linearized representations under elastic Riemannian metrics, has been used for solving regression and testing under appropriate models. Anuj presented some recent extensions of this work involving morphological analysis of tree-like structures such as neurons.

In the afternoon session II of February 4, Wei Pan presented ”Testing for group differences in brain functional connectivity” [21]. There have been evidence showing that that altered brain functional networks are associated with neurological illnesses such as Alzheimer’s disease. Exploring brain networks of clinical populations compared to those of controls would be a key inquiry to reveal underlying neurological processes related to such illnesses. Standard approaches for comparing networks includes mass-univariate test and deriving some network summary statistics, like clustering coefficient. Mass-univariate tests can be low powered for multiple weak signals since the dimensionality of networks is usually high, and deriving network summary statistics is not easy and oversimplified. Wei proposed a global test. The proposed tests combine statistical evidence against a null hypothesis from multiple sources across a range of plausible tuning parameter values reflecting uncertainty with the unknown truth. The proposed tests are not only easy to use, but also highly powered robustly across various scenarios. The usage and advantages of these novel tests are demonstrated on an Alzheimer’s disease dataset and simulated data. Russel Shiohara from University of Pennsylvania presented ”Two-Sample Tests for Connectomes using Distance Statistics”. Russel proposed statistical methods for quantifying variability in a population of connectomes using general representations. The methods used generalized variances for complex objects based on distance statistics. Methods were developed for two-sample testing at the whole connectome and the subnetwork levels and the asymptotic properties of the test statistics were studied. These methods was applied in a connectomic study of autism spectrum disorders using DTI.

Tingting Zhang from University of Virginia and Martin Linquist from Johns Hopkins University were the two presenters of afternoon session III of February 5. Tingting Zhang’s presentation was on ”Bayesian Inference for High-Dimensional ODE Models with Applications to Brain Connectivity Studies”. Tingting proposed a widely applicable high-dimensional ordinary differential equations (ODE) model to explore connectivity among multiple small brain regions [22]. The new model, called the modular and indicator-based dynamic directional model (MIDDM), uses indicators to represent significant directional interactions among brain regions and features a cluster structure, which consists of modules of densely connected brain regions. A Bayesian hierarchical model was developed to make inferences about the MIDDM and also to provide a new statistical approach to quantify ODE model uncertainty that arises from the inherent inadequacy of the ODE model for a complex system. The proposed Bayesian framework to an auditory electrocorticography dataset to identify significant clusters and directional effects among different brain regions. Martin Linquist’s presentation was on ”Dynamic Connectivity: Pitfalls and Promises”. To date, most resting state fMRI studies have assumed that the functional connectivity between distinct brain regions is constant across time. However, recently, there has been increased interest in quantifying possible dynamic changes in FC during fMRI experiments, as it is thought this may provide insight into the fundamental workings of brain networks. Martin proposed a dynamic conditional correlations (DCC) model to quantify the dynamic change of brain connectivity [23]. DCC is a multivariate GARCH model. The study of dynamic correlations actually increases the number of data points, so there is critical need to use summary statistics that can be used to find meaningful individual differences. The average dynamic correlation and the variability in dynamic correlation in each stage can be used. Also, one can find some connectivity ”state” matrices, which are connectivity patterns that subjects tend to return to during the course of an experiment, to compute the dwell time each subject spends in a given state. The standard approach towards determining coherent brain states across subjects is to perform clustering on the results of the dynamic connectivity analysis. Martin evaluated the reproducibility of metrics computed from dynamic FC, and moderately strong reproducibility of the average correlation was found.

Day 5: February 5, 2016

In the morning session of February 5, there were three presenters from University of Alberta, Giseon Heo, Matthew Brown and Dana Cobzas. Giseon Heo’s presentation was on ”Persistent homology: an approach for high dimensional data analysis”. Topological data analysis (TDA) has been popularized since its development in early

2000, and it has shown its effectiveness in discerning true features from noise in high-dimensional data. Giseon introduced persistent homology, a particular branch of computational topology and discussed how it can be incorporated to classical statistics and techniques in machine learning [24]. Matthew Brown's presentation was on "Opening the analysis black box: Improving robustness and interpretation". One primary purpose of neuroimaging data analysis is to abstract away most of the dimensionality and complexity in the data by extracting just a small number of significant patterns from it.This analysis involves a long chain of steps that interact with the data at various points. In practice, the analysis can fail at various steps due to a host of reasons such as the influence of noise, bad convergence in some optimization algorithm, and so on. However, the final output of the analysis often provides no indication that such failures have occurred. Another important consideration is that the analysis often abstracts away too much of the structure in the neuroimaging data. Matthew discussed several approaches for delving into what the data analysis is doing to allow for improved robustness through quality assurance checking as well as improved interpretation through consideration of important patterns in the data that often go unnoticed. Dana Cobzas's presentation was on "Sparse classification for significant anatomy detection in a group study". Dana presented a new framework for discriminative anatomy detection in high dimensional neuroimaging data [25]. Current methods for identifying significant regions related to a group study typically use voxel-based mass univariate approaches. Those methods have limited ability to identify complex population differences because they do not take into account multivariate relationships in data. High dimensional pattern classification methods aim to optimally perform feature extraction and selection to find a set of features that differentiate the groups. Dana proposed a sparse classification method that identifies anatomical regions that are both discriminative and clinically interpretable. Results on synthetic and real MRI data of multiple sclerosis patients and age- and gender-matched healthy controls show superior performance of our method in detecting stable and significant regions in a statistical group analysis when compared to a generative sparse method and to a voxel-based analysis method.

Scientific Progress Made

Much progress has been made in this workshop. We summarize the comments from some of the workshop participants on this regard.

John Aston, Cambridge University: Just a quick email to say thanks so much for all your organization last week. The workshop was great, and Banff was really fun.

Brain Hobbs, University of Texas M.D. Anderson Cancer Center: Great conference. Thank you for all of your efforts in effectuating and facilitating my participation.

Clay Holroyd, University of Victoria: Many thanks, [the organizers] for organizing the meeting, and for inviting me. I enjoyed it.

Dana Cobzas, University of Alberta: I felt I like to leave a note also. Thanks a lot to [the organizers] for giving me the opportunity to attend such a good workshop. I learned a lot, and now have a pile of papers to read. The friendly atmosphere encouraged me to talk with researched that I would have never approached otherwise. And of course is always special to be at the Banff centre.

Daniel Rowe, Marquette University: Thanks for organizing the workshop.I really enjoyed it.

Jeffrey Morris, University of Texas M.D. Anderson Cancer Center: Yes, thank you to the organizers for a great meeting!

Joerg Polzehl, Weierstrass Institute for Applied Analysis and Stochastics: thanks also from me. I really enjoyed the program, meeting all of you and of course the fantastic environment BIRS provides.

Leixi Li, University of California, Berkeley: I'd like to echo what Tingting and Hernando said. It was a great workshop, and I enjoyed it a lot. Thanks all the organizers, and particularly, Hongtu and Linglong, for great leadership! And I look forward to the next workshop, and would be happy to contribute in whichever way to make it happen.

Michele Guindani, University of Texas M.D. Anderson Cancer Center: I would also like to thank you for organizing such an interesting workshop. It was the best workshop I also have ever attended.

Hernando Ombao, University of California, Irvine: Thanks to all the organizers especially Hongtu and Linglong for their leadership.It's never too early to plan for the next one! This was the best workshop I haveattended. I like our spirited, honest and respectful discussions.

Stephen Strother, Baycrest/University of Toronto: Dear Nassif - This was one of the best workshops I have attended in quite a few years, particularly for size and the time to discuss the content in some depth, all complimented and enhanced by the facilities and location. All thanks to our organizers, and the excellent BIRS environment. I will definitely keep an eye out for more relevant BIRS meetings.

Tingting Zhang, University of Virginia: Thank Linglong and Hongtu so much for organizing this wonderful workshop. I had a great time there, getting inspired by many great talks, meeting and learning from cheerful friends, while enjoying delicious food and beautiful views there.

Vikas Singh, University of Wisconsin-Madison: I wanted to send an email to congratulate you on such an awesome meeting. To be honest, I've rarely attended a set of sessions that were so interesting with a group of amazing friends and colleagues. Thanks much for asking me to be a part of this get together.

Participants

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Chapter 4

Vertex Algebras and Quantum Groups (16w5070)

February 7 - 12, 2016

Organizer(s): Yuly Billig (Carleton University), Michael Lau (Laval University), Antun Milas (SUNY-Albany), Kailash Misra (North Carolina State University)

Overview of the Field

During the twentieth century, the theory of Lie algebras, both finite- and infinite-dimensional, has been a major area of mathematical research with numerous applications. In particular, during the late 1970s and early 1980s, the representation theory of Kac-Moody Lie algebras (analogs of finite-dimensional semisimple Lie algebras) generated intense interest. The early development of the subject was driven by its remarkable connections with combinatorics, group theory, number theory, partial differential equations, and topology, as well as by its deep links with areas of physics, including integrable systems, statistical mechanics, and two-dimensional conformal field theory. In particular, the representation theory of an important class of infinite-dimensional Lie algebras known as affine Lie algebras led to the discovery of both vertex algebras and quantum groups in the mid-1980s. Motivated by the appearance of vertex operators in the construction of the Moonshine module, Borcherds introduced vertex operator algebras as part of his proof of the Moonshine Conjectures in 1986. (He received the Fields Medal for this accomplishment in 1998.) These are the precise algebraic counterparts to chiral algebras in 2-dimensional conformal field theory as formalized by Belavin, Polyakov, and Zamolodchikov. Among other applications, these algebras play a crucial role in string theory, integrable systems, M-theory, and quantum gravity. In 1985, the interaction of affine Lie algebras with integrable systems led Drinfeld and Jimbo to introduce quantized universal enveloping algebras, or quantum groups, associated with symmetrizable Kac-Moody algebras. These are q-deformations of the universal enveloping algebras of the corresponding Kac-Moody algebras, and like universal enveloping algebras, they carry an important Hopf algebra structure. The abstract theory of integrable representations of quantum groups, developed by Lusztig, illustrates the similarity between quantum groups and Kac-Moody Lie algebras. The theory of canonical bases for quantum groups has provided interesting insights to the representation theory of quantum groups. This has led to many applications in other areas of mathematics and mathematical physics.

Both vertex algebras and quantum groups are deeply intertwined with conformal field theory and integrable systems, though the exact nature of their relationship with each other remains somewhat mysterious. The main objective of the proposed meeting is to bring together experts in both of these domains in hopes of illuminating some of these connections.

Recent Developments

Motivated by developments in vertex algebras and quantum groups, Borcherds, Etingof-Kazhdan and E. Frenkel-Reshetikhin introduced certain deformations of vertex algebras called quantum vertex (or chiral) algebras. This promising idea was extended by Haisheng Li through several important publications. Interestingly, almost all of the most crucial questions about quantum vertex algebras remain unanswered. For instance, quantum affine algebras have been associated to weak quantum vertex algebras by Li in a certain conceptual way, but it still remains to construct the corresponding weak quantum vertex algebras explicitly and prove that they are quantum vertex algebras. Closely related to these developments, are recent breakthroughs in the area of quantum toroidal algebras, Yangians, quantum W-algebras and related structures. We expect that the proposed conference will provide an important and timely boost to research in this area.

It has been known since the early 1990s that rational vertex algebras coming from affine Lie algebras and quantum groups at roots of unity are closely related at the level of tensor categories. This connection often goes under the name of Kazhdan-Lusztig correspondence. Although the correspondence is functorial, it requires semisimplification at the level of quantum groups. Recently, motivated by a line of work in physics, it was conjectured that the category of modules for a certain irrational vertex algebra, called the triplet algebra, is equivalent to the representation category of the small quantum group associated to sl_2, without any semisimplification. The equivalence is expected to persist at the level of other vertex (super)algebras constructed via screening operators, called W-algebras, and small quantum groups in general. Moreover, the recent important work of Huang, Lepowsky and Zhang on logarithmic tensor product theory of vertex algebras provides a nice framework for studying these irrational vertex algebras from the categorical perspective. This workshop will report on progress in the area, as well as on connections with other areas of mathematics such as modular forms.

Presentation Highlights

Henning Haahr Andersen. *Tilting modules for quantum groups at roots of unity.*

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra denote by $U_v = U_v(\mathfrak{g})$ the associated quantized enveloping algebra over $\mathbb{Q}(v)$. We set $A = \mathbb{Z}[v, v^{-1}] \subset \mathbb{Q}(v)$ and let U_A denote the Lusztig A-form of U_v defined via quantum divided powers. Then for any field K and any parameter $q \in K$
$\{0\}$ we shall consider $U_q = U_A \otimes_A K$ (with K being an A-algebra via $v \mapsto q$) and its module category $\mathcal{C}_q$ consisting of all finite-dimensional U_q-modules (of type 1).

We identify $X = \mathbb{Z}^n$ ($n =$ rk $\mathfrak{g}$) with the characters of U_q^0 (the "Cartan" subalgebra of U_q) and $X^+ = \mathbb{Z}^n_{\geq 0}$ with the set of dominant characters.

For each $\lambda \in X^+$ we have four modules in $\mathcal{C}_q$:

$\nabla_q(\lambda)$, the dual Weyl module

$L_q(\lambda) = \mathrm{soc}\nabla_q(\lambda)$, the simple module

$\Delta_q(\lambda) = \nabla_q(\lambda)^*$, the Weyl module

$T_q(\lambda)$, the indecomposable tilting module,

all having λ as their unique highest weight. We have $\iota^\lambda : \Delta_q(\lambda) \hookrightarrow T_q(\lambda)$, $\pi^\lambda : T_q(\lambda) \twoheadrightarrow \nabla_q(\lambda)$ with $c^\lambda = \pi^\lambda \circ \iota^\lambda \neq 0$.

Questions: Q1. What is $\mathrm{ch}L_q(\lambda)$?

Q2. What is $\mathrm{ch}T_q(\lambda)$?

Remarks: a) If char$K = 0$ and ord $(q) = \ell$ then (with some slight conditions on ℓ) Q1 was solved in the early 1990's by Kazhdan and Lusztig plus Kashiwara and Tanisaki. Moreover, Q2 was solved a few years later by Soergel (with similar conditions on ℓ).

b) If char $K = p > 0$ and $q = 1$ then it is known that when $p \geq 2h - 2$ (h is the Coxeter number), then Q2 implies Q1. Actually, only a tiny (finite) part of Q2 is needed for this.

c) Very recently (December 2015) Riche and Williamson have (when char $K = p$ and $q = 1$) proposed a conjecture for Q2 and proved this conjecture for type A_n and $p > n$.

In an effort to understand the homomorphisms between tilting modules Stroppel, Tubbenhauer and I proved the following results (2015):

Theorem 1. Let $M, N \in \mathcal{C}_q$ and suppose M has a Δ_q-filtration and N has a ∇_q-filtration. If for each $\lambda \in X^+$ $\{f_j^\lambda | j = 1, \ldots, m_\lambda\}$, resp. $\{g_i^\lambda | i = 1, \ldots, n_\lambda\}$ is a basis for $Hom_{U_q}(Mn\nabla_q(\lambda))$, resp. $Hom_{U_q}(\Delta_q(\lambda), N)$, then the set $\{c_{ij}^\lambda = \overline{g}_i^\lambda \circ \overline{f}_j^\lambda | \lambda \in X^+, i, j\}$ is a basis for $Hom_{U_q}(M, N)$. Here $\overline{f}_j^\lambda : M \to T_q(\lambda)$, resp. $\overline{g}_i^\lambda : T_q(\lambda) \to N$ is any homomorphism such that $\pi^\lambda \circ \overline{f}_j^\lambda = f_j^\lambda$, resp. $\overline{g}_i^\lambda \circ \iota^\lambda = g_i^\lambda$.

Theorem 2. If T is a tilting module then $End_{U_q}(T)$ is a cellular algebra.

Proof: Take $T = M = N$ in Theorem 1. Then c_{ij}^λ is a cellular basis.

Example. If $\mathfrak{g} = sl(V)$ and V_q is the corresponding U_q-module for $U_q = U_q(sl(V))$ then $End_{U_q}(V_q^{\otimes^d})$ is cellular for all d.

Daniel Nakano. *Cohomology and support theory for quantum groups.*

Quantum groups are a fertile area for explicit computations of cohomology and support varieties because of the availability of geometric methods involving complex algebraic geometry. In this talk I will present results which illustrate these strong connections between the combinatorics, geometry and representation theory. We set the following notation.

- G: a simple, simply connected algebraic group over $\mathbb{C}$
- $\mathfrak{g} = \mathrm{Lie}\, G$
- h: Coxeter number
- ζ: primitive ℓth root of unity
- $\mathbb{U}_\zeta(\mathfrak{g})$: quantized enveloping algebra specialized at ζ
- $\mathbb{U}(\mathfrak{g})$: ordinary universal enveloping algebra
- $U_\zeta(\mathfrak{g})$: Lusztig $\mathcal{A}$-form specialized at ζ (distribution algebra)
- $u_\zeta(\mathfrak{g})$: small quantum group (f.d. Hopf algebra)
- $X = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_n$: the weight lattice, where $\omega_i \in \mathbb{E}$ are the fundamental weights.
- $X_+ = \mathbb{N}\omega_1 + \cdots + \mathbb{N}\omega_n$: the dominant weights
- $L_\zeta(\lambda)$: simple module $U_\zeta(\mathfrak{g})$ of highest weight $\lambda \in X_+$
- $\nabla_\zeta(\lambda) = \mathrm{ind}_{U_\zeta(\mathfrak{b})}^{U_\zeta(\mathfrak{g})}\lambda$: induced module where $\lambda \in X_+$
- $\Delta_\zeta(\lambda)$: Weyl module where $\lambda \in X_+$
- $T_\zeta(\lambda)$: tilting module of high weight $\lambda \in X_+$

In a fundamental computation, Ginzburg and Kumar demonstrated when $\ell > h$, the cohomology ring identifies with the coordinate algebra of the nilpotent cone of the underlying Lie algebra g=Lie(G).

Theorem. [Ginzburg-Kumar, 1993] Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathcal{N}$ be the nilpotent cone. If $l > h$ then

(a) $\mathrm{H}^{odd}(u_\zeta(\mathfrak{g}), \mathbb{C}) = 0$;

(b) $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[\mathcal{N}]$.

When $\ell \leq h$, we calculated the cohomology ring and showed that the odd degree cohomology vanishes and in most cases $\mathrm{H}^{2\bullet}(u_\zeta(\mathfrak{g}), \mathbb{C}) = \mathbb{C}[G \cdot \mathfrak{u}_J]$ where $G \cdot \mathfrak{u}_J$ is the closure of some Richardson orbit (see [BNPP]). Our calculations heavily used the fact that

$$R^n \mathrm{ind}_{P_J}^G S^\bullet(\mathfrak{u}_J^*) = 0$$

for $n > 0$ (Grauert-Riemenschneider Vanishing Theorem). It is an open problem to prove this vanishing results over fields of characteristic $p > 0$.

With the finite generation of the cohomology ring one can define support varieties for quantum groups. Given $M \in \mathrm{mod}(u_\zeta(\mathfrak{g})$, let $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$ be the support variety of M. In general one can consider the "naive functor" which takes $M \in \mathrm{mod}(U_\zeta(\mathfrak{g}))$ to G-equivariant coherent sheaves on the nilpotent cone (when $l > h$) by using the cohomology of the small quantum group. This functor was lifted to an equivalence of derived categories in the work of Arkhipov, Bezrukavnikov and Ginzburg (ABG).

Theorem. [Arkhipov-Bezrukavnikov-Ginzburg, 2003] Let $l > h$. There exists the following equivalences of derived categories

$$D^b(U_\zeta(\mathfrak{g})_0^{\mathrm{mix}}) \cong D^b(Coh^{G\times\mathbb{C}^*}(G\times_B \mathfrak{n})) \cong D^b(Perv^{\mathrm{mix}}(\mathrm{Gr})).$$

Bezrukanikov used the ABG equivalence to compute the support varieties of tilting modules. For the induced/Weyl modules the support varieties were computed by Ostrik and Bendel-Nakano-Pillen-Parshall. The calculations for the simple modules employed the deep fact that the Lusztig Character Formula holds for quantum groups when $\ell > h$, and the positivity of the coefficients of the (parabolic) Kazhdan-Lusztig polynomial for the affine Weyl group. We present this in the following theorem.

Theorem. [Drupieski-Nakano-Parshall, 2012] Let $\ell > h$ and $\lambda \in X^+$. Choose $J \subseteq \Delta$ such that $w(\Phi_\lambda) = \Phi_J$ for some $w \in W$. Then

$$\mathcal{V}_{u_\zeta(\mathfrak{g})}(L_\zeta(\lambda)) = G\cdot \mathfrak{u}_J.$$

At the end of my talk a number of open problems described below were presented.

1) Calculate $\mathrm{Ext}^n_{U_\zeta(\mathfrak{g})}(L(\lambda), L(\mu))$ for all $\lambda, \mu \in X_+$.

From the proof of the Lusztig Character Formula, this is known when λ and μ are regular weights. What about singular weights?

2) Is there a “rank variety” description of $\mathcal{V}_{u_\zeta(\mathfrak{g})}(M)$?

3) Let $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d$ be the corresponding untwisted affine Lie algebra, and $\tilde{\mathfrak{g}}$ be the subalgebra $\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \subseteq \hat{\mathfrak{g}}$. For $\kappa \in \mathbb{C}$ we let $\mathcal{O}_\kappa$ be the full subcategory of all $\tilde{\mathfrak{g}}$-modules, M, for which the central element c acts by κ and M satisfies category $\mathcal{O}$-type finiteness conditions. For a certain κ, there exists an equivalence of tensor categories

$$F_\ell : \mathcal{O}_\kappa \to \mathrm{mod}(U_\zeta(\mathfrak{g}))$$

a) Is there any new information about support varieties and cohomology that can be gained by using this equivalence of categories?

b) How does the twisting of modules under the Frobenius morphism behave under this equivalence?

Evgeny Mukhin. *Trivial models with non-trivial Bethe Ansatz.*

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra with simple roots $\alpha_1, \dots, \alpha_r$. Let L_λ be an irreducible $\mathfrak{g}$-module of highest weight λ. Consider $V = L_{\lambda_1} \otimes \dots \otimes L_{\lambda_n}$, and let $(z_1, \dots, z_n) \in \mathbb{C}^n$, $z_i \neq z_j$.

Define Gaudin Hamiltonians:

$$H_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{\Omega^{(i,j)}}{z_i - z_j},$$

where $\Omega^{(i,j)}$ is the canonical element of $U\mathfrak{g} \otimes U\mathfrak{g}$ acting in i, j factors.

Example. For gl_{r+1}: $\Omega = \sum\limits_{i,j=1}^{r+1} e_{ij} \otimes e_{ji}$.

We have $[H_i, H_j] = 0$.

Problem. Find common eigenvectors and eigenvalues of H_i.

Since H_i commute with $U\mathfrak{g}$ diagonal action, we can restrict to V_λ^{sing}-subspace of V of singular vectors of weight λ. Let $\lambda = \lambda_1 + \dots + \lambda_n - \sum\limits_{i=1}^{r} \ell_i \alpha_i$, $\ell_i \in \mathbb{Z}_{\geq 0}$.

The problem is solved by Bethe Ansatz method. Let $t = (t_i^{(j)}, j = 1, \dots, r, i = 1, \dots, \ell_j)$ be auxiliary variables. One can explicitly define $W(t, z) \in V_\lambda$ called weight function and

$$\Phi = \prod \left(t_i^{(k)} - t_j^{(s)}\right)^{(\alpha_k, \alpha_s)} \prod \left(t_i^{(k)} - z_s\right)^{-(\alpha_k, \lambda_s)}.$$

Then one proves

Theorem. If

$$\begin{cases} \dfrac{\partial_{t_i^{(j)}} \Phi}{\Phi} = 0 \quad \text{for all } i, j, \end{cases}$$

then $W(t, z)$ is an eigenvector of H_i and $W(t, z) \in V_\lambda^{sing}$.

What if $\dim(V_\lambda^{sing}) = 1$?

Message. Bethe Ansatz is still interesting!

Reasons:

- can be solved explicitly
- $t_i^{(j)}$ are zeros of interesting polynomials (such as Jacobi or Jacobi-Pi neiro polynomials)
- can be used to find sufficient amount of Bethe equation (4) in more general situation
- leads to explicitly computable multidimensional integrals (sl_2 gives the famous Selberg integrals)
- give meaningful recursions (???).

Fedor Malikov. *Strong homotopy chiral algebroids.*

Let A be $\mathbb{C}$-algebra. A Picard-Lie A-algebroid is an exact sequence

$$0 \longrightarrow A \longrightarrow L \longrightarrow T_A \longrightarrow 0,$$

where L is a Lie algebroid, i.e., an A-module and a Lie algebra, and the arrows preserve all the structure involved. The category of Picard-Lie A-algebroids is a torsor over the abelian group in categories $\Omega_A^{[1,2>}$. This result gives classification of algebras of twisted differential operators (TDO) over A, provided A is smooth.

If A is not smooth, then, following an idea of V. Hinich, we define an ∞-Picard-Lie algebroid, by using the exact sequence above but allowing the Lie algebras involved to be replaced with Lie$_\infty$-algebras. We prove that the arising groupoid is a torsor over $L\Omega_A^{[1,2>}$, the analogously truncated Illusie cotangent complex [Ill1, Ill2].

Our main result is that an analogous discussion allows to extend the result of [GMS] to obtain and classify ∞-chiral algebroids over a dga A. The definition of the latter is similar to the one just described except that the algebra A is replaced with the commutative chiral algebra $J_\infty A$, modules with chiral modules, and Lie algebras, such as T_A, with Lie*-algebras, such as $J_\infty T_A$, and then with Lie$^*_\infty$ algebras. The result is a groupoid that is a torsor over $L\Omega_{J_\infty A}^{[1,2>}(J_\infty T_A)$, where the latter is the homotopy version of the Chevalley-De Rham complex that arises in the Beilinson-Drinfeld compound pseudo-tensor category [BD].

Chongying Dong. *On orbifold theory.*

Orbifold theory studies a vertex operator algebra V under the action of a finite automorphism group G. The well-known Orbifold Theory Conjecture says that if V is rational then 1) V^G is rational, 2) every irreducible V^G-module occurs in an irreducible g-twisted V-module for some $g \in G$.

The appearance of the twisted modules is the main feature of the orbifold theory. In the case that V is the vertex operator algebra associated to the highest weight module for an affine Kac-Moody algebra, the g-twisted modules are exactly the modules for the twisted affine Kac-Moody algebras.

Here are our main results: Assume 1) V is a rational vertex operator algebra and G is a finite automorphism group of V, 2) the weight of every irreducible g-twisted module ($g \in G$) is positive except for V itself. Then every irreducible V^G-module occurs in an irreducible g-twisted V-module for some $g \in G$.

This result reduces the orbifold theory conjecture to the rationality of V^G. The main tool we use to prove the result is the modular invariance of trace functions by Zhu, Dong-Li-Mason. The assumption that any irreducible g-twisted V-module has a positive weight except for V itself holds for the most well known rational vertex operator algebras. This assumption allows us to use the tensor product and related results given by Huang and Dong-Li-Ng.

We also find an interesting formula on the global dimension of V. The global dimension is defined to be the sum of squares of the quantum dimensions of inequivalent irreducible V-modules. It turns out the global dimension of V is a sum of the squares of the quantum dimensions of inequivalent irreducible g-twisted V-modules for any finite order automorphism g.

Here are some important open problems in the orbifold theory:

Conjecture 1. If V is rational and G is a finite automorphism group of V then V^G is rational.

Conjecture 2 . If V is rational then V is g-rational for any finite order automorphism g.

Conjecture 3. If V is rational and G is a finite automorphism group of V, then there exists a finite-dimensional semi-simple Hopf algebra A such that the module category of V^G and the module category of A are equivalent as tensor categories.

Terry Gannon. (joint with T. Creutzig) *The theory of C_2-cofinite VOAs.*

The theory of rational VOAs is quite well understood.

1. The corresponding category of modules has finitely many simples, has direct sums, is semisimple, has tensor product, is rigid. Such a category is called a *fusion category*. The category of finite-dimensional representations over $\mathbb{C}$ of a finite group is a prototypical example of a fusion category. The category of rational VOA is even better: it is a *modular tensor category* (MTC). This means it is a fusion category which is ribbon and satisfies $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{op}$, where $Z(\mathcal{C})$ is Drinfeld double. A modular tensor category has finite-dimensional representations of all surface mapping class groups, e.g. $SL_2(\mathcal{Z})$.

Conjecture. Every MTC is a category of modules of a rational VOA.

Fusion categories can be studied by subfactor methods, and we find that we seem to know perhaps 0% of all fusion categories. Taking their doubles give MTCs, which conjecturally correspond to VOAs. So it is tempting to guess that we know 0% of rational VOAs. In other words, we are missing general construction.

2. Zhu proved that the character $Z_\mu(\tau) = \mathrm{Tr}_\mu q^{L_0 - c/24}$ of rational VOAs form a vector-valued modular form of weight 0 for $SL_2(\mathbb{Z})$. There is a generalization, where $u \in V$ are inserted, giving vector-vlaued modular forms of higher weight.

3. Verlinde's formula expresses the tensor product coefficients $M_i \otimes M_j \cong \bigoplus_k N_{ij}^k M_k$ in terms of the matrix $S = \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ coming for the $SL_2(\mathbb{Z})$-multiple in # 2:

$$N_{ij}^k = \sum_\ell \frac{S_{i\ell} S_{j\ell} \overline{S}_{k\ell}}{S_{1\ell}}.$$

The "elementary" part of this is that Verlinde's formula holds for S associated to the Hopf link in the MTC (this was proved by Turaev). The deep part is that S also appears in # 2.

If we drop semisimplicity for the definition of a rational VOA, we get the class of C_2-cofonite VOAs. These should be a gentle generalization of rational VOAs. We should expect that almost all C_2-cofinite VOAs are not rational. But there are only 2 families that we studied: W_p triplet models, and symplectic fermions SF_d. More families of examples are needed!

1. Huang / Miyamoto proved that the category of C_2-cofinite VOAs have finitely many simples, each has projective cover, all Hom spaces are finite, every module has finite composition series, it is an abelian tensor category. Missing: rigidity. Apart from rigidity, it is a *finite tensor category*, the nonsemisimple version of fusion category. Also it is braided. Tempting to guess that nonsemisimple version of MTC is a ribbon finite tensor category satisfying $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{op}$. A big question is to relate this to Lyubashenko's work, which builds representations of mapping class groups from a closely related category.

Theorem. If V is C_2-cofinite, rigid, not rational, then V has uncountably many indecomposables for infinitely many composition series lengths.

For example, W_p has 4 families of indecomposables for each composition length, each parametrized by $\mathbb{CP}^1$ (tame-type representations). SF_d for $d > 1$ has wild representation type.

Expectation: almost every C_2-cofinite VOA has wild representation type – their indecomposables will never be classified.

Subfactor methods should extend to finite tensor categories, but may not be effective.

2. The characters of C_2-cofinite VOAs can be defined, but look like weight-0 piece + weight-1 piece $+ \ldots +$ weight-N piece. When $SL_2(\mathbb{Z})$ is applied to them, the weight k piece picks up τ^k. So $\mathbb{C}$-span of characters are not closed under $SL_2(\mathbb{Z})$. To get a vector-valued modular form, we have to define new pseudo-characters with τ, q exponents.

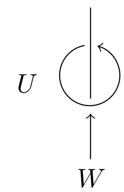
Physics approach (e.g. Runkel): Insert intertwining vertex operator of type $\binom{M}{P_V, M}$, where P_V is projective cover. At least for symplectic fermions, this gives missing pseudocharacters.

Associative algebra approach (Miyamoto): use symmetric linear functionals, etc. This always works but is not very effective. Might be overkill: Arike-Nagatomo showed for symplectic fermions it gives dimension $\geq 2^{2d-1}+3$, but the modular completion of characters is $d+4$-dimensional.

Big question: Can physics approach be reconciled with Miyamoto's? Is Miyamoto's overkill?

3. Search for categorical analogue of Verlinde. Block diagonalise fusion matrices for Grothendieck ring: e.g. for W_p get 2 1×1 blocks, $p-1$ 2×2 blocks; for SF_d get 2 1×1 blocks, 1 2×2 blocks. These blocks coincide with subrep of open Hopf links: let P be projective, $\Phi_{w,p} \in Z(End(P))$. $\Phi_{w,p}$ define representation of fusion ring, subreps are those blocks.

$\Phi_{U,W}$ =



Tempting to guess this always happens for C_2 cofinite VOAs. No analogue of this for full tensor product ring.

Dražen Adamović. *Conformal embeddings and realizations of certain simple W-algebras.*

In this talk we present certain new results and constructions related with conformal embeddings, extension theory of vertex algebras, and with W-algebras appearing in logarithmic conformal field theory.

First we discuss notion of simple current modules for affine VOA $V_k(\mathfrak{g})$ in the category KL_k. We present a result (D. Adamović, O. Perse [1]) on new simple current modules for $V_{-1}(sl_n)$. Then motivated by a conformal embedding $V_k(gl_n)$ into $V_k(sl_{n+1})$, for $k = -\frac{n+1}{2}$ [AKMPP1], we present a new conjectural list of simple current modules.

More precisely, for $i \in \mathbb{Z}$, let $M_{k,i} = L_{sl(n)}(\lambda_{k,i})$ where $\lambda_{k,i} = (k-i)\Lambda_0 + i\Lambda_1$ $(i \geq 0)$, $\lambda_{k,i} = (k+i)\Lambda_0 - i\Lambda_n$ $(i < 0)$.

Conjecture Let $k = -\frac{n+1}{2}$ $(n \geq 4)$. In the category KL_k of $V_k(sl(n))$–modules, the following fusion rules holds: $M_{k,i} \times M_{k,j} = M_{k,i+j}$.

Next we present new results on conformal embeddings of affine vertex algebras into W-algebras:

Theorem[AKMPP2]. Let $k = -\frac{2}{3}(n+2)$. Then we have conformal embedding $V_{k+1}(gl(n))$ in $W_k = W_k(sl(n+2), f_\theta)$. Assume that $n \geq 3$. Then $W_k = \bigoplus_{i \in \mathbb{Z}} W_k^{(i)}$ and each $W_k^{(i)}$ is irreducible $V_{k+1}(gl(n))$–module.

We show that results on conformal embeddings can imply new construction of extension of affine VOAs. A particular emphasis is put on the conformal embedding $V_{k+1}(gl_n)$ in $W_k(sl_{n+2}, f_\theta)$, for $k = -\frac{2}{3}(n+2)$.

A complete reducibility result for this conformal embedding is presented.

Next we show that vertex algebras $V_{-\frac{3}{2}}(sl_3)$ and $W_{-\frac{8}{3}}(sl_4, f_\theta)$ are related with vertex algebras appearing in logarithmic conformal field theory. Explicit realization of $W_{-\frac{8}{3}}(sl_4, f_\theta)$ from [AKMPP2] was discussed. We also present a result from [1] that parafermionic vertex algebra $K(sl_3, -\frac{3}{2})$ is isomorphic to the W–algebra $W^0_{A_2}(p)$, for $p = 2$, where $W^0_{A_2}(p)$ is a higher rank generalization of the triplet vertex algebra $W(p)$ from [AM]. We discuss a conjecture that $W^0_{A_2}(p)$ is a C_2–cofinite, non-rational vertex algebra.

Valerio Toledano Laredo. *Yangians, quantum loop algebras and elliptic quantum groups.*

The Yangian $Y\mathfrak{g}$ and quantum loop algebra $U_q(L\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ share very many similarities, and were long thought to have the same representations, though no precise relation between them existed until recently.

I will explain how to construct a faithful functor from the finite-dimensional representations of $Y\mathfrak{g}$ to those of $U_q(L\mathfrak{g})$. The functor is entirely explicit, and governed by the monodromy of the abelian difference equations determined by the commuting fields of the Yangian. It yields a meromorphic, braided Kazhdan-Lusztig equivalence between finite-dimensional representations of the $Y\mathfrak{g}$ and of $U_q(L\mathfrak{g})$.

A similar construction yields a faithful functor from representations of $U_q(L\mathfrak{g})$ to those of the elliptic quantum group $E_{q,\tau}(\mathfrak{g})$ corresponding to $\mathfrak{g}$. This allows in particular a classification of irreducible finite-dimensional representations of $E_{q,\tau}(\mathfrak{g})$, which was previously unknown.

This is joint work with Sachin Gautam (Perimeter Institute).

Naihuan Jing (joint work with N. Rozhkovskaya). *Vertex operators and Giambelli identities.*

1. Giambelli identity.

Classical Jacobi-Trudi identity says that for any partition $\lambda = \lambda_1 + \ldots + \lambda_\ell$ $(\lambda_1 \geq \ldots \geq \lambda_\ell > 0)$ the Schur function s_λ is the determinant of certain homogeneous symmetric function $h_n := s_{(n)}$:

$$s_\lambda = det(h_{\lambda_i - i + j})_{\ell\times\ell}.$$

In a series of classical papers of Weyl in 1920's, he gave similar determinant formula for the character of simple modules of classical Lie simple algebras (in types B, C, D), e.g.,

$$sp_\lambda = \frac{1}{2} det(h_{\lambda_i - i + j} + h_{\lambda_i - i - j + 2}),$$

where h_n is the character of simple module corresponding to $\square\square\cdots\square$.

Our second motivation comes from Okounkov-Olshanski formula for the shifted Schur function

$$s^*_\lambda = \frac{det(x_i + n - i|\lambda_j + n - j)}{det(x_i + n - i|n - j)},$$

where $(x|k) = x(x-1)\ldots(x-k+1)$ for $k \geq 1$; and $\frac{1}{(x+1)\ldots(x+(-k))}$ for $k < 0$.

We introduce the so-called generalized Schur function s_λ (still the same notation) for any composition $\lambda = \lambda_1 + \ldots + \lambda_\ell$ in the ring $B = \mathbb{C}[h_1^{(0)}, h_2^{(0)}, \ldots]$ and $B^{\leq n}$ = subspace of deg $\leq n$. In the general generators $h_k^{(r)} \in B^{\leq r+k}$, $h_{-k}^{(k)} = 1$ and $h_k^{(r)} = 0$ for $k + r < 0$. We define

$$s_\lambda \doteq det\left[h^{(j-1)}_{\lambda_i - i + 1}\right].$$

Then we can define the generalized elementary symmetric function $e_a^{(p)}$ and the essential part is

$$e_a^{(p)} = det\left[h^{(-p+j)}_{p+1-i}\right], p > a,$$

and other trivial cases for $p \leq a$. Then one can introduce the hook Schur function

$$s_{(m|n)} = \sum_{p=0}^{n} (-1)^p h^{(p)}_{m+1} e^{(-p)}_{-n}.$$

Proposition (Newton-like formula).

$$\sum_{p=-\infty}^{\infty} (-1)^{a-p} h_b^{(p)} e_a^{(-p)} = \delta_{ab},\ a, b \in \mathbb{Z}.$$

Theorem (Giambelli-like). For every partition $(\alpha|\beta) = (\alpha_1 \ldots \alpha_r|\beta_1 \ldots \beta_r)$ in Frobenius notation of a partition, and every $n \geq \ell(\alpha|\beta)$,

$$s_{(\alpha|\beta)} = det\left[s_{(\lambda_i - i|n-j)}\right]_{n\times n} = det\left[s_{(\alpha_i|\beta_j)}\right]_{r\times r}.$$

2. Vertex operator realization of generalized Schur functions.

Define the action of ψ_k, ψ_k^* on $B = \underset{m\in\mathbb{Z}}{\oplus} B^{(m)}$, $B^{(m)} = z^m\mathbb{C}[s_\lambda]$:

$$\psi_k(s_\lambda z^m) = s_{(k-m-1,\lambda)} z^{m+1},$$

$$\psi_k^*(s_\lambda z^m) = \sum_{t=1}^{\infty}(-1)^{t+1}\delta_{k-m-1,\lambda_t-t}s(\lambda_1+1,\ldots,\lambda_t+1,\lambda_{t+1},\ldots)z^{m-1}.$$

Theorem. 1. The operators ψ_k, ψ_k^* satisfy the Clifford algebra relations

$$\{\psi_k, \psi_\ell\} = \{\psi_k^*, \psi_\ell^*\} = 0,\ \{\psi_k, \psi_\ell^*\} = \delta_{k,\ell}.$$

2. For any partition λ, with $\mu = \lambda'$, the dual of λ,

$$\psi_{\lambda_\ell+\ell}\ldots\psi_{\lambda_1+1}\mathbf{1} = s_\lambda z^\ell,$$

$$\psi^*_{-\lambda_1-\ell+1}\cdots\psi^*_{-\lambda_\ell}\mathbf{1} = (-1)^{|\lambda|}s_\mu z^{-\ell}.$$

(which imply that the module of the Clifford algebra is simple).

We then can write down the vertex operator realization of the generalized Schur functions.

Haisheng Li. *q-Virasoro algebra and affine Lie algebra.*

This talk is based on a joint work with H. Guo, S. Tan and Q. Wang. The so-called q-Virasoro algebra is an infinite-dimensional Lie algebra introduced by Belov and Chaltikian many years ago. In this talk, I will first review a theory of equivariant quasi modules for vertex algebras, which was developed by the author. Then I will explain how one can associate vertex algebras to the q-Virasoro algebra and show that the q-Virasoro with q a root of unity of order $2\ell+1$ is actually isomorphic to the affine Kac-Moody Lie algebra of type $B_\ell^{(1)}$.

First, we review the theory of quasi modules for vertex algebras. Let W be an arbitrary vector space over $\mathbb{C}$. Set $\mathcal{E}(W) = Hom(W, W((x)))$. A subset U of $\mathcal{E}(W)$ is said to be quasi-local if for any $a(x), b(x) \in U$ there exists a non-zero polynomial $p(x,z)$ such that

$$p(x,z)a(x)b(z) = p(x,z)b(z)a(x).$$

Assume $a(x), b(x) \in U$ are quasi-local. Define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}(a(x), z)b(x) = p(x+z, z)\left(p(x_1, x)a(x_1)b(x)\right)_{x_1=x+z}.$$

Theorem. Every quasi-local subset U of $\mathcal{E}(W)$ generates a vertex algebra $\mathcal{W}$ with W as a quasi module.

Now consider the q-Virasoro algebra $\mathcal{D}$, which is a Lie algebra with generators $D^\alpha(n)$ for $\alpha, n \in \mathbb{Z}$ and C, satisfying certain relations. For each$\alpha \in \mathbb{Z}$, form a generating function

$$D^\alpha(x) = \sum_{n\in\mathbb{Z}} D^\alpha(n)x^{-n-1}.$$

The fact is that these generating functions are quasi-local.

Theorem. [Guo-Li-Tan-Wang] $\mathcal{D}$ is isomorphic to the covariant Lie algebra $\widehat{\mathfrak{g}}/S$ of the affine Lie algebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a certain Lie algebra with an automorphism group S

Theorem. [Guo-Li-Tan-Wang] The category of restricted $\mathcal{D}$-modules of level ℓ is isomorphic to the category of equivariant quasimodules for the vertex algebra $V_{\widehat{\mathfrak{g}}}(\ell, 0)$.

Theorem. [Guo-Li-Tan-Wang] Assume that q is a root of unity of order $2\ell+1$. Then $\mathcal{D}$ is isomorphic to the affine Kac-Moody algebra of type $B_\ell^{(1)}$.

Iana I. Anguelova. *Towards quantum chiral algebras.*

In 1988 I. Frenkel and N. Jing constructed the vertex representations of the quantum affine algebras, starting with the quantum Heisenberg operators $a_{ik}, k \in \mathbb{Z}$ satisfying the relations ($\gamma = e^{tc/2} = q^c$)

$$[a_i(m), a_i(n)] = \delta_{i,-m}\frac{1}{mt^2}(q^{mA_{ij}} - q^{-mA_{ij}})(q^m - q^{-m}),$$

and defining the vertex operators

$$Y_i^{\pm}(z) = exp\Big(\pm t \sum_{n\geq 1} \frac{q^{\mp \frac{n}{2}}}{q^n - q^{-n}} a_i(-n) z^n \Big) \times exp\Big(\mp t \sum_{n\geq 1} \frac{q^{\mp \frac{n}{2}}}{q^n - q^{-n}} a_i(n) z^{-n} \Big) a_i^{\pm 1} z^{\pm a_i(0)+1}.$$

These vertex operators (together with $\Phi_i(z)$ and $\Psi_i(z)$) define a representation of the quantum affine algebras at $c = 1$.

The fundamental problem posed by Igor Frenkel then is to formulate and develop a suitable theory of quantum vertex algebras (quantum chiral algebras) incorporating as examples the Frenkel-Jing quantum vertex operators. As one works towards an answer to this problem one of the big questions that needs to be addressed at the start is: do we require Completeness with respect to Operator Product Expansions (OPEs) for our quantum chiral algebras? I.e., do we require that the coefficients in the OPEs are vertex operators in the **same** (quantum) vertex algebra? Here, as in physics, the term "*chiral algebra*" will refer to the fact that we do require completeness with respect to the OPEs. For example, in the quantum affine case (FJ) we have OPEs such as:

$$Y_i^{\mp}(z) Y_i^{\pm}(w) \sim \frac{q^2}{q^2-1} \frac{w : Y_i^{\pm}(qw) Y_i^{\mp}(w) :}{z - qw} - \frac{w : Y_i^{\pm}(q^{-1}w) Y_i^{\mp}(w) :}{(q^2-1)(z - q^{-1}w)}$$

This OPE reflects the fact that for the quantum affine algebras we have poles in the OPEs not only at $z = w$, but in fact at $z = q^n w, n \in \mathbb{Z}$ (here $q \in \mathbb{C}$). We observe that both in this case (trigonometric) and in the elliptic case (such as the deformed Virasoro chiral algebra of E. Frenkel and N. Reshetikhin), it is clear that the OPE completeness requires field descendants of the type $\mathbf{a}(\gamma \mathbf{z})$ for $\gamma \neq 1$. This has profound consequences: in particular if both the fields $a(z)$ and $a(\gamma z)$ are to be incorporated in the **same chiral algebra structure**, this would result in the state-field correspondence becoming non-invertible! Recall the **state-field correspondence** is a map from the space of states W to the space of fields V, given by $W \ni a \mapsto a(z) = Y(a, z) \in V$ (bijection for super vertex algebras). Its inverse map is the **field-state correspondence**, a map from the space of fields V to the space of states W on which the fields act, defined by $V \ni \tilde{a}(z) \mapsto a := \tilde{a}(z)|0\rangle|_{z=0} \in W$. If both the fields $a(z)$ and $a(\gamma z)$ have to be incorporated in the same chiral algebra , then the field-state correspondence map will send the different fields $a(z)$ and $a(\gamma z)$ to the same state element $a \in W$:

$$a(\gamma z)|0\rangle|_{z=0} = a(z)|0\rangle|_{z=0} = a \in W.$$

Thus the space of fields V will be a (ramified) cover of the space of states W. Our proposed definition of a chiral algebra with Γ-type singularities reflects this. A particular "baby" class of chiral algebras with Γ-type singularities is the class where the group of singularities is the roots of unity group $\Gamma = \{1, \varepsilon, \dots, \varepsilon^{N-1}\}$ and the R-matrix is $R(a \otimes b) = (-1)^{p(a)p(b)} a \otimes b$ (still trivial). We called this particular subclass of chiral algebras "twisted vertex algebras" (IA). They represent the simplest nontrivial examples of chiral algebras with OPE singularities located at points besides $z = w$. Nevertheless they are interesting in its own accord as they reflect the cases of the boson-fermion correspondences of types B, C and D, which are all isomorphisms of twisted vertex algebras. The details on the bosonizations of types B, C and D can be found in a series of my papers, which have delineated these correspondences and some of their properties.

In conclusion, we would like to thank the organizers and BIRS for the extremely productive Workshop, and we wanted to remark that we truly appreciated the great atmosphere at the Banff Centre.

Andrew Linshaw. *Orbifolds and cosets via invariant theory.*

Given a vertex algebra $\mathcal{V}$ and a group of automorphisms $G \subset \text{Aut}(\mathcal{V})$, the invariant subalgebra $\mathcal{V}^G$ is called an *orbifold* of $\mathcal{V}$. Similarly, given a vertex subalgebra $\mathcal{A} \subset \mathcal{V}$, the subalgebra $\text{Com}(\mathcal{A}, \mathcal{V}) \subset \mathcal{V}$ which commutes with $\mathcal{A}$ is called a *coset* of $\mathcal{V}$. Many interesting vertex algebras can be constructed either as orbifolds or cosets, or as extensions of these structures. It is widely believed that nice properties of $\mathcal{V}$ such as strong finite generation, C_2-cofiniteness, and rationality, will be inherited by $\mathcal{V}^G$ and $\text{Com}(\mathcal{A}, \mathcal{V})$ if G and $\mathcal{A}$ are also nice, but so far few general results of this kind are known.

In recent work, we have established the strong finite generation of orbifolds of free field algebras under arbitrary reductive automorphism groups. This should be viewed as a vertex algebra analogue of Hilbert's theorem that if a reductive group G acts on a finite-dimensional complex vector space V, the ring $\mathbb{C}[V]^G$ of invariant polynomial functions is finitely generated. The proof of our result makes use of the structure and representation theory of $\mathcal{V}^{\text{Aut}(\mathcal{V})}$ when $\mathcal{V}$ is a free field algebra, together with the fact that all $\mathcal{V}^{\text{Aut}(\mathcal{V})}$-modules appearing in the decomposition of $\mathcal{V}$ have the C_1-cofiniteness property according to Miyamoto's definition.

In joint work with T. Creutzig, we have shown that a broad class of cosets of affine vertex algebras inside larger vertex algebras can be described by passing to a certain limit, which is an orbifold of a free field algebra. These cosets depend continuously on the central charge, and a strong generating set for the limit gives rise to a strong generating set for the coset at generic points in the family. In this way, we have established the strong finite generation of a broad class of cosets, and in many instances we can give an explicit minimal strong generating set.

In this conference we discussed these results, as well as recent joint work with T. Arakawa, T. Creutzig, and K. Kawasetsu in which we extended our methods to orbifolds and cosets of the minimal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f_{\text{min}})$ associated to a simple Lie algebra $\mathfrak{g}$ and its minimal nilpotent element f_{min}. For example, we obtained the following description of the coset of the affine vertex algebra generated in weight one inside $\mathcal{W}^k(\mathfrak{g}, f_{\text{min}})$, when $\mathfrak{g}$ is either $\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$.

Theorem. For $n \geq 3$ and for generic values of k, the coset $\mathcal{C}^k = \text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{min}}))$ is of type $\mathcal{W}(2, 3, \dots, n^2 - 2)$. In other words, a minimal strong generating set consists of a field in each weight $2, 3, \dots, n^2 - 2$.

Theorem. For $n \geq 2$ and for generic values of k, the coset $\mathcal{C}^k = \text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, f_{\text{min}}))$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$.

We also discussed some cases where the set of nongeneric values of the central charge (where our description of the coset does not work) can be described explicitly. This allowed us to describe the coset of the *simple* affine vertex algebra inside the simple minimal $\mathcal{W}$-algebras in several examples.

Nicolas Guay. *Twisted Yangians for symmetric pairs of types B,C,D and their representations.*

This is joint work in progress with Vidas Regelskis and Curtis Wendlandt.

Yangians are one of the important families of affine quantum groups and they afford a rich representation theory. Theoretical physics is a source of motivation for their study, one reason being that representations of Yangians lead to solutions of the quantum Yang-Baxter equation, another one being that Yangians control certain symmetries of integrable systems. For integrable systems with boundaries, it turns out that so-called twisted Yangians are sometimes relevant for the study of their symmetries. In the mathematical literature, the twisted Yangians that have been the most studied were introduced by G. Olshanski about twenty-five years ago and are associated to the symmetric pairs $(\mathfrak{gl}_n, \mathfrak{so}_n)$ and $(\mathfrak{gl}_n, \mathfrak{sp}_n)$. Over the years, their representation theory has been investigated in several papers of S. Khoroshkin, A. Molev, M. Nazarov, V. Toledano Laredo and others. Our ongoing project is to introduce similar twisted Yangians for other classical symmetric pairs, study their properties and develop their representation theory.

Given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and an involution ρ of $\mathfrak{g}$, the pair $(\mathfrak{g}, \mathfrak{g}^\rho)$ is called a symmetric pair. We are interested mainly in the cases when $\mathfrak{g}$ is $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ and $\mathfrak{g}^\rho$ is isomorphic to $\mathfrak{gl}_{\frac{N}{2}}$ (if N is even) or $\mathfrak{so}_p \oplus \mathfrak{so}_{N-p}$ or $\mathfrak{sp}_p \oplus \mathfrak{sp}_{N-p}$ for some $0 \leq p \leq N$. The involution ρ can be extended to the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ via $\rho(X \otimes t^k) = \rho(X) \otimes (-t)^k$ and the invariant subalgebra $\mathfrak{g}[t]^\rho$ is called a twisted current algebra.

The twisted Yangian $Y^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ is a deformation of the enveloping algebra of $\mathfrak{g}[t]^\rho$. It can be defined as a coideal subalgebra of the Yangian $Y(\mathfrak{g})$ of $\mathfrak{g}$ via the RTT-presentation of $Y(\mathfrak{g})$. Actually, we first obtain the extended twisted Yangian $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ as a coideal subalgebra of the extended Yangian $X(\mathfrak{g})$ of $\mathfrak{g}$ and $Y^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ can be obtained either as a subalgebra of $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ or as a quotient by an ideal generated by central elements. We prove that, equivalently, $X^{tw}(\mathfrak{g}, \mathfrak{g}^\rho)$ can be defined by generators satisfying the reflection equation and a symmetry relation. These basic results parallel those for Olshanski's twisted Yangians and we are able to recover these when $\mathfrak{g} = \mathfrak{so}_3 \cong \mathfrak{sl}_2$ or $\mathfrak{g} = \mathfrak{sp}_2 = \mathfrak{sl}_2$.

A natural first step in the study of representations of twisted Yangians is to classify their irreducible finite dimensional modules. For $Y(\mathfrak{g})$, these modules are classified by certain tuples of polynomials. This is still true for twisted Yangians. To obtain a classification theorem, we start by proving that finite dimensional irreducible modules are highest weight modules, hence are quotients of Verma modules. It is thus sufficient to determine when such a module is non-trivial and when its non-zero irreducible quotient is finite dimensional. The proof of the criterion for finite dimensionality can be split into two parts: the first one consists of using induction via a reduction on the rank of $\mathfrak{g}$; the second step is to give a direct proof when the rank of $\mathfrak{g}$ is minimal. In some cases, this second part follows immediately from know results for the twisted Yangians of $\mathfrak{sl}_n$.

Kiyokazu Nagatomo. *Vertex operator algebras, minimal models and modular linear differential equations.*
On this occasion I discuss two ways to characterize the minimal series of Virasoro vertex operator algebras L_c with the central charge c and 4 simple modules. Let V be a VOA. We propose two characterizations which are about characters of V-modules. More explicitly one is that the second and third coefficients of the character of V are 0 and 1, respectively, and other is that the second coefficients of characters of V-modules except V are all 1. Then we show that the first condition confines to $L_{-46/3}$, $L_{-3/5}$ and the two VOAs which are extension of $L_{-114/7}$ and $L_{4/5}$ by their simple modules. The second conditions also gives $L_{-46/3}$, $L_{-3/5}$, and the lattice VOA V_L with $L = \mathbb{Z}\alpha$ and $\langle \alpha, \alpha \rangle = 6$.

Vidas Regelskis. *Classification of trigonometric reflection matrices.*
This is a joint work with Bart Vlaar. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra and let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involutive Lie algebra automorphism. Let $\mathfrak{k} = \{x \in \mathfrak{g} : \theta(x) = x\}$ denote the theta fixed subalgebra. Assuming that θ is of a second kind, that is $\dim(\theta(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$, where $\mathfrak{b}^+$ is the standard Borel subalgebra, there exists a quantum group analogue $B^{\mathbf{c},\mathbf{s}} = B^{\mathbf{c},\mathbf{s}}(\theta)$ of the universal enveloping algebra $U(\mathfrak{k})$, which is a left coideal subalgebra of the Drinfeld-Jimbo quantized universal enveloping algebra $U_q(\mathfrak{g})$. In a suitable setting, the algebras $B^{\mathbf{c},\mathbf{s}}$ are in a bijection with Satake diagrams. In his talk, we focused on the case when $\mathfrak{g}$ is an untwisted affine Lie algebra of classical type: $\widehat{\mathfrak{sl}}_N$, $\widehat{\mathfrak{so}}_N$ or $\widehat{\mathfrak{sp}}_N$. In this setting the natural (vector) representation of $U_q(\mathfrak{g})$ on the N-dimensional vector space is also an irreducible representation of its coideal subalgebras $B^{\mathbf{c},\mathbf{s}}$ indexed by affine Satake diagrams. We explain a method to obtain solutions of the Sklyanin's twisted and untwisted reflection equations. In this approach reflection matrices are defined as the intertwiners of the natural representation of $U_q(\mathfrak{g})$ restricted to a given subalgebra $B^{\mathbf{c},\mathbf{s}}$. In particular, he described a class of so-called quasistandard reflection matrices, that have elegant representation-theoretic properties.

Simon Wood. *The rationality of $N = 1$ minimal models through symmetric polynomials.*
Let $V(c)$ be the universal $N = 1$ Virasoro vertex algebra at central charge $c \in \mathbb{C}$. Unless

$$c = c_{p_+,p_-} = \frac{3}{2} - 3\frac{(p_+ - p_-)^2}{p_+ p_-},$$

for some $p_+, p_- \geq 2$, $\gcd\{\frac{p_- - p_+}{2}, p_+\}$, the vertex algebra $V(c)$ is simple as a module over itself. However, $V(c_{p_+,p_-})$ is not simple and has a unique non-trivial proper ideal I_{p_+,p_-} generated by a singular vector η of conformal weight $\frac{1}{2}(p_+ - 1)(p_- - 1)$. The quotient vertex algebras $M(p_+, p_-) = V(c_{p_+,p_-})/I_{p_+,p_-}$ are called the $N = 1$ minimal model vertex algebras and they are rational, that is, c_2-cofinite and all their modules are semi-simple. The purpose of this talk is to sketch a new proof of the module classification of $M(p_+, p_-)$.

The main idea is to invoke Zhu's algebra. The Zhu algebra $A(V)$ of a vertex algebra V is the associative algebra of zero modes of V acting on vectors (in any module) that are annihilated by modes which lower conformal weight. The simple V-modules are in a 1-1 correspondence with the simple $A(V)$-modules.[1] Additionally the Zhu algebra $A(V)$ of a vertex algebra V can be constructed as a vector space quotient of the vertex algebra.

The image $A(I)$ of an ideal $I \subset V$ is an ideal of the Zhu algebra. In particular, $A(V(c_{p_+,p_-})) \cong \mathbb{C}[T]$ where T is the image of the conformal vector and the ideal is generated by the image of singular vector η. This image must therefore be some polynomial $f(T)$, which yields the presentation $A(M(p_+, p_-)) \cong \mathbb{C}/\langle f(T) \rangle$. The allowed conformal weights of highest weight vectors for the $N = 1$ minimal model vertex algebra are therefore the roots of $f(T)$.

In practice it is very hard to determine the explicit formulae for singular vectors which would be required for computing $f(T)$. This problem can be neatly circumvented by using free field realisations to construct the

[1] Some mild qualifiers need to be added to make this statement fully rigorous, but they shall not concern us here.

singular vectors as module homomorphism images of certain highest weight vectors. In particular, these free field realisations yield formulae for singular vectors in terms of the combinatorial data of symmetric functions which can also be used to evaluate the action of the zero modes of singular vectors and thereby determine the roots of $f(T)$.

Anton M. Zeitlin. *Towards the continuous Kazhdan-Lusztig correspondence.*

Recently, continuous series of unitary representations of $U_q(sl(2,\mathbb{R}))$ closed under the tensor product were introduced, thus generating the "continuous" tensor category. These representations, known as "modular double" representations, extensively studied by L. Faddeev, K. Schmüdgen and J. Teschner, are constructed via the unitary representations of the quantum plane and do not have the classical limit. The tensor category of representations of the quantum plane is quite subtle and is related to the properties of the quantum dilogarithm. However, it was shown by I. Ip that one can obtain the tensor category of $ax+b$-group, the affine group of a line (below it is denoted as G) as a classical limit of the tensor category of the quantum plane. Also, the basic structures from the quantum Teichmüller theory can be derived from the tensor products of the representations of the quantum plane, which is very important for the proper understanding of Chern-Simons theory associated with $SL(2,\mathbb{R})$ (see below for more details).

An important problem to think about is the construction of the affine analogues of the aforementioned continuous tensor categories. This can lead to generalization the Kazhdan-Lusztig equivalence of categories to the case of the quantum plane and $U_q(sl(2,\mathbb{R}))$. In other words, we propose that there exists a braided tensor category of representations of the affine algebra $\widehat{sl}_k(2,\mathbb{R})$, equivalent to the braided tensor category of the mentioned above continuous series of $U_q(sl(2,\mathbb{R}))$. One fact supporting this conjecture is that there exists a category of representations of the Virasoro algebra related to the Liouville theory, which is equivalent to the mentioned category of representations of the Virasoro algebra. Therefore, the corresponding category of representations of the affine algebra $\widehat{sl}_k(2,\mathbb{R})$ is a missing piece in this equivalence of categories.

However, a first natural problem to solve is to find unitary representations for the $\widehat{ax+b}_k$, the loop version of the $ax+b$-group with central extension (with the central charge k) for the a-subgroup and compare them and their tensor structure with the representation theory of the quantum plane.

One can construct the unitary representations of $\hat{G}$, which naturally generalize the irreducible representations of G. The corresponding representation space is the L^2 space with respect to the Wiener measure. These representations of $\hat{G}$ are labeled by a certain function. It is proven that certain representations in this class (e.g. when this function is constant) are irreducible.

It is known that the unitary representations of G are closed under the tensor product and there are three types of simple objects in the category of unitary representations of G. It appears that their tensor products decompose as the direct integrals of these "simple" objects. Using the principles discussed in the beginning of this section, one can hope to have the braided tensor category for $\hat{G}$, where the braiding is related to the value of the central charge. I also hope to obtain a differential equation governing the intertwining operators, i.e. the analogue of the Knizhnik-Zamolodchikov equation.

However, the results related to the $\widehat{ax+b}_k$-group are only a part of the main task, namely the construction of the continuous series of unitary representations of $\widehat{sl}_k(2,\mathbb{R})$.

Unfortunately, the standard approach of inducing representation of $\widehat{sl}_k(2,\mathbb{R})$ would not fit the construction, since the resulting modules appear to be nonunitary. Together with Igor Frenkel we used the results obtained for $\hat{G}$ to construct new modules for $\widehat{sl}_k(2,\mathbb{R})$ by means of the Wakimoto-type formalism, using the "currents" corresponding to the Lie algebra elements of $\widehat{ax+b}_k$ and the infinite dimensional Heisenberg algebra free fields. It turned out that the correlators of the resulting $\widehat{sl}_k(2,\mathbb{R})$-currents, defining the pairing in the representation diverge, and therefore we had to describe the scheme of eliminating those divergencies. This led to a very interesting graphical formalism, similar to Feynman diagram technique, where divergences corresponded to 1-loop graphs. The regularization scheme involved dependence on the infinite family of parameters: one parameter for each loop with a given number of vertices. One can generalize this construction to the higher rank case, using the similar procedure, that works for the quantum groups, introduced by I. Frenkel and I. Ip. A necessary question which immediately can be asked is as follows: for which values of regularization parameters the resulting representations will be unitary? Answering this question will involve studying the resulting bilinear form and analyzing it, using the graph formalism, that was a cornerstone of the definition of those representations. However, we do believe that finding the representations forming a braided tensor category will single out the necessary family of unitary representations. Therefore the next question to ask is: what are the intertwining operators for a suggested tensor category? One of the ways of doing that is to construct them in a similar fashion as in the Virasoro case given in the papers of J. Teschner.

Currently, the study of Chern-Simons theories with noncompact gauge groups became very important from the point of view of both mathematics and physics. In particular, it was argued since the late 80s that the canonical quantization of $SL(2,\mathbb{R})$ Chern-Simons theory is connected with both the quantum Teichmüller theory, which is related to the representation theory of the quantum plane, and Liouville theory, related to representation theory of $U_q(sl(2,\mathbb{R}))$. It is known that Liouville theory can be obtained from the $SL(2,\mathbb{R})$ WZW theory by means of the Drinfeld-Sokolov reduction. At the same time, the study of Chern-Simons theory with compact gauge group G showed, that its space of states in the presence of Wilson lines is isomorphic to the space of conformal blocks of the WZW model associated with G. Therefore, we may expect that this rule works in the noncompact case too, and the construction of the continuous series of unitary representations of $\widehat{sl}_k(2,\mathbb{R})$ will provide an important link between the quantum Teichmüller theory, $SL(2,\mathbb{R})$ Chern-Simons, $SL(2,\mathbb{R})$ WZW model and its reduction, the Liouville theory.

Yaping Yang. *Cohomological Hall algebras and affine quantum groups.*

It is a classical theorem of Ringel that the Hall algebra of a quiver Q is isomorphic to the positive half of the quantum enveloping algebra of the Lie algebra associated to Q. In this talk, I will talk about an affine analogue of the above theorem.

The goal of my talk is to give a geometric construction of the affine quantum groups: Yangian, quantum loop algebra and the elliptic quantum groups. For each quiver Q and each cohomology theory A, I will introduce the cohomological Hall algebra (CoHA), as the A-cohomology of the moduli of representations of the preprojective algebra of Q. This generalizes the K-theoretic Hall algebra of commuting varieties defined by Schiffmann-Vasserot. I will describe a family of representations of this CoHA coming from A-homology of Nakajima quiver varieties.

When A is the usual cohomology, the (extended) preprojective CoHA is isomorphic to the Borel subalgebra of Yangian, which is a deformation of the current algebra of the Lie algebra of Q. The preprojective CoHA action on Nakajima quiver varieties is compatible with the actions of Yangian constructed by Nakajima, Varagnolo and Maulik-Okounkov. When A is the K-theory and elliptic cohomology, the (extended) preprojective CoHA is expected to be the Borel subalgebra of quantum affine algebra and elliptic quantum group respectively.

The construction of CoHA works for any cohomology theory. In particular, we obtain new affine quantum groups corresponding to arbitrary cohomology theories.

This talk is based on my joint work with Gufang Zhao.

Alex Weekes. *Highest weights for some algebras constructed from Yangians.*

There is a philosophy, as outlined for example by Braden, Licata, Proudfoot and Webster, that one can do "Lie theory" for quite general Poisson varieties. Here, the role of the enveloping algebra of a Lie algebra is played by a deformation quantization of the algebra of functions on our variety. There are interesting notions of highest weight theory and of a category O.

In this talk we describe work on such a theory for algebras called truncated shifted Yangians. These algebras conjecturally quantize slices to Schubert varieties in the affine Grassmannian. We outline the construction of these algebras, the combinatorics of their highest weight theory, and some connections to the geometry of Nakajima quiver varieties. This is joint work with Joel Kamnitzer, Peter Tingley, Ben Webster and Oded Yacobi.

Jethro van Ekeren. *Fusion for principal W-algebras.*

I describe joint work with T. Arakawa in which we determine modular properties of characters of certain non-C2 vertex algebras, and apply this to establish the fusion rules of principal affine W-algebras originally computed by Frenkel, Kac and Wakimoto.

Xiao He. *Reduction by stages for affine W algebras.*

We are considering the reduction by stages in the affine W-algebras level. In the finite W-algebra case, as the finite W-algebras are the quantizations of the Slodowy slices and there are reduction by stages for Slodowy slices, so sometimes we can lift this reduction by stages procedure to the algebraic level. The affine W-algebras are quantizations of the Arc spaces of the Slodowy slices and we still have the reduction by stages for these Arc spaces of Slodowy slices, so we believe that in some cases we still can lift this reduction by stages procedure to the algebraic level. This kind of reduction by stages can relates the W-algebras more closely and help us to understand better the structures also the representations of W-algebras.

Open Problems Session

Shashank Kanade.

In a joint paper with Matthew C. Russell, we found six new conjectural partition identities of Rogers-Ramanujan-Capparelli type. Of those six conjectures, three are directly related to the representation theory of affine Kac-Moody algebras, and the rest may not have much to do with the representation theory. Below, I'll describe the first three of our conjectures.

Given a non-negative integer n, we say that $n = l_1 + l_2 + \cdots + l_k$ is a partition of n if each l_i is a positive integer with $l_1 \geq l_2 \geq \cdots \geq l_k$. We say that a partition satisfies Condition * if $l_i \geq l_{i+2} + 3$ for all i and if $l_i - l_{i+1} \in \{0, 1\}$ implies that $3|l_i + l_{i+1}$.

Here are the conjectures. For all n the following three identities hold.

1. Partitions of n in which each part is congruent to $\pm 1, \pm 3$ mod 9 are equinumerous with the partitions of n that satisfy Condition *.

2. Partitions of n in which each part is congruent to $\pm 2, \pm 3$ mod 9 are equinumerous with the partitions of n that satisfy Condition * and moreover have their smallest part at least 2.

3. Partitions of n in which each part is congruent to $\pm 3, \pm 4$ mod 9 are equinumerous with the partitions of n that satisfy Condition * and moreover have their smallest part at least 3.

These identities have been verified up to partitions of $n = 1500$.

Consider the affine Lie algebra $g = D_4^{(3)}$. It has three integrable modules of level 3. It is known using the Weyl-Kac character formula and the Lepowsky-Milne numerator formula that the principally specialized characters of these modules are closely related to the q-series one obtains by finding the generating functions of partitions that satisfy the mod 9 congruence conditions. Let $F = \prod_{m \equiv \pm 1 (\mathrm{mod} 6), m \geq 1} (1 - q^m)^{-1}$. Then:

$$\begin{aligned}
\chi(L(\Lambda_0 + \Lambda_1)) &= F \cdot \prod_{m \equiv \pm 1, \pm 3 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1} \\
\chi(L(3\Lambda_0)) &= F \cdot \prod_{m \equiv \pm 2, \pm 3 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1} \\
\chi(L(\Lambda_2)) &= F \cdot \prod_{m \equiv \pm 3, \pm 4 (\mathrm{mod} 9), m \geq 1} (1 - q^m)^{-1}.
\end{aligned}$$

The remaining 3 identities don't have symmetric congruence conditions and therefore may not be directly related to the representation theory of affine Kac-Moody algebras.

Further details can be found in paper [KR].

Anton M. Zeitlin.

It is known that the quantum groups for the split reductive real Lie algebras have representations via positive unbounded self-adjoint operators in Hilbert space. These are known as modular double representations (results of L. Faddeev, J. Teschner, I.B. Frenkel and I. Ip). In the case of sl(2,R) it is known that they form a continuous braided tensor category: tensor product of two representations labeled by the continuous parameter decompose as a direct integral of such representations. It is also known that in this particular case there is an equivalent braided tensor category for corresponding Virasoro algebra (arising from the so-called Liouville theory). At the same time, in the paper of I.B. Frenkel and A.M. Zeitlin, the category of representations (continuous series) for affine sl(2,R) is proposed, which is expected to give rise to the braided tensor category for affine sl(2,R), equivalent to the above two. There is also a relation of those tensor categories to SL(2,R) Chern-Simons theory and quantum Teichmueller spaces.

Some open problems are:

1) Establish the correspondence between the proposed three categories of sl(2,R).

2) Construct the tensor product decompositions for modular double representations of quantum groups of higher rank.

3) Construct the equivalent to 2) braided tensor categories for W-algebras and corresponding affine algebras.

4) Find the relation of 2), 3) to Chern-Simons theories for higher rank real reductive Lie groups and higher Teichmueller theory.

Moonshine Seminar

After dark, the workshop participants met for an informal "Moonshine Seminar" with the following talks:

Fedor Malikov. *Introduction to chiral de Rham complex.*

Terry Gannon. *Mathieu Moonshine.*

Thomas Creutzig. *Supernatural VOA.*

The seminar focused on the following

Mathieu Moonshine Conjecture. Let V be the sheaf cohomology of the chiral de Rham complex over a K3 surface. Then V is a super vertex operator algebra admitting the action of the Mathieu group M_{24} by automorphisms. The action of M_{24} fixes an $N = 4$ superconformal subalgebra in V.

Elliptic genus as a topological invariant of manifolds was introduced by Ochanine [O] and Witten [W]. Eguchi, Ooguri and Tachikawa conjectured in [EOT] that for K3 surfaces elliptic genus admits the action of Mathieu M_{24} group, fixing the $N = 4$ superconformal subalgebra. Recently Bailin Song [S] showed that H^0 of the chiral de Rham complex over a Kummer surface is the $N = 4$ superconformal vertex algebra. Terry Gannon showed in [25] that twisted elliptic genus of a K3 surface does decompose into an infinite sum of characters of M_{24}.

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Chapter 5

Synchronizing smooth and topological 4-manifolds (16w5145)

February 21 - 26, 2016

Organizer(s): Matthew Hedden (Michigan State University), Jennifer Hom (Georgia Tech), Kent Orr (Indiana University), Mark Powell (UQAM)

Overview of the Field

Much is still unknown about 4-dimensional manifolds. Indeed, while the 3-dimensional Poincaré conjecture was settled in the affirmative by Perelman at the turn of the 21st century, the smooth 4-dimensional Poincaré conjecture remains open. Furthermore, the question of classifying the set of smooth structures on a 4-manifold remains a mystery; there is no smooth 4-manifold for which the set of smooth structures is completely known. Knot and link concordance has proved to be a powerful tool for understanding 4-manifolds; for example, any topologically but not smoothly slice knot gives rise to an exotic $\mathbb{R}^4$.

A primary goal of this workshop was to bring together researchers in smooth and topological 4-manifolds in order to facilitate interactions between experts in both areas. The list of invited participants included both well-established senior experts in the field together with rising new talent, including a large number of postdocs and graduate students. We made a particular effort to ensure that all of the young mathematicians present were given the opportunity to present.

From the smooth perspective, Heegaard Floer homology has been particularly successful in yielding new insights into concordance. Several new invariants and results were presented at this workshop, particularly the talks of Feller, Hendricks, Levine, Sato, Stipsicz, and Wang. In particular, the Upsilon invariant of Ozsváth-Stipsicz-Szabó has yielded many new results (e.g. [1], [2], [3]) and the $\underline{V}_0$ and $\overline{V}_0$ invariants of Hendricks-Manolescu seem well-positioned to yield new results as well.

The global structure of the knot concordance group can be understood by considering geometric filtrations related to gropes and Whitney towers, and the closely related solvable filtration. After a survey on this theory by Jae Choon Cha, recent progress on our understanding of these filtrations was presented by Taehee Kim, relating to a filtration of concordance classes of knots that models the doubly slice question, and Chris Davis presented work on the difference between 0.5 and 1-solvability. Slava Krushkal presented recent progress on the surgery conjecture in 4-manifolds. This conjecture is key to understanding topological manifolds with arbitrary fundamental groups, and the techniques used to study it have strong connections to link concordance.

David Gay, Rob Kirby, Jeff Meier, Juanita Pinzón Caicedo, and Alex Zupan made up a strong contingent of experts in the recently blossoming theory of trisections of 4-manifolds. This was an opportunity, through a series of excellent talks, for many not directly involved with the development of the theory to learn about recent advances, and for the practitioners to get together. This is an exciting theory which enables one to understand 4-manifolds in terms of curves on surfaces, and it is hoped that it will enable us to attack some of the hard problems in 4-manifold topology that have hitherto proved intractable.

The conference helped fuel a great deal of intra- and inter- action involving those currently thinking about these different aspects of knot concordance and 4-manifolds.

Presentations

- **Jae Choon Cha** opened the workshop with an excellent overview of topological concordance of knots and links. He discussed disk embedding in dimension 4 and filtrations arising from gropes and Whitney towers. His talk also featured numerous open problems (see Section 5), which helped to kick off a week of fruitful interactions and discussions amongst the participants.

- **Christopher Davis** spoke on 1-solvability and genus one algebraically slice knots; this is joint with Taylor Martin, Carolyn Otto, and Jung Hwan Park. In the 1990's Cochran Orr and Teichner introduced a filtration of knot concordance indexed by half integers (the solvable filtration.) Since then this filtration has been a convenient setting for many advances in knot concordance. There are now many results in the literature demonstrating the difference between the nth and $(n.5)$th terms in this filtration, but none regarding the difference between the $(n.5)$th and $(n+1)$st. Davis, Martin, Otto, and Park prove that every genus one (0.5)-solvable knot is 1-solvable. They also provide a new sufficient condition for a high genus (0.5)-solvable knot to be 1-solvable and give some possible candidates for knots which are (0.5)-solvable but not 1-solvable.

- **Andrew Donald** spoke on a slicing obstruction from the 10/8 theorem. This is joint with Faramarz Vafaee. A smooth knot slicing obstruction can be derived from Furuta's 10/8 theorem using 0-surgery on knots. They show that this detects torsion elements in the concordance group and can be used to find topologically slice knots which are not smoothly slice.

- **Peter Feller** spoke on joint work with David Krcatovich in which they use the Upsilon invariant to provide bounds on cobordisms between knots that "contain full-twists". They recover and generalize a classical consequence of the Morton-Franks-Williams inequality for knots: positive braids that contain a full twist realize the braid index of their closure. They also provide inductive formulas for the Upsilon invariants of torus knots and compare the Upsilon function to the Levine-Tristram signature profile.

- **Stefan Friedl** spoke on a conjectural "if and only if criterion" for topological concordance to the unknot and the Hopf link; the conjecture was originally stated by Peter Teichner and Friedl in 2004. In his talk, Friedl provided evidence for the conjecture and reported on rather preliminary work with Patrick Orson on extending this conjecture to the concordance to the Hopf link.

- **Kristen Hendricks** gave a talk on involutive Heegaard Floer homology, in which she and Ciprian Manolescu use the conjugation symmetry on the Heegaard Floer complexes to define a three-manifold invariant. Within this package of invariants are two new invariants of homology cobordism, one of which (unlike other invariants arising from Heegaard Floer homology) detects non-sliceness of the figure-eight knot. These homology cobordism invariants give rise to knot concordance invariants by considering surgery along the knot; the computation of these knot invariants depends on the knot Floer complex CFK^∞, together with an endomorphism ι_K.

- **Francesco Lin** gave a talk on Pin(2) monopole Floer homology, the Morse-theoretic analogue of Manolescu's Pin(2)-equivariant Seiberg-Witten-Floer homolog. It can be used to provide an alternative disproof of the longstanding Triangulation Conjecture. He also discussed some computational tools for the theory.

- **Taehee Kim** spoke on joint work with Jae Choon Cha on unknotted gropes and Whitney towers in 4-space. Gropes and Whitney towers are primary tools for studying 4-dimensional topology. As an effort to understand gropes and Whitney towers via the structure of their complements, they introduce notions of unknotted gropes/Whitney towers in 4-space. This is motivated by Freeman's result that an embedded 2-sphere in 4-space is topologically unknotted if its complement has infinite cyclic fundamental group. As an application, they establish grope and Whitney tower bi-filtrations of knots in 3-space by taking a slice of unknotted gropes/Whitney towers. Using the amenable signature theorem by Cha, which is based on the work of Cha and Orr, they prove that these bi-filtrations have rich structures.

- **Slave Krushkal** gave two talks, the first on 1/2 -π_1-null surgery kernels, and the second on a homotopy$^+$ solution to the A-B slice problem, both joint work with Mike Freedman. It has been known for a long time that π_1-null surgery kernel imply surgery (pg 94 Freedman-Quinn book) and also that weaker grope based kernels are"universa" for surgery (if you can solve these problems, you can solve all unobstructed surgery problems.) Freedman and Krushkal have shown that a kind of kernel"half way betwee" the two is still universal.

  Four-dimensional surgery is a fundamental technique underlying geometric classification results for topological 4-manifolds. It is known to work in the topological category for a class of "good" fundamental groups. This result was originally established in the simply-connected case by Freedman in 1981, and it is currently known to hold for groups of subexponential growth and a somewhat larger class generated by these. The A-B slice problem is a reformulation of the surgery conjecture for free groups, which is the most difficult case. In this talk Krushkal showed that the A-B slice problem admits a link-homotopy+ solution. The proof relies on geometric applications of the group-theoretic 2-Engel relation. He also discussed implications for the surgery conjecture.

- **Adam Levine** spoke on satellite operators and piecewise-linear concordance. He shows that there exists a knot in a homology sphere Y, which is the boundary of a contractible 4-manifolds, such that K does not bound a piecewise-linear disk in any homology 4-ball bounded by Y. His proof relies on a computation of the concordance invariants τ and ε using bordered Floer homology, which shows that a certain satellite operator does not induce a surjection on the knot concordance group.

- **Lukas Lewark** spoke on joint work with Peter Feller on upper bounds for the topological slice genus of knots. In 1981, Freedman proved that knots with trivial Alexander polynomial bound a locally flat disc in the four-ball. As a consequence, Feller showed that the degree of the Alexander polynomial constitutes an upper bound for the topological slice genus of a knot. Lewark and Feller proved a stronger bound, which is still determined solely by the knot's Seifert form. Their work leads to upper bounds for the slice genus of torus knots (Baader, Feller, Lewark, Liechti) and two-bridge knots (Feller, Mccoy), and for the stable slice genus of alternating knots (Baader, Lewark).

- **Andrew Lobb** gave a talk on Khovanov-Rozansky smooth sliceness obstructions; this is joint work with Lukas Lewark. Rasmussen's invariant from perturbed Khovanov cohomology is a concordance homomorphism to the integers which also gives a lower bound on the smooth slice genus. Khovanov-Rozansky $sl(n)$ cohomology generalizes Khovanov cohomology (which appears as the case $n = 2$) and perturbations of it give rise both to a slew of concordance homomorphisms which are also lower bounds as well as to lower bounds which are not equivalent to concordance homomorphisms. For the case $n = 2$ there is essentially only one perturbation, while already perturbations of the case $n = 3$ exhibit complicated behavior.

- **Jeff Meier**, **Juanita Pinzón Caicedo**, and **Alex Zupan** all gave talks on trisections of 4-manifolds. Pinzón Caicedo spoke on joint work with Nick Castro in which they develop a definition of relative trisections for 4-manifolds with boundary, and prove a uniqueness result in terms of stabilizations. Meier spoke on joint work with Zupan, where they adapt the theory of trisections to the relative setting of knotted surfaces in the four-sphere. Their theory serves as a four-dimensional analogue to bridge splittings of classical knots and links: every such surface admits a decomposition into three standard pieces called a bridge trisection. Zupan spoke on joint work with Meier and Trent Schirmer. They show that a given link has Stable Generalized Property R if and only if a certain infinite family of induced trisections is nonstandard.

- **Matthias Nagel** spoke on unlinking information from 4-manifolds. He explained how to obtain lower bounds on unlinking numbers through 4-manifold techniques using a generalization of a theorem of Cochran-Lickorish. He demonstrated the method using links from Kohn's table whose unlinking numbers have only recently been determined through these methods.

- **Daniel Ruberman** spoke on two results related to 4-manifolds with boundary. The first, joint with Dave Auckly, Hee Jung Kim, and Paul Melvin, is a construction of diffeomorphisms of finite order on the boundary of certain contractible manifolds that change their smooth structure relative to the boundary. Tange has recently announced a similar result. They show in fact that for any finite group G acting on the 3-sphere, there is a G-action on the boundary of a contractible manifold, such that every element changes the smooth structure relative to the boundary. Their construction initially produces reducible boundaries, and then they show how to make these hyperbolic. The second set of results, joint with Arunima Ray, is concerned with two analogues of Dehn's lemma for 4-manifolds. They give examples of a reducible 3-manifold Y bounding a 4-manifold W that does not split smoothly as a boundary-connected sum, even though the reducing sphere in Y is null-homotopic in W. By a different construction, they find a contractible 4-manifold W with boundary a 3-manifold Y containing an essential torus that doesn't bound (smoothly, in one version; topologically in another version) a solid torus in W.

- **Kouki Sato** gave a talk on Heegaard Floer correction terms of 1-surgeries along $(2, q)$-cablings. The Heegaard Floer correction term (d-invariant) is an invariant of rational homology 3-spheres equipped with a Spin^c structure. In particular, the correction term of 1-surgeries along knots in the 3-sphere is a ($2\mathbb{Z}$-valued) knot concordance invariant d_1. In this work, Sato estimates d_1 for the $(2, q)$-cable of any knot K. This estimate does not depend on the knot type of K. If K belongs to a certain class which contains all negative knots, then equality holds. By using this estimate, Sato obtain two corollaries. One of the corollaries shows that the relationship between d_1 and the Heegaard Floer tau invariant is very weak in general. The other one gives infinitely many knots which cannot be unknotted either by only positive crossing changes or by only negative crossing changes.

- **Minkyoung Song** gave a talk on invariants and structures of the homology cobordism group of homology cylinders. The homology cobordism group of homology cylinders is enlargement of both the mapping class group and the concordance group of string links in homology $D^2 \times I$. Song studies the structure of the group via a filtration of extended Milnor invariants combined with Johnson homomorphisms, and also obtains deeper information invisible to previously known invariants by employing Hirzebruch-type intersection form defect invariants

- **Laura Starkston** spoke on line arrangements in the topological, smooth, and symplectic categories. This is joint work with Danny Ruberman. A complex line arrangement is a collection of complex projective lines in $\mathbb{CP}^2$ which may intersect at points of multiplicity greater than two. The combinatorial arrangements which can be geometrically realized and their space of realizations have been studied classically. They define symplectic, smooth, and topological versions of complex line arrangements in $\mathbb{CP}^2$, and studied their realizability. While one might hope that these more flexible categories allow us to realize any combinatorics, they showed that there are obstructions to topological realizations of many combinatorial arrangements. Many open questions remain about realizability in different categories.

- **András Stipsicz** gave a talk on the Upsilon invariant, which provides a homomorphism from the smooth knot concordance group to the group of piecewise-linear functions from $[0, 2]$ to $\mathbb{R}$. The invariant comes from applying a 1-parameter family of linear transformations to the knot Floer bifiltered chain complex CFK^∞.

- **Shida Wang** spoke on semigroups of iterated torus knots and the Upsilon invariant. He discussed the usage of semigroups and some subtleties in the computation of the Upsilon invariant for torus knots. He also gave some nontriviality results on the kernel of the Upsilon invariant.

Recent Developments and Open Problems

The following open problems were discussed at the workshop.

Jae Choon Cha's talk

1. The π_1-null disc lemma for free groups, or perhaps for amenable groups.

2. Are good boundary links freely slice?

3. Whitehead double conjecture: A Whitehead double of a link is freely slice if and only if the link is homotopically trivial.

4. Is the map from $\mathcal{C}$ to the concordance of knots in $\mathbb{Z}$-homology 3-spheres an injection?
5. Develop a homology surgery machinery in dimension 4.
6. Construct higher order invariants of boundary links beyond those determined by the map to the free group.
7. Is there finite order in $\mathcal{C}$ other than the order 2 elements given by negative amphichiral knots?
8. Do all algebraic order 4 knots have infinite order in the knot concordance group?
9. Can the amenable signature theorem be used for non-solvable groups to obstruct slicing, or perhaps bounding certain types of Whitney towers?
10. Is the intersection of the grope, Whitney tower or n-solvable filtration trivial?
11. Does homology cobordism of zero surgeries preserving the homotopy classes of the meridians imply concordance?

Jae Choon Cha problem session

1. Is it true that K slice if and only if $BD_n(K)$ is slice for all n? It is known that $BD_n(K)$ is $\mathbb{Q}$-slice (bounds a disc in a rational homology 4-ball). Then $BD_{n-1}(K)$ is rationally slice. Thus $K\#K^r$ is $\mathbb{Q}$-slice.
2. He drew a knot a bit like a twist knot with an a-fold clasp and $-a$ twists. With $a = 1$ it is the figure eight. Now let K_a be the $(2, 1)$ cable of that knot. It is known that K_a is $\mathbb{Q}$-slice. The answer is probably no because Miyazaki, using Casson and Gordon, showed that K_1 is not homotopy ribbon.

   Also, M_{K_a} is 0-surgery cobordant to $S^1 \times S^2$ via W such that $H_1(W) = \mathbb{Z} \oplus \mathbb{Z}_2$ and $H_2(W) = \mathbb{Z}$. The meridians are not homotopic. When you find a new concordance invariant please compute it on a couple of these knots and email Jae Choon.

Jeff Meier

1. Is there a unique surface with bridge trisection number 3?
2. Is a knot of a connect sum of n copies of $\mathbb{RP}^2$s in S^4 with $\pi_1 = \mathbb{Z}/2$ unknotted i.e. (topologically) equivalent to the standard embedding of such a surface?

Mark Powell

1. Are the iterated graded quotients of the bipolar filtration of topologically slice knots non trivial, or even better of infinite rank?

Adam Levine

1. If K_1 is concordant to K_2, then for all n, $S^3_n(K_1)$ is homology cobordant to $S^3_n(K_2)$. Is the converse true?

   Note that for all n, there exists K_1 and K_2 such that $S^3_n(K_1)$ is homology cobordant to $S^3_n(K_2)$ but K_1 and K_2 are not concordant. (Cochran-Franklin-Hedden-Horn for $n = 0$). But you have to change the knots for different n. Challenge: find a single pair that works for multiple n.

   Let P be a winding number 1 satellite operator. Let $P(K, t)$ be the t-twisted satellite of K. Suppose that $P(U, -n)$ is slice. Then for any knot K, $S^3_n(K)$ is homology cobordant to $S^3_n(P(K))$.
2. Can we find P such that $P(U, -n)$ is slice for all n, but $P \cup \eta$ (where η is the meridian of the solid torus in which P sits) not concordant to the Hopf link?

Matt Hedden

1. Same question as Adam for branched covers of knots. If all the branched covers are homology cobordant then are the knots concordant?

2. Follow up from Kent (see below, Matt stood up once after Adam and once after Kent). Does there exist a P in a solid torus such that $P\colon \mathcal{C} \to \mathcal{C}$ is a non-trivial homomorphism, not equal to the identity nor to zero? He would conjecture no.

Kent Orr

1. Related to work of Davis and Ray. Rough question: can one put a structure on the "family of satellite operators" so that $\mathcal{C}^{top}$ is a "module" over this structure. One has to also consider string links, by the work of John Burke and Diego Vela.

2. Cochran-Harvey-Leidy. Fractal structure conjecture. The primary decomposition of the iterated quotients of the n-solvable filtration gives evidence. Can we define interesting metrics that enable us to make the fractal structure precise? Does the family of satellite operators have a fractal structure?

Danny Ruberman

1. Let Y be an integral homology 3-sphere that bounds a contractible 4-manifold W, i.e., $\partial W = Y$. Suppose that K in Y is a knot. Does K bound an embedded disc in W? (No locally flat requirement.)

   Recall the proof that any homology sphere bounds a contractible 4-manifold. Take a product $Y \times I$ then do surgery to make it a homology cobordism with $\pi_1 = 0$. Stack infinitely many and then use the proper h-cobordism theorem to recognise the end as $S^3 \times \mathbb{R}$ and add in a point.

2. Given (Y, K) as above, does there exist a 1-connected h-cobordism X from Y to Y containing a concordance from K to K?

3. Real version. Suppose Y has a $\mathbb{Z}_n$ action with fixed set a knot K. Does the $\mathbb{Z}_n$ action extend over $W \simeq *$? (Yes if $\mathbb{Z}_n$ acts freely.)

4. (Also from Adam Levine's talk.) Is every knot in an integral homology sphere topologically concordant in a homology cobordism to a knot in S^3?

Shida Wang

1. In his talk, Shida Wang conjectured that L-space iterated torus knots are the only L-space knots whose set of exponents of the Alexander function are closed under addition. Based on conversations with David Krcatovich, Wang tried more potential counterexamples to his conjecture.

Slava Krushkal

1. Are Whitehead doubles of all homotopically trivial links are topologically slice?

Scientific Progress Made

- Dave Gay and Rob Kirby finished the last piece of a puzzle and proved that a trisection of a finitely presented group determines a unique smooth, closed, oriented 4-manifold, in analogy with Heegaard splittings which can be regarded as a bisection of a 3-manifold group. This was a question that Dave Gay asked at an open problems session, and Gay writes that "the initial burst of discussion that followed from me asking the question really helped push us to figure this out. One nice nugget of a corollary is that the smooth 4-dimensional Poincare conjecture can now be stated as a purely group theoretic question."

- Alexander Zupan and Jeff Meier clarified a picture of a potentially non- standard trisection of S^4 while working late into the night at Banff. Meier also began a correspondence with Kouki Sato regarding the calculation of correction terms for certain 3-manifolds obtained as surgery on links of two components.

- Paul Melvin and Danny Ruberman found time to discuss their work on equivariant corks, and to make progress on sharpening the results that Ruberman spoke about in his lecture. Laura Starkston and Ruberman spent considerable time finishing the details of a 'complexification' result for real pseudoline configurations; this was part of Starkston's lecture at the conference. They were able to improve their result to show that a topological configuration of circles in the real projective plane can be isotoped so that it is the intersection of the real projective plane with a collection of symplectic 2-spheres in the complex projective plane.
- Lukas Lewark and Andrew Lobb benefitted in particular from the discussions of the Upsilon invariant, which have influenced their current research.
- The work in Shida Wang's talk was posted on the arXiv [5] after he collected people's feedback in the workshop. After Shida Wang's talk, David Krcatovich asked how to determine whether a numerical semigroup arises from that of an iteratered torus knot. In their further discussion, David Krcatovich wondered whether the Upsilon function of any knot is a linear combination of that of L-space iterated torus knots. These problems are still under investigation and may begin a new collaborative project.
- Peter Feller and Lukas Lewark continued on our project about upper bounds for the topological slice genus.
- Kristen Hendricks and Jen Hom conjectured a formula for the behavior of ι_K under connected sum. Further discussions after the workshop between Hendricks and Ian Zemke refined this conjecture, and Zemke is currently writing up a proof of the conjecture.
- Francesco Lin benefitted from discussions with Matt Hedden and Mark Powell, resulting in the [4], where he studies the behavior of Manolescu's correction terms under certain Dehn surgeries, with applications to homology cobordism, Seifert fibered surgeries and concordance invariants.

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Chapter 6

Gauge/Gravity Duality and Condensed Matter Physics (16w5067)

February 28 - March 4, 2016

Organizer(s): Johanna Erdmenger (Max Planck Institute, Munich), Moshe Rozali (University of British Columbia)

Overview

String theory is a promising candidate for a unified theory encompassing both quantum physics and gravity. Recently string theory has emerged as a more general mathematical and physical framework to describe physical phenomena in a much wider range of applications. This is due to the new concept of gauge/gravity duality which has been derived within string theory.

Gauge/gravity duality provides a map by which strongly coupled quantum field theories are related to classical gravity theories. This provides an exciting new approach for describing strongly coupled quantum systems, for which conventional descriptions are scarce since the standard approach of considering quasi-particle excitations in an expansion about non-interacting theories does not apply.

Strongly coupled systems are ubiquitous in condensed matter physics. They are present for instance in high-Tc superconductors, in quantum Hall systems, in systems that exhibit parity breaking and in non-equilibrium and time-dependent configurations.

One example of the systems to be studied are quenches of quantum systems which lead to a time evolution to a new configuration. Gauge/gravity duality maps this scenario for instance to collapsing matter configurations in gravity, which eventually lead to the formation of a black hole. This example illustrates how gauge/gravity duality leads to unexpected new connections between previously unrelated areas of theoretical physics.

The aim of the meeting is to bring together leading international experts from both gauge/gravity duality and condensed matter physics to explore how the new approach may be best adapted to suitable open questions in condensed matter physics. Conversely, it will be investigated how the needs for new approaches in condensed matter physics point at desirable generalizations of gauge/gravity duality and even at a better understanding of its mathematical foundations. This may lead to further insight into quantum gravity theories more generally.

Presentation Highlights

Following are summaries of some of the talks in the workshop and the issues they raise, including future directions of research.

Thermodynamics of polarized relativistic matter (Pavel Kovtun)

Understanding transport in matter subject to external fields requires that we understand its thermodynamics first. The talk described the free energy of equilibrium relativistic matter subject to external gravitational and electromagnetic fields, to one-derivative order in the gradients of the external fields. The free energy allows for a straightforward derivation of bound currents and bound momenta in equilibrium. At leading order, the energy-momentum tensor admits a simple expression in terms of the polarization tensor. At one-derivative order, bulk thermodynamics is characterized by eight scalar functions of the external fields in 2+1 dimensions, and by twenty-one scalar functions of the external fields in 3+1 dimensions.

A Precision Test of AdS/CFT with Flavor (Andreas Karch)

In the work Karch presented he and his collaborators (Brandon Robinson and Christoph Uhlemann, both also from the University of Washington) put AdS/CFT dualities involving probe branes to a precision test.

Such probe brane embeddings are the starting points of many constructions that embed interesting field theories, with potential applications to condensed matter physics questions, directly into string theory. Such "top-down" constructions have several advantages over the more ad-hoc bottom-up constructions that just postulate a gravitational action. In particular, the detailed dictionary between gravity side and field theory is known. Since both field theory and gravity dual are explicitly known, one can put the putative duality to stringent tests.

To perform this test, they constructed on the holographic side a new class of supersymmetric D7-brane embeddings into $AdS_5 \times S^5$, which allow to describe N=4 SYM coupled to massive N=2 supersymmetric flavors on S4. With these embeddings they compared holographic results to a field theory analysis of the free energy using supersymmetric localization. Exact agreement has been demonstrated.

The fluid manifesto: topological sigma models and dissipative hydrodynamics (Mukund Rangamani)

Mukund discussed the Wilsonian effective field theories in mixed states of a quantum field theory. The ideas was to take the Schwinger-Keldysh construction seriously and extract from the fundamental definition of the path integral a set of topological symmetries which are inherent in the construction. He demonstrate that the field redefinitions of the Schwinger-Keldysh construction lead to a pair of BRST charges. Furthermore, in thermal density matrices the KMS condition can be encoded into a second pair of BRST charges. Computing the algebra of the quartet of BRST charges he finds a structure that is well known in the topological field theory literature, called the extended equivariant cohomology algebra. The symmetry being gauged in this context is thermal diffeomorphisms, owing to the fact that the KMS condition relates operators separated along the thermal circle. A simple toy model which illustrates these structures is the linear dissipative system that is captured by Langevin dynamics.

The real pay-off comes when we turn to analyzing hydrodynamics. He shows that one can write a Landau-Ginzaburg like sigma model for hydrodynamics which incorporates the basic principles we have extracted from the underlyingSchwinger-Keldysh formalism. We argue that hydrodynamics should be viewed as a sigma model, with the low-energy variables being the Nambu-Goldstone bosons for broken global diffeomorphism (and flavour symmetries in presence of charges). Exploiting the topological symmetries unearthed in theSchwinger-Keldysh constructed he constrained the low energy theory effectively to obtain a clean action principle for dissipative hydrodynamics. He went on to show that the theory satisfies various constraints such as the second law of thermodynamics, by virtue of the Jarzynski work-energy relation appearing as a Ward identity of our symmetries. The key to the construction is to maintain manifest covariance with respect to the topological symmetries which involves introducing various ghost fields. These ideas, are expected to have a bearing on other areas especially physics of black holes, which via the fluid/gravity correspondence is related to hydrodynamics.

Stokes equations on black hole event horizons (Jerome Gauntlett)

In seeking applications of holography to real world systems, the thermoelectric DC conductivity is a very important quantity to focus on. Jerome showed, universally within holography, that the DC conductivity can be obtained by solving a generalised system of Stokes equations (time independent and linearised Navier-Stokes equations) on the horizon of the black hole spacetime that is dual to the field theory of interest. This is an exact results and does not rely on making any approximations, such as a hydrodynamic limit.

Jerome discussed how the equations can be solved in closed form for one-dimensional lattices. He also showed how when the disorder is weak, the equations can be solved perturbatively and obtained some universal results, including a kind of generalised Wiedemann-Franz Law that all holographic systems must satisfy.

A simple model of momentum relaxation in Lifshitz holography (Tomas Andrade)

Tomas expanded the holographic studies of momentum relaxation to include non-relativistic scaling symmetries in the ultraviolet. He did so by constructing black branes with Lifshitz asymptotics dressed with axions which explicitly depend on the boundary directions. Such configurations arise as analytic solutions of the Einstein-Proca theory coupled to massless scalar fields in arbitrary dimensions. Studying linear perturbations on these backgrounds, he conclude that there is a dual Ward identity which accounts for the dissipation of momentum in the system. In addition, he numerically compute the frequency dependent thermal conductivity of the branes and verify that its DC limit is finite.

Holography and the nature of strange metal entanglement (Jan Zaanen)

Zaanen reported on recent developments in main stream condensed matter where AdS/CMT inspired developments are slowly getting on steam. The highlight is discovery of hydrodynamical flows in various electron systems in the form of three back-to-back publications in Science in march 2016. Also the first indications of holographic strange metal behaviours have been seen in photoemission experiments. Zaanen stressed that the AdS/CMT community should focus on the influence of periodic potentials on holographic fermions since this has a substantial potential to impact greatly on empirical condensed matter physics.

Black Holes from Quantum Quenches (Julian Sonner)

The contradiction between black holes and local quantum field theory, manifested in the information paradox and related puzzles, is sharpest for transient black holes that form by collapse, slowly evaporate, and eventually disappear. The goal of Julian Sonners presentation was thus to describe a first-principles CFT calculation, drawing on techniques and ideas from condensed-matter physics, corresponding holographically to the spherical collapse of a shell of matter in three dimensional quantum gravity. It was shown how in field theory terms, one can analytically follow the equilibration process, from early times to thermalization, of a CFT in its groundstate with a sudden injection of energy at time t = 0. By formulating a continuum version of Zamolodchikovs monodromy method to calculate conformal blocks at large central charge c, a framework was outlined to compute a general class of probe observables in the collapse state, incorporating the full backreaction of matter fields on the dual geometry. The talk concluded with two illustrative applications: firstly calculating a scalar field two-point function at time-like separation and secondly the time-dependent entanglement en- tropy of an interval, both showing thermalization at late times. These results turn out to be in perfect agreement with previous gravity calculations in the AdS3-Vaidya geometry. Information loss appears in the CFT as an explicit violation of unitarity in the 1/c expansion, restored by nonperturbative corrections.

Hydrodynamic theory of quantum fluctuating superconductivity (Blaise Gouteraux)

A hydrodynamic theory of transport in quantum mechanically phase-disordered superconductors is possible when supercurrent relaxation can be treated as a slow process. We obtain general results for the frequency-dependent conductivity of such a regime. With time-reversal invariance, the conductivity is characterized by a Drude-like peak, with width given by the supercurrent relaxation rate. Using the memory matrix formalism, we obtain a formula for this width (and hence also the dc resistivity) when the supercurrent is relaxed by short range Coulomb interactions. This leads to a new effective field theoretic and fully quantum derivation of a classic result on flux flow resistance. With strong breaking of time-reversal invariance, the optical conductivity exhibits what we call a ‘hydrodynamic supercyclotron’ resonance. We obtain the frequency and decay rate of this resonance for the case of supercurrent relaxation due to an emergent Chern-Simons gauge fi eld. The supercurrent decay rate in this ‘topologically ordered superfluid vortex liquid’ is determined by the conductivities of the normal component of the liquid. Our work gives a controlled framework for low temperature metallic phases arising from phase-disordered superconductivity.

The holographic Weyl semi-metal (Karl Landsteiner)

Weyl semimetals are an exciting new class of 3D materials with exotic transport properties. They are characterised by pointlike singularities in the Brillouin zone at which conduction and valence bands touch. Around these points the electronic quasiparticle excitations can be described by either left- or right-handed Weyl spinors. At strong coupling such semiclassical reasoning based on fermionic quasiparticles might not be available. The question arises then if it is possible to construct a model at strong coupling that has the essential physical properties of a Weyl semimetal, in particular, if there exists any strongly coupled model in which a quantum phase transition between a topological and a topologically trivial state persists even in the absence of the notion of singularities in the band structure? A tool to answer these questions is the AdS/CFT correspondence. In my talk I present a holographic model of a Weyl semi-metal and argue that the intrinsic anomalous Hall effect can be used as an order parameter to detect the topologically non-trivial phase.

Integrable one-point functions of AdS/dCFT from matrix product states (Charlotte Kristjansen)

One-point functions of certain non-protected scalar operators in the defect CFT dual to the D3-D5 probe brane system with k units of world volume flux can be expressed as overlaps between Bethe eigenstates of the Heisenberg spin chain and a matrix product state. We present a closed expression of determinant form for these one-point functions, valid for any value of k. The determinant formula factorizes into the k=2 result times a k-dependent prefactor. Making use of the transfer matrix of the Heisenberg spin chain we recursively relate the matrix product state for higher even and odd k to the matrix product state for k=2 and k=3 respectively. We furthermore find evidence that the matrix product states for k=2 and k=3 are related via a ratio of Baxter's Q-operators. The general k formula has an interesting thermodynamical limit involving a non-trivial scaling of k, which indicates that the match between string and field theory one-point functions found for chiral primaries might be tested for non-protected operators as well. We revisit the string computation for chiral primaries and discuss how it can be extended to non-protected operators.

Fractional Wigner Crystal in the Helical Luttinger Liquid (Niccolo Traverso)

An electron confined to one spatial dimension does not have many options. It can have spin up or down and it can go right or left. In helical systems the possibilities are further reduced: the spin projection is tied to the direction of motion. This phenomenon is called spin-momentum locking and takes place at the edges of two-dimensional topological insulators. Its consequences are exciting for spintronics: since spin up and down electrons counter-propagate, spin transport can be easily generated by charge currents. During the talk, I have addressed question: Are charge density and spin correlations influenced by spin-momentum locking? For weak interactions, the density correlations are featureless and the density is not affected by impurities, while spin correlations are well represented by a planar spin helix that can be pinned by magnetic impurities (EPL 113 37002 (2016)). For strong interactions, non-helical one-dimensional systems are characterized by the formation of the so called Wigner molecule. This strongly correlated state can be modelled as a chain of electrons, free to oscillate around their equilibrium positions (charge excitations), and interacting with their nearest neighbours via a weak residual antiferromagnetic coupling (spin excitations). Such a state cannot represent the strongly interacting sector of the helical Luttinger liquid since spin-momentum locking forbids the separation between spin and charge low energy excitations. Indeed, in the helical case, a Wigner oscillation of fractional charges e/2 built on a strongly anisotropic spin wave (PRL 115, 206402 (2015)) emerges in the strongly interaction regime. The interaction process which is responsible for such an effect is correlated two-particle backscattering, which amounts to convert two right (left) movers into two left (right) movers, and is relevant, in generic helical Luttinger liquids, whenever axial spin symmetry is broken in the bulk of the topological insulator. From the methodological point of view, the weakly interacting regime is well captured by the Luttinger liquid theory, while the strongly interacting sector is analysed by means of semiclassical quantization of the sine-Gordon model at finite soliton density

Quantum Chaos and Thermalization Through Eigenstates (Anatoli Polkovnikov)

In this talk Anatoli reviewed basic notions of classical and quantum chaos in single particle systems and in particular the relation of quantum chaos and the random matrix theory (RMT). In particular I briefly reviewed the Berry conjecture on the structure of chaotic eigenstates near the classical limit, Berry-Tabor and BGS conjectures on the level statistics in generic integrable and chaotic systems and the ideas of Wigner relating spectra of complex systems to those of random matrices. Then I discussed the eigenstate thermalization hypothesis (ETH), introduced by J. Deutch and M. Srednicki, as a natural extension of RMT. I showed implications of quantum chaos and ETH to entanglement in chaotic systems, delocalization of wave functions in the eigenstate basis leading to familiar thermodynamic relations including fluctuation theorems and in some cases allowing one to extend them beyond linear response. I briefly outlined ideas on steady states in non-ergodic integrable systems with an extensive number of integrals of motion introducing ideas of the generalized Gibbs ensemble. Finally I discussed in some level of detail many-particle periodically driven Floquet systems and their close analogy to the many-body localized systems.

Fish and Hydrodynamics in a Large Number of Dimensions (Chris Herzog)

A simple setting in which to study non-equilibrium physics is a Riemann problem where the initial conditions consist of a planar interface, to the right of which the system is kept at temperature T_R and to the left at a different temperature T_L. In the context of conformal field theory in two space-time dimensions, Bernard and Doyon (2012) argued for the emergence of a non-equilibrium steady state whose properties are strongly constrained by the underlying scale and Lorentz invariance. More recently, other authors have tried to extend their work to conformal field theories in larger numbers of dimensions. Through the AdS/CFT correspondence there are further interesting connections to gravity and black hole dynamics.

An interesting limit to study is when the number of dimensions grows large. The dual black hole is described exactly by a pair of second order, nonlinear differential equations:

$$\partial_t e - \partial_\zeta^2 e = -\partial_\zeta j \ , \qquad \partial_t j - \partial_\zeta^2 j = -\partial_\zeta \left(\frac{j^2}{e} + e \right) \ .$$

In the "ideal limit where the second order terms can be dropped, the system of equations supports shock and rarefaction waves. The allowed shock and rarefaction solutions are governed by fish-like curves in the energy density and current. These curves provide a sort of phase diagram for the system.

Top-down holographic Fermi surfaces (Steve Gubser)

Gubser summarized work on holographic Fermi surfaces focusing on constructions in type IIB string theory dual to N=4 super-Yang-Mills theory at finite chemical potential. Results from an assortment of supergravity fermions in a variety of backgrounds suggest a "boson rule" for the existence or non-existence of Fermi surfaces. This rule hinges on the properties of the bosonic component of the composite operator dual to the supergravity fermion. Gubser also explained a "fermion rule" which predicts when Fermi momenta will be anomalously small compared to the chemical potential. A Luttinger count based on a particular example suggests that a large majority of the R-charge carried in an anti-de Sitter black hole comes from fermions in the dual field theory.

The calculations that support these conclusions boil down to the propagation of free fermions in curved geometries. The geometries are analytically known black holes in five-dimensional gauged supergravity. The equations of motion for the fermions come from the spin-1/2 part of the gauged supergravity action, and they are variants of the Dirac equation that include gauge couplings, Pauli couplings, and spatially variable mass terms. All the concrete results come from numerical solutions to these modified Dirac equations. The focus is on near-extremal geometries whose horizon entropy vanishes in the extremal limit.

Horizon as dynamical phase transition (Sung-Sik Lee)

We show that renormalization group flow can be viewed as a gradual wave function collapse, where an initial state associated with the action of field theory evolves toward a final state that describes an IR fixed point. The process of collapse is described by the radial evolution in the dual holographic theory. If the theory is in the same phase as the assumed IR fixed point, the initial state is smoothly projected to the final state. On the other hand, the initial state can not be smoothly projected to the final state, if the system is in a different phase. Obstructions to smooth projection appear as dynamical phase transitions, which in turn give rise to horizons in the bulk geometry. We demonstrate the connection between critical behavior and horizon in an example, by deriving the bulk metrics that emerge in various phases of the U(N) vector model in the large N limit based on the holographic dual constructed from quantum renormalization group.

Discussion on holographic disorder and instabilities led by Sera Cremonini and Daniel Arean

Daniel and I led a discussion on breaking symmetries in holography and on the resulting phenomenology. One of the topics discussed in some detail was that of instabilities associated with geometries that exhibit scaling behavior. In particular, most of the focus was on the various infrared phases of gravitational solutions describing non-relativistic systems which don't preserve hyperscaling. Recently, the effort has been on better understanding the role of broken translational invariance, but more generally a rich structure of phases is observed in the deep IR, including ones with emergent conformal symmetry. One of the questions that was posed is whether certain RG flows connecting UV to IR CFTs but traversing an intermediate non-relativistic, hyperscaling violating regime can be understood in a more generic fashion, by thinking along the lines of possible extensions of holographic c-theorems. While the latter generically fail when Lorentz invariance is not preserved along the flow, there may be special conditions on the geometry which guarantee (in certain cases) monotonic c-functions.

The second main topic that was brought up was the breaking of translational invariance, and in particular the implications for possible holographic bounds on the conductivity and on the shear viscosity to entropy ratio. For the latter, broken translational invariance leads to temperature dependence, and in certain models this seemingly causes the shear viscosity to entropy ratio to approach zero at very low temperatures. Questions were raised regarding the precise connection between the shear viscosity and correlators of the holographic stress tensor, when momentum is dissipated and the standard relation between the horizon and the boundary shear metric fluctuations is modified in a crucial way.

Anomaly matching and locality in warped conformal field theory (Kristan Jensen)

Warped conformal field theories (WCFTs) are exotic, chiral, but non-Lorentz-invariant two-dimensional theories. Little is known about them. They have been conjectured to be the theories dual to gravity on so-called warped AdS3 spacetimes, which appear in the near-horizon limit of many black holes including near-extremal Kerr. Perhaps the defining feature of a WCFT is its global symmetry algebra. What happens is that a WCFT is invariant under translations and right-dilatations; the right translation is enhanced to a right-moving Virasoro algebra, and the left translation to a right-moving abelian Kac-Moody algebra, rather than to a left-moving Virasoro algebra as in conventional two-dimensional CFT.

The punchline of my talk was that, using a combination of free-field and non-perturbative arguments, WCFTs are generically non-local, and in particular the gravity dual to warped AdS3 is “semi-local in the sense of Iqbal/Liu/Mezeis 2011 paper.

My argument had three components. The first was to show that all free WCFTs admit an infinite number of exactly marginal, non-local deformations, and so free local WCFTs are infinitely fine-tuned. The second was to classify the ‘t Hooft anomalies of a local WCFT, and to show that these are not matched by their conjectured holographic duals. The third was to consider fluctuations on top of warped AdS3 backgrounds, which it turns out have much in common with fluctuations in the $AdS_2 \times R^{d-1}$ throat of extremal charged black branes. Namely, fluctuations are dual to operators with a momentum-dependent operator dimension. This is the hallmark of a “semi-local theory, as defined by Iqbal, Liu, and Mezei.

Outcome of the Meeting

The meeting had very productive interactions between the two communities represented, those interested in condensed matter physics, and those trained in string theory and its methods. Besides the good mix represented in the talks and the issues they raised, many informal discussions were taking place. We expect those discussions to promote further collaborations between these communities, which is essential to the future of this interdisciplinary endeavour. We also are looking forward to possibly meeting again in Banff for another installment in the series of these highly productive meetings.

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Chapter 7

Homological Mirror Geometry (16w5062)

March 6 - 11, 2016

Organizer(s): Matthew Ballard (University of South Carolina), Colin Diemer (University of Alberta), David Favero (University of Alberta)

Overview of the Field

Introduced in 1994 by Maxim Kontsevich [K95], Homological Mirror Symmetry (HMS) proposes a means of describing the mirror symmetry phenomenon originating in string theory by the identification of appropriate categorical structures. Roughly, this means identifying the (derived category of) coherent sheaves on an algebraic variety with a Fukaya category of Lagrangians on a "mirror" symplectic manifold or, in notation, $D^b\mathrm{coh}(X) \cong \mathrm{Fuk}(X^\vee)$. In addition to giving a rigorous mathematical interpretation of ideas emerging from physics, proving the conjecture has required substantial developments in algebraic geometry, symplectic topology, homological algebra, and other disparate fields of mathematics. The aim of this workshop was to investigate cutting edge developments in both fields which are inspired by interactions with HMS. In some cases, this involves direct proofs of cases of HMS, but the workshop also focused on mathematics which has been inspired by the structures appearing in HMS. Let us briefly introduce the themes which were the focus topics of the workshop. A more detailed discussion of the content of the presentations and the surrounding discussions and conversations will follow in the later sections.

Autoequivalences of Derived Categories Inspired by Mirror Symmetry: this topic was represented in the talks by Rina Anno, Paul Aspinwall, Will Donovan, and Timothy Logvinenko. If one has a HMS equivalence of the form $D^b\mathrm{coh}(X) \cong \mathrm{Fuk}(X^\vee)$, then one in particular has an identification of the respective groups of autoequivalences of the categories. Early in the history of HMS, it was recognized by Seidel [S99] that there are natural autoequivalences of the Fukaya categories of symplectic manifolds coming from Dehn twists around Lagrangian spheres. The mirror autoequivalences acting on the derived category of coherent sheaves on a mirror variety were first studied by Seidel-Thomas [ST01], yet still remain somewhat mysterious and do not always have an obvious interpretation in terms of classical projective geometry (i.e. do not correspond to a geometrically induced automorphism of $D^b\mathrm{coh}(X)$). There has been substantial recent interest in generalizing the Seidel-Thomas auotequivalences as a means of enhance our understanding of the structure of derived categories. These studies largely center around Rina Anno's notion of a "spherical functor", whose foundations have recently been formalized [AL13]. In the reverse direction, birational equivalences such as flops often induce autoequivalences of $D^b\mathrm{coh}(X)$, e.g. via "flop-flop" functors [BO95]. The corresponding mirror theory and associated symplectic operations remains largely unexplored..

Grade Restriction Windows and Geometric Invariant Theory: this topic was represented in the talks of

Gabriel Kerr, Daniel Halpern-Leistner and Kentaro Hori. Physical arguments of Herbst, Hori, and Page in 2008 [HHP08] suggested a method of understanding the derived categories of geometric invariant theory (GIT) quotients motivated via their study of the mirror geometry. Roughly, they considered transitions which mathematicians would identify as variations of wall-crossings in GIT and, by studying the behavior of these objects under parallel transport in the mirror symplectic (or Kähler) moduli space, developed rules which identify which objects in the equivariant derived category of a space descend to objects in the equivariant derived category of a GIT quotient. Implementation of this theory in concrete mathematical terms has been an active area of research, with notable contributions from several workshop participants. First studied in the mathematical literature by Segal [S09] and developed further by Donovan-Segal [14] the theory was interpreted by Halpern-Leistner as a categorification of the classical notion of Kirwan surjectivity in equivariant cohomology [H-L15]. Parallel work by Ballard, Favero, and Katzarkov [BFK15] exhibited the theory as a powerful method for constructing preferred decompositions of derived categories (e.g. exceptional collections). The theory remains very much an active area with many applications still under construction. A particular topic of interest among several workshop participants is further applications to wall-crossing equivalences. For example, the "flop-flop" functors mentioned above are not yet completely implemented in the theory, despite their intrinsic geometric importance, although important developments have been obtained by Donovan and Segal [14] and Halpern-Leistner and Shipman [H-LS13], and a fairly complete understanding for toric wall-crossings was given by Herbst and Walcher [HW09].

Geometry of the Gauged Linear Sigma Model (GLSM): this topic was represented in the talks of Nick Addington, Shinobu Hosono, and Paul Aspinwall. Introduced by Witten, the gauged linear sigma model is a supersymmetric quantum field theory originally introduced to understand the "Landau-Ginzburg/Calabi-Yau" correspondence. The correspondence has several deep manifestations in the current mathematical literature; one avatar of particular relevance to the workshop is Orlov's interpretation [O09] (developed further in independent work of Isik [I1] and Shipman [S12]) which described the derived category of a hypersurface or complete intersection in terms of graded matrix factorizations. This identification can be understood as a phase transition in the corresponding GLSM. Indeed, the work of Herbst, Hori, Page mentioned above [HHP08] also included a framework for understanding the LG/CY correspondence via grade restriction rules - this was confirmed in the mathematical literature both in the proof of Shipman and the work of Ballard, Favero, Katzarkov. Beyond the LG/CY correspondence, the underlying geometry of these phase transitions has been the source of some truly fascinating developments. Indeed, many of the topics mentioned above (flop-flop equivalences, spherical functors, grade restriction windows, ...) all fit into the framework of the GLSM. A crucial theme of this workshop was extracting yet more geometry from phase transitions in the the GLSM.

Bridgeland Stability: this topic was represented in the talks of Arend Beyer, Ludmil Katzarkov, and Yokinobu Toda. The Bridgeland stability manifold is a deep construction originally designed to construct mathematically the "stringy Kähler moduli space" of a triangulated category. Roughly, the Bridgeland stability manifold encodes the non-commutative deformations of a category, and thus a huge amount of geometry, and should control many aspects of the deformation theory of the categories involved in HMS. It's initial motivation follows heavily from homological mirror symmetry, as it is expected to be the "correct" (from a physical perspective) mirror moduli space of symplectic (A-model) mirrors. Explicit constructions of even individual points or open subsets of a stability manifold are notoriously difficult, although there as been dramatic progress in the last few years.

Recent Developments and Open Problems

The major recent developments and open problems presented in the workshop revolved around the connection between birational geometry and symplectic automorphisms in mirror symmetry and derived categories.

Provide a mathematical framework for Hori's generalized Grade Restriction Rules: In an extremely fundamental development for derived categories [O09], Orlov provided a deep correspondence between B-branes on LG-models and on algebraic varieties. This mathematical development was later explained in the physics literature by Herbst-Hori-Page [HHP08] by introducing the concept of a "Grade Restriction Rule". As a consequence of this idea from high-energy theoretical physics, in independent work Ballard-Favero-Katzarkov, and Halpern-Leistner, developed a unifying relationship between derived categories and variation of geometric invariant theory quotients. However, the relationship was limited to the types of Grade Restriction Rule in the Herbst-Hori-Page framework and has failed to explain, for example, the famous relationship between Pfaffian and Grassmanian Calabi-Yau 3-folds. In his talk, Hori explained a new, mathematically precise, Grade Restriction Rule, motivated physically. An open problem is to explain and expand the mathematical nature of this rule and use it to extend the work of Ballard-Favero-Katzarkov and Halpern-Leistner.

Find an action of the fundamental group of the Kähler moduli space or space of Bridgeland Stability Conditions of X on the derived category: HMS predicts that symplectic automorphisms should manifest in the derived category of mirror algebraic varieties. More precisely, given an algebraic variety X, let $X^\vee$ be the mirror. Symplectic automorphisms of $X^\vee$ give autoequivalences of $D^b(X)$. One way to produce symplectic automorphisms of $\hat{X}$ is take the monodromy around loops in the complex structure moduli space. This should produce a faithful action of the fundamental group of this moduli space on $D(X)$. A lot of progress in this direction was presented during this workshop by Daniel Halpern-Leistner, Paul Aspinwall, Rina Anno, Timothy Logvinenko, and Will Donovan. A lot of progress has been made in this direction, however, it remains open how to do this more generally, especially using Bridgeland's model for this moduli space.

Use tilting to construct new examples of Bridgeland Stability Conditions in all dimensions: Bridgeland stability is a very important concept in derived categories and mirror symmetry, serving as a mathematically well-defined notion of the stringy Kähler moduli space which is discussed in high-energy theoretical physic circles. Bayer-Macri-Toda have recently made many breakthroughs in constructing Bridgeland stability conditions which was presented by Bayer during the workshop. In [BMT14], they have also presented a conjectural program for constructing these stability conditions more generally. The program involves generalizing the classical Bogomolov-Gieseker inequality and using tilts of classical stability conditions to get ones on derived categories. This is very exciting, as, for one, it shows that the moduli space of stability conditions is non-empty. In particular, Bridgeland's notion of a stringy Kähler moduli space is not trivial. Participants Benjamin Schmidt, Barbara Bologenese, Arend Bayer, and Yukinobu Toda are working in this direction.

Develop semi-orthogonal decompositions as a phenomenon in homological mirror symmetry, especially their relationship with mirror constructions to variation of geometric invariant theory quotients: Semi-orthogonal decompositions are a way to simplify derived categories into more managable pieces or at least establish interesting relationships between different derived categories. In many ways, they are predicted by the structure of the mirror and Kontsevich's homological mirror symmetry conjecture. The appearance of semi-orthgonal decompositions in algebraic and symplectic geometry can, therefore, be viewed as an additional structure one should attach to the homological mirror symmetry conjecture. This conjectural enhancement of homological mirror symmetry was fleshed out in [BDFKK15]. Developments in this direction were discussed by Gabriel Kerr and Nick Addington.

Presentation Highlights

· Monday's talks focused on grade restriction windows and autoequivalences induced by birational transformations and wall-crossings.

Kentaro Hori (IPMU): Grade Restriction Rules from Hemisphere Partition Functions

The workshop began with a talk by Prof. Kentaro Hori. He spoke on in progress work with R. Eager, J. Knapp, and M. Romo describing the underlying physics of grade restrictions rules. As discussed above, Hori's ongoing contributions to this topic have drawn substantial interest from mathematicians working in derived categories, as such his talk was crucial for setting the tone for many of the subsequent talks. Hori began with a review of the GLSM and the data which goes into defining the particular (2,2) supersymmetric quantum field theory under study, which roughly consists of a representation of a gauge group G and a G-invariant superpotential W. Hori reviewed his classical work with Tong [HT07] describing the structures behind the Rødland (or Pfaffian-Grassmannian) model [R00], historically the first example of a non-birational derived equivalence. The talk then emphasized the played by the Hemisphere partition function which is roughly the exact partition function of the GLSM, and how its convergence dictates the grade restriction rules. The talk concluding with some explicit examples where full exceptional collections can be extracted by these physical methods, which appeared to be beyond the scope of the current state of rigorous mathematical proofs.

Daniel Halpern-Leistner (Columbia): Magic windows and representations of generalized braid groups on the derived category of a GIT quotient

Prof. Halpern-Leistner spoke on recent work with Steven Sam which studies certain cases where the behavior the derived category under a GIT wall-crossing can be understood combinatorially. The particular setup involves considering symmetric representations, which are roughly linear representations of a reductive group whose associated GIT chamber structure has a zonotopal structure. This corresponds to the "local" case of a very large class of wall-crossings. In this situation all wall-crossings are equivalences, and the talk explained how the corresponding derived categories are generated by certain strong exceptional collections of line bundles ("magic windows"); these sets of line bundles have appeared implicitly in many recent works, for example those of workshop participants W. Donovan and E. Segal concerning Grassmannian flops [14]. The zonotopal GIT chamber determines a complexified hyperplane arrangement whose fundamental groupoid should, assuming homological mirror symmetry, act faithfully on these derived categories, as it constitutes the mirror Kähler moduli space. The talk showed how this representation can be constructed explicitly (without invoking HMS).

Official Abstract: One consequence of the homological mirror symmetry conjecture predicts that many varieties will have "hidden symmetries" in the form of autoequivalences of their derived categories of coherent sheaves which do not correspond to any automorphism of the underlying variety. In fact the fundamental groupoid of a certain "complexified Kaehler moduli space" conjecturally acts on the derived category. When the space in question is the cotangent bundle of a flag variety, actions of this kind have been studied intensely in the context of geometric representation theory and Kahzdan-Lusztig theory. We establish the conjectured group action on the derived category of any variety which arises as a symplectic or hyperkaehler reduction of a linear representation of a compact Lie group. Our methods are quite explicit and essentially combinatorial – leading to explicit generators for the derived category and an explicit description of the complexified Kaehler moduli space. The method generalizes the work of Donovan, Segal, Hori, Herbst, and Page which studies grade restriction rules in specific examples associated to "magic windows." Based on joint work with Steven Sam.

Nick Addington (University of Oregon): Complete intersections of unequal degrees

A fundamental theorem of Orlov [O09] asserts that if X_d is a hypersurface in $\mathbb{P}^d$ with degree $d < n + 1$ (i.e. the Fano case), then the derived category $D^b(X)$ admits a semiorthogonal decomposition into exceptional objects (coming from twists of $\mathcal{O}(1)$) and the orthogonal category given by graded matrix factorizations of the defining polynomial f of X_d. A prototypical example due to Kuznetsov is the derived category of a cubic fourfold in which the orthogonal component can be viewed as the derived category of K3 category [K10]. A similar result can be described for complete intersections of equal degrees, with the orthogonal component consisting of, roughly, a family or bundle of matrix factorization categories. For example, the intersection of two quadrics in $\mathbb{P}^5$ is a Calabi-Yau threefold and such a description of its derived category was given at least implicitly in classic work of Bondal-Orlov [BO95]. Prof. Addington's talk reported on joint work with conference participant Paul Aspinwall investigating the case of complete intersections of unequal degrees. The main moral of the talk was that is profitable to view the component orthogonal to the exceptional collection as a noncommutative resolution in the sense of van den Bergh [VDB05]. The main theorem announced stated that if $f_1, \ldots, f_k$ are homogenous polynomials with degrees d_i and subject to the restriction $d_1 > \max\{d_2, \ldots, d_k\}$, then the orthogonal component is a categorical resolution of the category of graded matrix factorizations of a singular hypersurface of degree d_1 (in an appropriate weighted projective space). The talk concluded by showing how this analysis can be applied to the study of extremal transitions of Calabi-Yau manifolds.

Official Abstract: For a Fano hypersurface in $\mathbb{P}^n$, the derived category decomposes into an exceptional collection and a category of matrix factorizations. For a complete intersection of k hypersurfaces of degree d, it decomposes into an exceptional collection and a sort of bundle of categories of matrix factorizations over $\mathbb{P}^{k-1}$. What about a complete intersection of hypersurfaces of unequal degrees $d_1 \ldots d_k$? Do we get a similar bundle over weighted $\mathbb{P}^{k-1}$, with weights $d_1 \ldots d_k$? Not really: it is better to view it as a categorical resolution of the category of matrix factorizations of some higher-dimensional, singular hypersurface. The prototypical example is Kuznetsov's degree-6 K3 surface resolving the category of matrix factorizations of a nodal cubic 4-fold. We will discuss several other examples and state some general results. This is joint work with Paul Aspinwall.

Will Donovan (IPMU): Twists and braids for general threefold flops

The pioneering example in the study of the relationships between birational geometry and derived categories is Bondal-Orlov's study of the Atiyah flop (essentially the simplest threefold flop) [BO95], and it associated autoequivalence (which is not trivial). Prof. Donovan spoke on joint work with Michael Wemyss studying the flop-flop equivalences for more general classes of threefold flops. As in the talk of Halpern-Leistner, one expects that these autoequivalences should correspond to monodromy transformations on a mirror symplectic manifold; thus, if one one studies multiple flops HMS predicts that this should correspond to an action of the fundamental group(oid) of an associated hyperplane or discriminantal arrangement. A basic example is if one flops two $(-1, -1)$ curves meeting along a point one obtains an action of the pure braid group PB_3 on the derived category. For a more general threefold flop, if one assumes Gorenstein singularities near the exceptional locus, it was observed by Reid that, upon taking a resolution of a generic hyperplane section, one may associate a marked Dynkin diagram to the flop. It is a folklore theorem from 1980's birational geometry that if one associates the root hyperplane arrangement to this Dynkin configuration, then the group of connected components of its complement is in bijection with the relative minimal models (roughly, all associated flops). The main theorem announced by Prof. Donovan was indeed a representation of this group of connected components on the autoequivalence group of the original threefold X, thus suggesting a very strong relationship between flops and derived autoequivalences, consistent with the conjectures of Bondal-Orlov. HMS predicts that the representation constructed by Donovan-Wemyss should be faithful, although this remains open.

Official abstract: When a 3-fold contains a floppable curve, there is an associated equivalence between the derived categories of the 3-fold and its flop. If the curve is reducible, there may exist multiple such flop functors, one for each irreducible component. I will explain joint work with Michael Wemyss, showing how this leads to new actions of braid-type groups on the derived category, and give an update on related results.

· Day 2 of the workshop focused more directly on mirror symmetry (both homological and "classical").

Charles Doran (University of Alberta and University of Maryland): Mirror Symmetry, Tyurin Degenerations, and Fibrations on Calabi-Yau Manifolds

Prof. Doran spoke on mirror constructions for Calabi-Yau threefolds equipped with a fibration structure in codimension one (for example, a Calabi-Yau threefold fibered in K3 surfaces). The principal observation in the talk was that the mirror to such fibration structures should be a particular type of degeneration which roughly consists of two copies of Fano varieties glued along a codimension one subscheme ("Tyurin degenerations"). Prof. Doran discussed motivations for this principle based on his previous work (with many collaborators) on classifying the types of variations of mixed Hodge structures which can occur for mirrors to Calabi-Yau threefolds, and sketched proofs for many classes of examples that the appropriately defined Hodge numbers agree with the mirror symmetry expectations. The talk included discussions of how this proposal fits with Batyrev-Borisov's combinatorial mirror symmetry for Calabi-Yau's which are complete intersections in toric varieties, as well as interplay with the compactification of the moduli space of K3 surfaces. The talk generated a substantial amount of audience discussion based on possible interpretations of Prof. Doran's results in homological mirror symmetry.

Official abstract: We present a new construction of mirror pairs of Calabi-Yau manifolds. On one side of the mirror correspondence are Calabi-Yau manifolds fibered in codimension one by Calabi-Yau submanifolds, for example elliptic fibered K3 surfaces or K3 surface fibered Calabi-Yau threefolds. On the other side are so-called Tyurin degenerations, i.e., smoothings of pairs of quasi-Fano varieties whose common intersection Calabi-Yaus are mirror to the fibers; these correspond to Type II Kulikov degenerations in the K3 surface case and Kawamata-Namikawa smoothings in the case of Calabi-Yau threefolds. Evidence that the construction produces mirror pairs comes from several directions: The fibered Calabi-Yaus are constructed by gluing the pair of Landau-Ginzburg models mirror to the pair of quasi-Fano varieties, and we establish mirror symmetry of Euler and Hodge numbers. Our construction is compatible with the Batyrev-Borisov mirror construction, wherein a bipartite nef partition produces the structures on both sides and the singular fibers of the fibration encode properties of the Landau-Ginzburg models mirror to the two quasi-Fano varieties. In the case of elliptic fibered K3 surfaces, the KSBA compactification of moduli of pairs suggests a broad correspondence between Type II degenerations of a lattice-polarized K3 surface and elliptic fibrations on its Dolgachev-Nikulin mirror. A complete classification of Calabi-Yau threefolds fibered by mirror quartic K3 surfaces leads to explicit constructions of candidate mirror threefolds and their Tyurin degenerations, showing that our construction is not limited to threefolds constructed as toric complete intersections. Finally, we show that in the context of homological mirror symmetry, non-commutative K3 fibrations should be mirror to Tyurin degenerations along loci in moduli disjoint from points of maximal unipotent monodromy.

Helge Ruddat (Mainz): Tropical descendent Gromov-Witten invariants

Prof. Ruddat spoke on joint work with Travis Mandel studying a tropical formulation of descendent Gromov-Witten invariants. Recall that a main theme in tropical geometry is, very roughly, to provide a calculus for doing enumerative calculations with curves in terms of piecewise linear "tropical" analogues - a process which often makes the underlying combinatorics more transparent. A landmark result in the field was Mikhalkin's tropical calculation of the genus zero Gromov-witten theory of $\mathbb{P}^2$, which was later generalized to arbitrary toric surfaces by Nishinou-Siebert. However, classical mirror symmetry in a strong form works at the level of "big" quantum cohomology, whose associated enumerative geometry is that of descendent invariants (roughly: the intersection theory of ψ-classes on the Moduli space of curves). Tropical descendent invariants have previously been studied by Markwig and Rau, who verified the equivalence of the classical and the tropical description in the case of $\mathbb{P}^2$. Prof. Ruddat announced a comparison of this tropical calculus with the rapidly emerging field of log Gromov-Witten invariants. The talk discussed the applications of this result to mirror symmetry, with log Gromov-Witten invariants playing a crucial role in the construction of Gross-Siebert mirrors.

Official abstract: Descendent Gromov-Witten invariants play a central role in canonical deformations of Landau-Ginzburg models as well as the multiplication rules of generalized theta functions, both relevant for (homological) mirror symmetry. In a joint work with Travis Mandel, I prove that tropical Gromov-Witten invariants with psi class conditions coincide with descendent log Gromov Witten invariants for smooth toric varieties whenever non-superabundance is given. We use toric degenerations a la facon de Siebert-Nishinou and we expect that our approach will be generalizable to Mumford or Gross-Siebert type degenerations.

Ludmil Katzarkov (University of Miami and University of Vienna): A categorical Donaldson-Uhlenbeck-Yau correspondence

Prof. Katzarkov's talk began with a departure from the title and abstract, wherein he offered a proposal to a question raised during Prof. Doran's earlier talk: namely how to understand the role of Tyurin degenerations in homological mirror symmetry. Katzarkov proposed interpreting the question in terms of Lagrangian skeleta (themselves a fairly recent proposal in mirror symmetry under heavy recent investigation by Nadler and Zaslow and collaborators in particular). In particular, Katzarkov proposed some elementary examples of Tyurin degenerations which could be understood combinatorially, and in principle which should give categorical equivalences. The main portion of Katzarkov's talk then concentrated on joint work with F. Haiden, M. Kontsevich, and P. Pandit which explores categorifications of the classical theorem(s) of Donaldson-Uhlenbeck-Yau which identify semistable holomorphic vector bundles on Kähler manifolds with Hermitian-Einstein connections. More precisely, Katzarkov proposed a definition of "DUY structure" on a category which conjecturally would identify the moduli space of semistable bundles with a Bridgeland moduli space of semistable objects. Indeed, this suggestion appears to be more in line with the original physical construction of the Bridgeland stability manifold as first constructed by Michael Douglas and conference participant Paul Aspinwall [AD04].

Official abstract: In this talk we will introduce the notion of Donaldson Uhlenbeck Yau correspondence. A connection with sheaves of categories will be discussed.

Shinobu Hosono (Gakushuin University): Conifold transitions in mirror symmetry of CICYs

Prof. Hosono spoke on his joint works with Hosono-Takagi which study (non-birational) derived equivalences generalizing the Rødland (Pfaffian-Grassmannian) model [R00]. Their recent works have been a fruitful testing ground for geometric techniques for proving derived equivalences. After a brief review of the Batyrev-Borisov mirror construction and Landau-Ginzburg models, Hosono introduced a particular model known as the Reye congruence which, roughly, is the set of lines in $\mathbb{P}^4$ which lie in a two dimensional family of quadrics. This set X is known to be a Calabi-Yau threefold, and admits a Landau-Ginzburg description via toric geometry. Hosono discussed the derived equivalent spaces (Fourier-Mukai partners) and various phase transitions which can be obtained from VGIT. Via mirror symmetry, there is a mirror family, and in this particular example the entire Kähler moduli space (including its discriminant locus) can be computed explicitly, and the large complex structure monodromy can also be computed. The calculations given in the talk agree with predictions from homological mirror symmetry, as well as allow from explicit calculation in "classical" mirror symmetry relating Gromov-WItten invariants with period integrals.

Official abstract: In a series of collaborations with Hiromichi Takagi, I have been studying certain complete intersection Calabi-Yau spaces, which are nicely related to determinantal varieties in projective spaces. After summarizing relations to the linear duality (due to Kuznetov), I will focus on the mirror symmetry of these Calabi-Yau spaces. In particular, I will describe conifold transitions explicitly for the case of mirror family obtained in our CMP paper (2014, vol.329, 1171–1218).

· Day 3 of the workshop focused on connections with physics.

Paul Aspinwall (Duke University): Mirror symmetry and discriminants

Prof. Aspinwall began his talk with a historical overview of mirror symmetry, with a focus on the developments in physics in the mid 90's, and in particular the works carried out by Aspinwall in various joint works with B. Greene, D. Morrison, and R. Plessar. The focus was on the (complexified) Kähler moduli space of a Calabi-Yau complete intersection, which roughly encodes the symplectic deformations of a mirror partner, and whose maximal degeneration points should correspond to different birational models under mirror symmetry. This overall structure has already appeared implicitly in the prior talks of W. Donovan, D. Halpern-Leistner and especially that of S. Hosono. Aspinwall drew attention to the discrimant locus of singular models which appear in the Kähler moduli space which, in the context of the GLSM, can be described at least implicitly via the techniques of Gelfand-Kapranov-Zelevinsky. Monodromy around a large complex structure limit induces a symplectomorphism (and thus an autoequivalence of the Fukaya category) which is mirror to a spherical functor on the correponding derived category of a given birational model (or matrix factorization category of the corresponding phase of the GLSM). This portion of the story is mostly well known to experts, and the philosophy is implicit in many of the talks at this workshop. However, Aspinwall made some conjectures which propose studying degenerations away from the large complex structure limit and deeper in the discriminant locus, which appear to be beyond the scope of the current mathematical literature; although in principle the general theory of spherical functors (cf. R. Anno's talk on Friday) should be able to accomodate such a theory. In particular, Aspinwall proposed an explicit subcategory given by the underlying toric geometry which should be the subcategory about which one "twists".

Official abstract: We analyze singularities in the parameter space of the gauged linear sigma model and show how they coincide with the GKZ A-determinant in the noncompact case. We show that this requires logarithmic coordinates to work correctly. The same analysis gives a natural picture for generic monodromy in the derived category around components of the discriminant in terms of specific spherical functors.

Eric Sharpe (Virginia Tech): Heterotic mirror symmetry

Prof. Sharpe spoke on the physics behind a more general version of mirror symmetry than that usually considered by mathematicians. Roughly, the mathematics literature focuses on models which come from quantum field theories with (2,2) supersymmetry, although physically only (0,2) supersymmetry is required to discuss mirror phenomena. Recall that "classical" mirror symmetry identified the complex deformations of a Calabi-Yau manifold X with the Kähler deformations of its mirror $X^\vee$, and thus implies an equivalence of Hodge numbers $h^{p,q}(X) = h^{n-p,q}(X^\vee)$. In (0,2) mirror symmetry the initial data consists of not just a Calabi-Yau manifold X, but the data of a stable vector bundle E on it, and the corresponding identification of bundle moduli implies an isomorphism of sheaf cohomologies $h^p(X, \wedge^q, E^*) = h^p(X^\circ, \wedge^q E^\vee)$. Explicit mathematical verification of even simple examples of this phenomenon is still under development. A crucial role has been the development of "quantum sheaf cohomology" which is a (0,2) analogue of ordinary quantum cohomology and which should encode the corresponding equivalence of complex and Kähler bundle moduli between a (0,2) mirror pair. Sharpe lectured on several classes of examples where one takes E to be the tangent bundle; two cases of particular interest were where $X = \mathbb{P}^1 \times \mathbb{P}^1$ and where X is a Grassmannian. The lecture generated a substantial amount of audience discussion on a hypothetical "(0,2) homological mirror symmetry".

Official abstract: In this talk we will describe progress towards a generalization of mirror symmetry pertinent for heterotic strings. Whereas ordinary mirror symmetry relates, in its simplest incarnations, pairs of Calabi-Yau manifolds, the heterotic generalization relates pairs of holomorphic vector bundles over (typically distinct) Calabi-Yau's, satisfying certain consistency conditions. We will also outline the corresponding analogue of quantum cohomology, known as quantum sheaf cohomology, describing results for deformations of tangent bundles of toric varieties and Grassmannians, and we will discuss (0,2) Landau-Ginzburg Toda-like mirrors to deformations of tangent bundles of products of projective spaces.

· Day 4 of the workshop was focused on stability conditions and notions of wall-crossing beyond those considered in previous talks.

Yokinobu Toda (IPMU): Wall-crossing formulas of higher rank DT invariants

Prof. Toda gave a very interesting talk on Donaldson-Thomas invariants. These invariants count curves on Calabi-Yau 3-folds. He described their relationship to Gromov-Witten invariants, another way of counting curves on Calabi-Yau 3-folds related to the classical enumerative mirror symmetry story. From 2008-2010, Bridgeland-Toda studied how DT-invariants change in a certain parameter space (wall-crossing formulas) and proved, for example, the DT/PT correspondence, and rationality of generating functions for these invariants. In this talk, Toda announced his results extending his work with Bridgeland for higher rank DT invariants.

Arend Bayer (Edinburgh: Stability conditions on surfaces: an update

Prof. Bayer spoke on recent advances in understanding the Bridgeland stability manifold of algebraic surfaces. Some of the first explicit examples of stability conditions were constructed by, of course, Bridgeland in his study of K3 surfaces, which proceeded by using the Harder-Narisimhan filtration on coherent sheaves on a surface to cosntruct a so-called "torsion pair", from which one can construct an explicit stability condition on $D^b\text{coh}(X)$. Extending these methods to threefolds is a very active problem. However, Bayer spoke on applications to some classical problems on surfaces, demonstrating the applicability of Bridgeland stability to concrete problems. To an algebraic surface, one may consider a Brill-Noether locus, which is roughly a moduli space of one-dimensional sheaves with fixed Euler characteristic and support on a fixed curve. This moduli space contains constructible subsets of sheaves with support on a fixed curve and having a prescribed number of global sections. Bayer showed in particular how a careful analysis of moduli spaces of Bridgeland semistable objects can lead to a new proof of the following classical result of Lazarsfeld: let (X, H) be a polarized K3 surface such that H^2 divides $H \cdot C$ for all curve classes C, if C' is any any smooth curve in the linear system $|H|$. Then the Brill-Noether locus has the expected dimension $g(r+1)(gd+r)$ where g is the genus, d is the degree H^2, and r is the number of sections.

Official abstract: I will give an update on applications of stability conditions for surfaces within algebraic geometry.

Gabriel Kerr (Kansas State): Mirror symmetry for elementary birational cobordisms

Prof. Kerr spoke on new results regarding a form of homological mirror symmetry for some toric varieties motivated by VGIT wall-crossings. Let $\mathbb{C}^*$ act on $V = \mathbb{C}^n$, and consider the GIT quotient $V//\mathbb{C}^*$. For example, one obtains (weighted) projective spaces as examples, and by varying the GIT quotient one may also view the standard Atiyah flop and other birational maps such as standard flips within such examples. Mirror symmetry for Fano toric varieties X of dimension n says that the mirror should be a Landau-Ginzburg model $w : (\mathbb{C}^*)^n \to \mathbb{C}$ where w is an explicit Laurent polynomial built from the vertices of the polytope defining X. Homological mirror symmetry says that there should be a derived equivalence $D^b(X) \cong \text{FS}(w)$ where $\text{FS}(w)$ denotes the Fukaya-Seidel category of vanishing cyles of w. In this form, relatively few instances of this theorem are completely proven. Indeed, though, Kerr outlined a proof for all examples of the above form. The principal idea was to make used of the semiorthogonal decompositions of $D^b(X)$ obtained by the grade restriction windows for the VGIT setup. This idea had been exploited in previous work, but only in very specific examples, see, e.g [BDFKK15]. For $\mathbb{C}^*$ acting on V, though, Kerr demonstrating that this structure is all very computable. A very careful analysis of the vanishing cycles of the mirror potential w then exhibits a similar structure. A subtle point is that it is not sufficient to simply match up the corresponding semiorthogonal components, but one must compute all homomorphisms and higher intersection products in order to compute the full dg structure on both sides and thus obtain a full proof of homological mirror symmetry for these examples.

Official abstract: It has been known since Bondal and Orlov's work on semi-orthogonal decompositions that for blow-ups, projective bundles and certain flips $f : X \dashrightarrow Y$, one may decompose the derived category of $D^b(X) = \langle D(Y), C\rangle$. In this talk I will describe the mirror LG model to C when f is a birational cobordism with trivial center. Diemer-Katzarkov-K. conjectured that this was a Fukaya-Seidel category FS (W) of a potential W from a higher dimensional pair of pants to the punctured plane. I will explain a recent proof of this conjecture. The classical version of HMS for weighted projective spaces of arbitrary dimension then will be observed as a corollary.

· Day 5 of the workshop focused on spherical functors.

Rina Anno (MIT): DG enhancements of derived categories of sheaves (Part I).

Timothy Logvinenko (Cardiff): P-functors (Part II).

Prof. Anno and Prof. Logvinenko gave a two part lecture series introducing the notion of P-functors. P-functors provide a beautiful way to categorify autoequivalences of the derived category appearing naturally in symplectic and hyperkähler geometry. They are an extension of the work of Seidel-Thomas [ST01] and Huybrechts-Thomas [HT06]. They explained the necessity of the DG category language in defining P-functors, their appearance in algebraic and symplectic geometry, and, how they lead to an action of a "Braid category" (which they defined) on the 2-category of derived categories of flag manifolds. Anno and Logveniko's work builded off previous work of Addington. Their lecture series on P-functors also tied in beautifully with the talks by Halpern-Leistner and Donovan who described similar phenomena, namely, braid group actions on derived categories and/or the 2-category of derived categories.

Official abstract: This talk is based on a joint work with T. Logvinenko, and gives some background for his talk on "P-functors". One of the major problems of working with triangulated categories is that the cone of a map between functors is not well defined, and in constructions such as that of P-twists, we need not just a cone, but a convolution of a three-term complex. In this talk, we will discuss Bondal and Kapranov's pretriangulated categories, where such convolutions exist naturally. We are also going to introduce the construction of a twisted line bundle over a DG category, a version of which is going to be instrumental in the definition of P-functors.

Official Abstract: $\mathbb{P}^n$ objects are a class of objects in derived categories of algebraic varieties first studied by Huybrechts and Thomas. They were shown to give rise to derived autoequivalences in a similar fashion to

Seidel-Thomas spherical objects. It was also shown that they could sometimes be produced out of spherical objects by taking a hyperplane section of the ambient variety. In this talk, based on work in progress with Rina Anno, we will first recall the basics on spherical and $\mathbb{P}^n$ objects, and then explain how to generalise the latter to the notion of P-functors between (enhanced) triangulated categories. We'll also discuss a closely related notion of a non-commutative line bundle over such category, inspired by a construction of Ed Segal.

Scientific Progress Made

The meeting collected a broad group of experts and early career mathematicians to focus on areas surrounding Homological Mirror Symmetry. The organizers made a concerted effort to invite smaller groups of existing collaborators and provide planned spaces in the schedule for these groups to focus on their work. Some particular groups:

- Rina Anno and Timothy Logvinenko
- Charles Doran, Tyler Kelly, and Ursula Whitcher
- Matthew Ballard and Tyler Kelly
- Will Donovan, and Ed Segal
- Yijia Liu and Andrew Harder
- Barbara Bolognese and Benjamin Schmidt
- Ludmil Katzarkov and Paul Horja
- Alessio Corti and Alexander Kaspryzk
- Nicholas Addington and Paul Aspinwall
- Gabriel Kerr and Ilia Zharkov

In addition, upon communication with participants after the completion of the conference, several people reported progress on new collaborations or emerging discussions which evolved during the week:

- Nick Addington: "I had a nice chat with Ed Segal about cubic 4-folds of discriminant 38."
- Will Donovan: "Toda and I discussed examples of rational curves with interesting non-commutative deformations, arising from work of Thomas (arXiv:math/9903034)."
- Paul Horja: "During the workshop, I worked on a joint project with Gabriel Kerr and Ludmil Katzarkov on Perverse Sheaves of Categories and Mirror Symmetry. I also had very fruitful discussions with Ed Segal and Will Donovan on the connection between global windows in VGIT, zonotopes and non-commutative resolutions."
- Tyler Kelly: "The workshop led to me having a conversation that helped find a bridge amongst symplectic geometry and algebraic geometry. Having people from both sides of mirror symmetry led to fruitful discussions between G. Kerr and myself, inspired by T. Logvinenkos talk."
- Helge Ruddat: "I had useful scientific exchange with Patrick Clarke on Frobenius manifolds and with Matt Ballard on log derived categories."
- Ed Segal: "This was a very productive workshop. I collaborated with Dan Halpern-Leistner developing some of our ongoing projects relating to derived categories and variation-of-GIT, I also had useful discussions with Kentaro Hori about some of his recent work in physics and how it relates to my mathematical work."

We are already aware of one paper whose creation owes substantial debt to the collaborative environment afforded by the workshop:

- Timothy Logvinenko: “I have worked with long-time collaborator Rina Anno on our ongoing project concerning P-functors, and more generally on the categorification of generalised braids, the project which the work on P-functors fits into and which we spoke on during the BIRS workshop.We have a pre-print titled ”P-functors”, shortly to appear on arXiv, and a large chunk of work on it was done during the BIRS workshop.”

Based on informal conversations with participants, we expect more such papers to appear imminently as many people are developing and finishing up projects over the summer.

Outcome of the Meeting

The workshop had 37 participants, ranging from Professors at Imperial and Duke to earliest career mathematicians (10 postdoctoral researchers and 2 graduate students). Among these were 5 female mathematicians. The scientific program consisted of 15 research lectures. Five talks were given by early career mathematicians. The participant list was international (UK, Japan, etc.) along with 6 Canadians. The schedule provided ample time for active research and structured mathematical discussion. Activity, in this regard, was high and organizers received compliments from participants for creating a productive atmosphere. The marvelous support and environment fostered by BIRS freed participants from day-to-day concerns and allowed for greater focus and productivity.

Participants

Addington, Nick (University of Oregon)
Aldi, Marco (Virginia Commonwealth University)
Anno, Rina (Massachusetts Institute of Technology)
Aspinwall, Paul (Duke University)
Ballard, Matthew (University of South Carolina)
Bayer, Arend (University of Edinburgh)
Bolognese, Barbara (University of Sheffield)
Clarke, Patrick (Drexel University)
Corti, Alessio (Imperial College London)
Diemer, Colin (University of Miami)
Donovan, Will (Tsinghua University)
Doran, Charles (University of Alberta)
Favero, David (University of Alberta)
Halpern-Leistner, Daniel (Columbia University)
Harder, Andrew (University of Miami)
Hori, Kentaro (Kavli Institute for the Physics and Mathematics of the Universe)
Horja, Paul (University of Miami)
Hosono, Shinobu (Gakushuin University)
Ingalls, Colin (Carleton University)
Karzhemanov, Ilya (IPMU)
Kasprzyk, Alexander (The University of Nottingham)
Katzarkov, Ludmil (University of Miami)
Kelly, Tyler (University of Cambridge)
Kerr, Gabriel (Kansas State University)
Liu, Yijia (McGill University)
Logvinenko, Timothy (Cardiff University)
Pearlstein, Gregory (Texas A&M)
Rizzardo, Alice (SISSA)
Rozhkovskaya, Natasha (Kansas State University)
Ruddat, Helge (Johannes-Gutenberg-UniversitŁt Mainz)
Sawon, Justin (University of North Carolina)

Schmidt, Benjamin (The Ohio State University)
Segal, Ed (Imperial College London)
Sharpe, Eric (Virginia Tech)
Toda, Yukinobu (University of Tokyo)
Whitcher, Ursula (American Mathematical Society)
Zharkov, Ilia (Kansas State University)

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Chapter 8

Beta Ensembles: Universality, Integrability, and Asymptotics (16w5076)

April 10 - 15, 2016

Organizer(s): Peter Forrester (University of Melbourne), Alice Guionnet (Massachusetts Institute of Technology), Brian Rider (Temple University), Benedek Valkó (University of Wisconsin - Madison)

Overview

Random matrix theory is a vibrant area of probability theory, with applications across mathematics, physics and engineering. The physically motivated β-ensembles (which can initially be viewed as models of a Coulomb gas) provide one-parameter families of particle systems that interpolate between the eigenvalue distributions of several of the classical models of random matrix theory (realized at $\beta = 1, 2, 4$). In recent years, the introduction of a range of new tools led to a period of intense research activity on the general beta ensembles, and our understanding of their properties continues at a fast pace. The wide range of new results naturally raise many open problems.

This meeting brought together 34 researchers from various groups working on different aspects of β-ensembles and related phenomena. Among the invited participants there were 7 postdocs and 3 Ph.D. students.

Scientific overview

The Gaussian invariant ensembles are one of the most studied models in random matrix theory. They were introduced in the 1950s by Wigner with the goal of modeling the energy levels of heavy atomic nuclei. The main idea was that the energy levels correspond to eigenvalues of a very complicated self-adjoint (or symmetric) operator, which we can be modeled by considering a large hermitian (or symmetric) matrix with iid complex or real standard normals (observing the appropriate symmetry). These models are called the Gaussian unitary (respectively orthogonal) ensembles (GUE and GOE), corresponding to the underlying invariance under conjugations by these symmetries. Dyson found that the joint eigenvalue density of the Gaussian ensembles is given by

$$\frac{1}{Z_{n,\beta}}\, e^{-\beta \sum_{k=1}^{n} \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}, \tag{8.0.1}$$

where β is 2 in the complex (Hermitian), and 1 in the real (symmetric) case. There is also a classical model for $\beta = 4$ with self-dual quaternion entries. Although the formula (8.0.1) is obtained as a joint eigenvalue density function of a random matrix, the density itself makes sense for any $\beta > 0$ with the appropriate normalizing constant $Z_{n,\beta}$. This family of distributions is called the Gaussian (or Hermite) β-ensemble. Note the structure of the density function: it contains a Vandermonde determinant raised to the power β and another component which can be thought of as a reference measure of iid normal distributions. It turns out that several other classical matrix models have joint eigenvalue densities with the same structure (Wishart, MANOVA, Haar unitary) for the special values $\beta = 1, 2, 4$. These models also have β-generalizations and so give rise to other beta-ensembles (e.g., the Laguerre, Jacobi, circular ensembles).

Dyson observed that these density functions can be thought of as a Boltzmann factor of a one-dimensional log-gas with logarithmic interaction and a certain background potential. The interaction corresponds to the Vandermonde term, while the background potential comes from the reference measure. Then the parameter β becomes the inverse temperature in the system. This observation lead to the study of β-ensembles using the methods of statistical mechanics. Several results about the classical $\beta = 1, 2$ and 4 cases were first discovered via this method by treating the general β case. A comprehensive overview of the field is covered in the recent monograph [18].

Starting with the work of Dyson, Gaudin and Mehta in the 60s and 70s and then with the ground breaking results of Tracy and Widom in the 90s building on the work of the Kyoto group in the 80s, the classical β cases turned out to be exactly solvable. The asymptotic local behavior of the eigenvalues of the Gaussian (and other classical) invariant ensembles has been fully described. (See [1], [28], [18] for an overview of these results.) Moreover, many of the limiting objects were soon after understood to capture the scaling limits of other integrable systems from combinatorics and statistical physics, most famously of various models of random growth composing the KPZ universality class [31]. In the meantime, the picture for the general beta case remained unclear.

The 2002 paper of Dumitriu and Edelman [15] introduced a key new tool into the study of general β-ensembles. There the authors showed that for any $\beta > 0$ one can construct a family of symmetric random tridiagonal matrices with independent entries so that the joint eigenvalue density is exactly the Gaussian β-ensemble (or β-Wishart/Laguerre ensemble). Similarly sparse structured matrix models were later found for the circular β-ensembles, including the Jacobi case [22].

Edelman and Sutton [17] then proposed that these tridiagonal models should be viewed as discretizations of random differential operators, and that the large dimensional limit of various local spectral statistics of the β ensembles could be obtained via the natural continuum limit of the given operator. This program was eventually carried out rigorously in [34], [32], [36], and [26]. The first two references provide random (Airy or Bessel type) differential operator descriptions of the "general β" Tracy-Widom edge distributions. The latter two gave (seemingly different) characterizations of the general β bulk process (for the Gaussian and circular ensemble respectively).

Importantly, these random operator and diffusion descriptions were novel even at the classical $\beta = 1, 2,$ and 4 values of the parameter. Furthermore, they have found basic applications in problems where direct analytic methods had appeared insufficient. To name a few examples: the proof of (a corrected version of) Dyson's conjecture for the bulk spacing distribution [37], sharp tail asymptotics of the edge laws [14] and unified small deviation bounds [27], and the resolution of the limit of the principal component for real spiked covariance matrices [6, 7].

The promise of these new methods prompted the 2009 AIM meeting *Brownian motion and random matrices* (http://aimath.org/pastworkshops/brownianrmt.html) organized by two members from the present list along with Bálint Virág (University of Toronto). Several of the problems and ideas proposed at that meeting have been taken up and in some cases successfully resolved, yet many basic problems remain and new directions have opened. In part, the BIRS workshop provided an important opportunity to see just how far the field has come in these few years and offered a forum to discuss the key open problems.

Recap of objectives

The main lines of proposed discussion for the BIRS workshop were the following.

1. **Universality**

   Spectacular progress has been made on the question of universality. That is, various of the general β limit distributions discovered via the random operator method have been proved to be universal for a wide class of Coulomb gas models on the line (where the classical Gaussian weight is replaced by a generic potential). The method of Krishnapur-Rider-Virág pushes forward the operator approach [23], while the robust proofs of Bourgade-Erdős-Yau [4, 5] realize in part Dyson's original vision. The transportation of measure approach of Bekerman-Figalli-Guionnet [2] introduces yet additional techniques. Still, basic questions regarding universality remain. As just one example, one imagines the "higher-order" Tracy-Widom laws (which arise in the case of non-regular potentials and have as yet only been rigorously addressed for $\beta = 2$ in [11]) have general β extensions, which are in turn universal in their context.

2. **Properties of the limit laws**

   Currently, the limiting point processes of the general β-ensembles are described via their counting function using coupled systems of differential equations, or as the spectra of certain random differential operators. These characterizations can be used to study various asymptotic properties of these processes (e.g. central limit theorem for the number of points, large deviation estimates of various quantities). Again though, there are many basic questions which are unanswered. The connection between the determinantal and Pfaffian descriptions of the classical limit processes and the characterizations of the general β processes is far from understood. In a second direction, the infinite systems of interacting diffusions introduced by Osada [30] and collaborators offer another setting to study the qualitative properties of the local laws. Last, new work on high-order expansions of the general β partition function and moment functionals (see e.g. [10], [19]) can provide insight given the topological information carried by the allied expansions at the classical values of β.

3. **Integrable systems**

   Another deep question is whether the general beta limit laws share a link to integrable systems as is famously the case for many of the classical random matrix limit laws. On one hand, the operator approach provides new partial differential description of many of these distribution functions (simultaneously for all beta). Whether or not the well known Painlevé property at $\beta = 1, 2, 4$ continues to hold is a tantalizing issue, while Rumanov has very recently made important progress in the case $\beta = 6$ [35]. In an entirely different direction, the theory of "Integrable Probability", in particular as introduced through the Macdonald Processes of Borodin-Corwin [8], are now known to include certain families of β-Jacobi ensembles [5].

4. **Numerics**

   Advances in software encompassing relevant special functions (such as multiple orthogonal polynomials and hypergeometric functions of matrix arguments) provide powerful new method for experimenting with beta ensemble distributions. Perhaps more importantly, the numerics have often guided new identities among these special functions (leading to exact finite dimensional formulas for certain beta function distributions) as well as introduced yet new sparse matrix models [13]. Finally, Edelman's method of "ghosts" [16] (with allied ideas appearing in the work of Forrester-Rains, see [20] for an application) offers the potential of a new algebraic framework in which to understand the beta ensembles.

Presentations

The workshop centered around 19 talks (of 50 minutes). Here we provide an overview of each, arranged loosely according to topic.

1. **Operator limits**

   Again, one of the primary tools for identifying the distributional limits for the beta-ensemble eigenvalues rests on proving the matrix models themselves have continuum (random) operator limits. We had three talks relating to quite different aspects of progress along these lines.

   José Ramírez spoke on the limiting smallest eigenvalue distributions (or hard edge laws) for sample covariance type matrices drawn from a spiked population [33]. Spiking here refers to a certain double scaling limit in which a fixed number of the coordinates of the (diagonal) population matrix tend to zero as the dimension goes to infinity. This complements (and through an additional limit recaptures) the spiked β soft edge laws due to Bloemendal-Virág [6, 7]. The hard edge limit laws are described in terms of a random integral operators, and partial differential equations satisfied by the corresponding distribution functions are derived as corollaries.

Igor Rumanov's talk showed how these partial differential equations can be used to give a Painlevé representation for the Tracy-Widom laws at $\beta = 6$, producing the first result of this kind for a β value which is not 1, 2 or 4.

Bálint Virág described recent results with Benedek Valkó on the Sine$_\beta$ operator, a first order vector-valued self-adjoint differential operator with spectrum given by the Sine$_\beta$ process, the bulk limit of the Gaussian β-ensembles. This work also provides a natural proof for the equivalence of the previous different-looking descriptions of the bulk limit through the Brownian carousel of Valkó-Virág and the diffusion type process discovered by Kilip-Stociu in the context of beta generalizations of unitary matrices. As an important side note, the resulting paper [38] was completed and posted on the ArXiv during the workshop.

2. **Log-correlated fields**

   Over the last several years there has been considerable interest in the properties of log-correlated random fields (the Gaussian Free Field and the Branching Random Walk providing two prime example), and in particular on the limiting distribution of the maximum which is supposed to have certain universal features. An important conjecture of Fyodorov and Simm is that the characteristic polynomial of various random matrix ensembles falls into this universality class.

   Both Elliot Paquette and Joseph Najnudel reported on progress on this conjecture, Paquette for the Gaussian Unitary Ensemble ($\beta = 2$) and Najnudel for the general beta circular ensembles.

   Christian Webb discussed a Stein's method approach to a central limit theorem for linear statistics of the beta circular ensembles [39].

3. β **ensembles in higher dimensions**

   While most of the progress thus far has focussed on β ensembles on the line (and their corresponding matrix models), it is natural to consider particle systems in two or higher dimensions with logarithmic (or similar) interaction. This is particularly relevant in dimension two in which case the ensemble is an honest Coulomb gas.

   Thomas Leblé gave an overview of recent results ([25, 26]) on large deviation principles of spatially averaged logarithmic and Riesz gases - a detailed asymptotics of the partition function being one approach to fluctuation results for linear statistics which is completely open in this context.

   Miika Nikula discussed an approach to rigidity estimates and local limit laws for 2-dimensional models using what is commonly referred to as the loop equations. This work has subsequently appeared in print [3].

4. **Sine$_\beta$ process**

   A major success of the operator (or diffusion) limit approach in random matrix theory is that it provides a tool to understand the qualitative features of the limiting spectral point processes. Laure Dumaz showed how the diffusion connected to the Brownian carousel can be used to prove that the high temperature ($\beta \to 0$) limit of the bulk β process is a homogeneous Poisson process. Diane Holcomb gave an overview of asymptotic results on low probability events (such as overcrowding of eigenvalues) connected to the Sine$_\beta$ process.

   In a different direction, Fumihiko Nakano showed how one can obtain the Sine$_\beta$ as the scaling limit of the spectrum of certain Schrödinger operators, and how this leads to another proof of the equivalence of the limit of the circular β-ensembles and the Sine$_\beta$.

5. **Numerics**

   Numerical analysis has long been a motivating source of random matrix theory. Indeed, the tridiagonal models of Dumitriu and Edelman are tied to the well known Householder transformations.

   In the opening talk of the workshop Alan Edelman revisited these now classical beta ensembles but through his idea of "ghosts". Ghosts offer the possibility of full matrix models for the beta ensembles. While not rigorously defined, Edelman demonstrated their effectiveness in computations.

Dumitriu meanwhile discussed the importance of the extreme eigenvalues of the beta Jacobi ensembles in various numerical applications, and also presented certain closed formulas for their distributions at finite dimension.

In a different direction, Govind Menon gave an overview on recent empirical studies [12] which suggest universality in the convergence times of various standard algorithms to compute eigenvalues. The beta ensembles provide important test cases in this work.

6. **Integrable systems**

   From the solvability of the unitary, orthogonal and symplectic ensembles to the appearance of the Painlevé functions in the limiting gap probabilities, there are intimate connections between random matrix theory and integrable systems.

   Alex Moll spoke about a law of large numbers and central limit theorem for the profile of partitions under Jack measures. The analysis rests on a connection to the quantum Benjamin-Ono equation [29].

   Karol Kozlowski discussed a Bethe Ansatz approach to the XXZ spin chain which has correlation functions that bare structural similarities to those of β ensembles.

7. **Dynamics, generalizations, and large deviations**

   As mentioned above, one of the most basic tools in random matrix theory is Dyson's Brownian motion: the eigenvalues of a Hermitian random matrix of independent Brownian motions form their own diffusion. While we do not have a stochastic matrix model for all beta (see the open problems below), one can write down a stochastic differential equation (with interaction) that has the correct eigenvalue law (as its stationary measure). Hirofumi Osada gave an overview of the theory required to construct these diffusions for infinitely many particles (eigenvalues).

   Moving away from the exact set up of the beta ensembles it is of interest to consider particle systems whose pairwise interaction is only "locally" coulombic. Starting with his thesis work (see [21]), Martin Venker has been one of the main researchers considering these questions. His talk summarized the current state of the art.

   Rounding out the talks, Alain Rouault showed how various sum rules (or "trace formulas") from the theory of orthogonal polynomials can be derived via large deviations estimates on standard beta ensembles.

Open problems

The workshop featured an open problem session Wednesday evening. All participants were invited to present a problem or topic for discussion. At the end of the of the session we also reviewed some the questions that were posed in the 2009 AIM workshop *Brownian Motion and Random Matrices* (and can be found at http://www.aimath.org/WWN/brownianrmt/brown on which there has been notable progress. The following is a sample of the open problems discussed.

1. M. Katori: Is there a stochastic (dynamic) version of the β-ghosts introduced by Edelman? Said differently, is it possible to generate a stochastic differential equation on tridiagonals whose fixed time marginals reproduce matrix models of Dumitriu and Edelman for say the β Hermite ensembles? This would in principle allow for a beta generalization of the important Airy process.

2. P. Koev: What algebraic properties should a "β-orthogonal matrix" satisfy? Granted that there is an honest object behind Edelman's ghosts, one imagines this can be diagonalized via conjugation by what could be viewed as a beta generalization of an orthogonal transformation. Can these β-orthogonal matrices be characterized in some independent way?

3. B. Virág: Consider the infinite version of the tridiagonal representation of the Gaussian β-ensemble and its spectral measure. Is the spectral measure absolutely continuous with respect to the Gaussian Chaos with the same dimension? (The Haussdorff dimension is expected to be $(1 - \frac{2}{\beta})_+$.) The spectral measure can be obtained as the limit of random measures with densities $\frac{1}{p_k(x)^2 + p_{k-1}(x)^2}$ where p_k is the k^{th} orthogonal

polynomial corresponding to the random three-term recursion determined by the entries of the tridiagonal matrix.

J. Najnudel: one can ask the same question for the circular β-ensemble and Verblunsky coefficients.

The presumed connection to the Gaussian chaos is a natural and deep question, and was already raised at the 2009 AIM meeting.

4. T. Leblé: What is the phase diagram of the Sine_β process? It is known that as $\beta \to 0$ the process converges to a homogeneous Poisson process while as $\beta \to \infty$ the process converges to the clock (or picket fence) process. Is there a phase transition for certain observables as β changes? A natural quantity to study would be the two-point correlation function.

5. I. Rumanov: Are there Painlevé representations for all Tracy-Widom(β)? Rumanov's progress on $\beta = 6$ is connected to an appropriate ansatz for the solution of the Lax pair of associated Quantum Painlevé II equation. The basic structure of this ansatz naturally extends to even integer values of beta. While pushing this idea forward remains a challenge, for other values of beta we still do not have a starting point for the analysis.

Conclusion

In all the workshop was a great success. The breadth of the lectures and open problems indicates a range of current activities with many questions remaining to be explored. We hope to establish regularly held workshops in the area.

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Chapter 9

Complex Analysis and Complex Geometry (16w5080)

May 1 - 6, 2016

Organizer(s): Finnur Lárusson (University of Adelaide), Norman Levenberg (Indiana University), Rasul Shafikov (University of Western Ontario) Alexandre Sukhov (Université des Sciences et Technologies de Lille)

Overview of the Field

Complex analysis and complex geometry form synergy through the geometric ideas used in analysis and analytic tools employed in geometry, and therefore they should be viewed as two aspects of the same subject. The fundamental objects of the theory are complex manifolds and, more generally, complex spaces, holomorphic functions on them, and holomorphic maps between them. Holomorphic functions can be defined in three equivalent ways as complex-differentiable functions, convergent power series, and as solutions of the homogeneous Cauchy-Riemann equation. The threefold nature of differentiability over the complex numbers gives complex analysis its distinctive character and is the ultimate reason why it is linked to so many areas of mathematics.

Plurisubharmonic functions are not as well known to nonexperts as holomorphic functions. They were first explicitly defined in the 1940s, but they had already appeared in attempts to geometrically describe domains of holomorphy at the very beginning of several complex variables in the first decade of the 20th century. Since the 1960s, one of their most important roles has been as weights in a priori estimates for solving the Cauchy-Riemann equation. They are intimately related to the complex Monge-Ampère equation, the second partial differential equation of complex analysis. There is also a potential-theoretic aspect to plurisubharmonic functions, which is the subject of pluripotential theory.

In the early decades of the modern era of the subject, from the 1940s into the 1970s, the notion of a complex space took shape and the geometry of analytic varieties and holomorphic maps was developed. Also, three approaches to solving the Cauchy-Riemann equations were discovered and applied. First came a sheaf-theoretic approach in the 1950s, making heavy use of homological algebra. Hilbert space methods appeared in the early 1960s and integral formulas around 1970 through interaction with partial differential equations and harmonic analysis. The complex Monge-Ampère equation came to the fore in the late 1970s with Yau's solution of the Calabi conjectures and Bedford and Taylor's work on the Dirichlet problem.

Today, as before, complex analysis and complex geometry is a highly interdisciplinary field. The foundational work described above has been followed by a broad range of research at the interfaces with a number of other areas, such as algebraic geometry, differential geometry, dynamical systems, functional analysis, homotopy theory, partial differential equations, and symplectic geometry. Complex analysts and complex geometers share a common toolkit, but find inspiration and open problems in many areas of mathematics.

Recent Developments and Open Problems

Holomorphic foliations

In recent years the theory of holomorphic foliations has become a central topic in geometric complex analysis and complex geometry. An important problem which has attracted the attention of several researchers concerns the existence of Levi-flat hypersurfaces in complex manifolds. Despite many attempts, the problem of the existence of a real analytic compact Levi-flat hypersurface in the complex projective plane still remains open. In higher dimensions this is a well-known result of Lins Neto [19].

Many interesting questions concern the local geometry of Levi-flat hypersurfaces near singular points. D. Burns and X. Gong [5] obtained a local classification of Levi-flat real analytic hypersurfaces near a Morse-type singularity. It is an open problem to obtain a similar classification for a wider class of singularities. Shafikov and Sukhov [22] proved that near nondicritical singularities, the Levi foliation extends as a singular holomorphic web to a neighbourhood of the singular point in the ambient space. Many interesting results and questions concerning the structure of Levi-degeneracy sets of CR manifolds are contained in the work of D. Cerveau and A. Lins Neto [6].

Another aspect of the theory involves the relationship between holomorphic foliations and the complex Monge-Ampére equation. The classical work of E. Bedford and M. Kalka in the late 1970s has had important applications in the recent work of Lempert and L. Vivas [18], and of Y. Rubinstein and S. Zelditch [21]. Lempert's approach (*Bull. Soc. Math. France,* 1981) to the construction of solutions of the complex Monge-Ampère equation was further developed by S. Donaldson [9], and more recently by G. Patrizio and A. Spiro [21], and in the almost complex setting by H. Gaussier and J. Joo [12]. More advanced analytic tools are used by V. Tosatti and B. Weinkove [23].

A series of papers by H. B. Lawson and F. R. Harvey (starting with *Amer. J. Math.,* 2009) is devoted to a geometric approach to the fundamental notion of plurisubharmonicity; they prove deep results on the Monge-Ampère equation on almost complex manifolds. This approach to plurisubharmonicity has already yielded a number of interesting applications, for example, Forstnerič and Drinovec Drnovšek in their presentations at the workshop gave a characterization of the minimal hull of compact sets in $\mathbb{R}^n$.

A very recent and highly active direction of research concerns the ergodic theory of holomorphic foliations, developed by T.-C. Dinh, V. Nguyên, and N. Sibony [8], and by C. Dupont and B. Deroin [7]. They have produced new and original constructions of invariant measures and Green currents associated to holomorphic foliations by analogy with discrete dynamical systems. This promising direction contains various natural and attractive questions. The interaction between holomorphic foliations and pluripotential theory is fundamental and will be a focus of activity in the coming years.

Elliptic complex geometry and Oka theory

Elliptic complex geometry is concerned with the flexible analytic geometry of complex affine spaces $\mathbb{C}^n$ and similar manifolds, opposite to the rigidity that characterizes hyperbolic complex manifolds. Three important classes of similar manifolds, in order of increasing size, are Stein manifolds with the density property, elliptic manifolds as defined by M. Gromov in a seminal paper of 1989, and Oka manifolds that have only recently emerged from the developments inspired by Gromov's paper.

The group of holomorphic automorphisms of $\mathbb{C}^n$, $n \geq 2$, is an infinite-dimensional group with a very rich structure. It has been intensively studied since a groundbreaking paper of E. Andersén and L. Lempert in 1992. Their work has been extended to Stein manifolds with the density property and applied to a range of embedding problems, in particular to the long-standing conjecture that every open Riemann surface can be embedded into $\mathbb{C}^2$. The flexibility of Oka manifolds is manifested in a tight connection between homotopy theory and complex analysis that has brought D. Quillen's theory of model categories into analysis for the first time in the work of F. Lárusson.

F. Forstnerič formally introduced Oka manifolds and Oka maps in 2009 and 2010, respectively, following his proof that more than a dozen Oka properties that had been under investigation for several years are all equivalent. A 500-page monograph of his, published in 2011 in Springer's Ergebnisse series, gives the first systematic exposition of much of the work that has been done in elliptic complex geometry and Oka theory over the past 25 years.

Recent work in elliptic complex geometry and Oka theory is varied and makes contact with several areas of mathematics. As the foundations of the area become more established, the focus of activity has shifted to applications and interaction with other areas.

Forstnerič and A. Alarcón have successfully applied concepts and methods from Oka theory to the theory of minimal surfaces: see for example [1]. In brand new work [11], Forstnerič and Lárusson have used convex integration theory and the theory of absolute neighbourhood retracts to prove a strong parametric h-principle for conformal minimal immersions and holomorphic null curves from a finitely-connected source. Among open problems in this area is how to treat more general sources; for those, Forstnerič and Lárusson proved a weak parametric h-principle.

F. Kutzschebauch and S. Kaliman continue to pursue a vigorous program of research into algebraic aspects of elliptic complex geometry, particularly the so-called algebraic density property. In very recent work [14], they have produced an effective sufficient condition for the algebraic volume density property and used it to show that certain homogeneous spaces satisfy the condition. The density property is very special, so new examples are of great interest. Kutzschebauch, R. Andrist and P.-M. Poloni have shown that certain Gizatullin surfaces have the algebraic density property [2]. Challenging problems remain open in this area, for example as simple a question as whether $(\mathbb{C}^*)^n$, $n \geq 2$, has the density property.

R. Lärkäng and Lárusson have made the first study of Oka theory for singular targets and in the process made a connection with the theory of affine toric varieties [17]. Their work shows that fundamental results of standard Oka theory break down when the targets are allowed to have mild singularities. To what extent a general Oka theory for singular targets can be developed is a challenging open question.

Holomorphic mappings

The theory of holomorphic mappings lies at the heart of the development of several complex variables in the second half of the twentieth century. In view of the classical result of Poincaré that the ball in $\mathbb{C}^n$, $n > 1$, is not biholomorphically equivalent to the polydisc, it is a fundamental problem to describe necessary and sufficient conditions for the equivalence of domains and the interaction between the equivalence of domains and their boundaries. Through attempts to better understand these questions, many important tools in complex analysis have been developed and sharpened, including metrics invariant under holomorphic maps, the theory of the Bergman projection and the $\overline{\partial}$-Neumann operator, the higher dimensional reflection principle, and Chern-Moser theory of local invariants of real hypersurfaces, to name a few.

In the spirit of the classical work of Bedford and Pinchuk in the nineties, K. Verma [24] generalizes the description of domains with noncompact automorphism groups in $\mathbb{C}^2$. It remains an open problem to classify domains with noncompact automorphism groups for a more general class of domains, say, pseudoconvex domains of D'Angelo finite type in $\mathbb{C}^n$. Several talks at the workshop directly or indirectly referred to this question.

I. Kossovkiy and R. Shafikov [26] recently gave an example of a family of nonminimal hypersurfaces in $\mathbb{C}^n$ which at a Levi degenerate point are formally CR equivalent but not biholomorphically equivalent, thus disproving a conjecture of Baouendi and Rothschild. Kossovskiy and B. Lamel [15] then constructed examples of smooth CR equivalences between Levi-degenerate hypersurfaces which are not analytic.

One of the open questions in CR geometry is the following: does there exist a compact, strictly pseudoconvex CR manifold in $\mathbb{C}^2$ that does not have any umbilical points? In his talk, P. Ebenfelt introduced a new approach to this problem, using a higher order version of Fefferman's Monge-Ampère operator (joint work with A. Zaitsev [10]).

Pluripotential theory and its applications

Pluripotential theory involves the study of plurisubharmonic functions on complex spaces. The theory of the complex Monge-Ampère operator on classes of plurisubharmonic functions on domains in $\mathbb{C}^n$ initiated by E. Bedford and B. A. Taylor was further developed by Z. Błocki, U. Cegrell, S. Kołodziej, and others. V. Guedj and A. Zeriahi [13] developed the notion of an ω-plurisubharmonic function on a compact Kähler manifold with a Kähler form ω. More recently, S. Boucksom, P. Eyssidieux, Guedj, and Zeriahi [4], advancing the interplay between pluripotential theory and complex geometry, laid down the foundations of pluripotential theory on compact complex manifolds in a general big cohomology class. A variational approach to solving degenerate Monge-Ampère equations in a big cohomology class on a compact Kähler manifold was developed by Berman, Boucksom, Guedj, and Zeriahi [3]. There are applications to topics as diverse as Kähler-Einstein metrics, Arakelov geometry, and equidistribution of zero sets of random sections of holomorphic line bundles.

Eyssidieux, Guedj, and Zeriahi developed an alternative approach to degenerate Monge-Ampére equations in terms of viscosity solutions, and compared viscosity concepts with pluripotential-theoretic ones. The purpose of Zeriahi's talk was to develop a viscosity theory for degenerate complex Monge-Ampère flows on compact Kähler manifolds. This was motivated by studying the time-asymptotic behavior of the Kähler-Ricci flow on mildly singular varieties. Their general theory allows them to generalize results of Song and Tian. Continuing on this geometric theme, Blocki discussed optimal regularity of geodesics in the space of Kähler metrics of a compact Kähler manifold (and on the space of volume forms on a compact Riemannian manifold) as they turn out to be solutions of homogeneous complex Monge-Ampère equations.

With respect to probabilistic results, Bayraktar presented several universality principles related to asymptotic zero distribution of random polynomials (as the degree goes to infinity), and, more generally, for random holomorphic sections of high powers $L^{\otimes n}$ of positive line bundles $L \to X$ for X a projective manifold endowed with a continuous metric. Roughly speaking, under natural assumptions, the asymptotic distribution is independent of the choice of probability law defining the random polynomials or sections.

As already mentioned, Forstnerič and Drinovec Drnovšek gave a characterization of the minimal hull of compact sets K in a minimally convex domain $D \subset \mathbb{R}^n$. This characterization was in the spirit of Poletsky's characterization of the polynomial hull of compact sets in $\mathbb{C}^n$: p belongs to the minimal hull of K with respect to D if and only if one can find a conformal minimal disk in D centered at p with most of its boundary near K. Poletsky raised the question of whether one could construct examples in the spirit of Stolzenberg-Wermer of a K with nontrivial minimal hull but with no minimal disks in its hull. The Poletsky theory relies on the fact that a plurisubharmonic function on a domain D is subharmonic on analytic disks in D. Poletsky's talk aimed at a deeper understanding of the space of analytic disks mapping into D.

Presentation Highlights

The presentations given at the workshop can be divided into several themes: geometry of real submanifolds (Drinovec Drnovsěk, Forstnerič, Gupta), geometry and dynamics of holomorphic maps (Arosio, Ebenfelt, Kossovskiy), automorphism groups (Andrist, Kaliman), function theory on complex manifolds (Bayraktar, Brudnyi, Chakrabarti, Kinzebulatov, Shcherbina), analytic discs (Bertrand, Poletsky), geometry of complex manifolds (Lärkäng, Merker, Prezelj, Wulcan), invariant metrics (Gaussier, Zimmer), Kähler metrics and Monge-Ampère equation (Błocki, Zeriahi). In addition, Bedford and Berteloot gave excellent survey talks that described mathematical ideas developed by Sergey Pinchuk who was honoured at this conference.

Speaker: **Rafael B. Andrist** (Bergische Universität Wuppertal)
Title: *The density property for Gizatullin surfaces of type* $[[0, 0, -r_2, -r_3]]$
Abstract: I will give a brief introduction to the density property for Stein manifolds, which is a notion to express that a manifold has "many" holomorphic automorphisms.

Although large classes of Stein manifolds with the density property are known, e.g. most of the homogeneous spaces of Stein Lie groups, they include only very few surfaces, namely $\mathbb{C}^2$, $\mathbb{C} \times \mathbb{C}^*$ and the smooth Danielewski surfaces. The lack of examples of such surfaces is due to the absence of the so-called "compatible pairs" of complete vector fields, which are usually the main tool for proving the density property.

Smooth Gizatullin surfaces provide a good class of candidates for surfaces with the density property, and they include the examples mentioned above. The next natural step is the investigation of Gizatullin surfaces of type $[[0, 0, -r_2, -r_3]]$, $r_2, r_3 \geq 2$, which can be described by the equations

$$\left\{ \begin{array}{rcl} yu & = & xP(x) \\ xv & = & uQ(u) \\ yv & = & P(x)Q(u) \end{array} \right.$$

in $\mathbb{C}^4$ with coordinates (x, y, u, v), where P and Q are polynomials of degree $r_2 - 1$ resp. $r_3 - 1$. We establish the density property for smooth Gizatullin surfaces of this type and describe a dense subgroup of the identity components of their holomorphic automorphism groups.

Joint work with Frank Kutzschebauch and Pierre-Marie Poloni.

Speaker: **Leandro Arosio** (Università di Roma "Tor Vergata")
Title: *Models for holomorphic self-maps of the unit ball*
Abstract: In order to study the forward or backward iteration of a holomorphic self-map f of a complex manifold X, it is natural to search for a semi-conjugacy of f with some automorphism of a complex manifold. Examples of this approach are given by the Schroeder, Valiron and Abel equation in the unit disc D. Given a holomorphic self-map f of the ball B^q, we show that it is canonically semi-conjugate to an automorphism (called a canonical model) of a possibly lower dimensional ball B^k, and this semi-conjugacy satisfies a universal property. This approach unifies in a common framework recent works of Bracci, Gentili, Poggi-Corradini, Ostapyuk.

This is done performing a time-dependent conjugacy of the autonomous dynamical system defined by f, obtaining in this way a non-autonomous dynamical system admitting a relatively compact forward (resp. backward) orbit, and then proving the existence of a natural complex structure on a suitable quotient of the direct limit (resp. subset of the inverse limit). As a corollary we prove the existence of a holomorphic solution with values in the upper half-plane of the Valiron equation for a hyperbolic holomorphic self-map of B^q.

Speaker: **Turgay Bayraktar** (Syracuse University)
Title: *Universality principles for random polynomials*
Abstract: In this talk, I will present several universality principles concerned with zero distribution of random polynomials or more generally random holomorphic sections of high powers $L^{\otimes n}$ of positive line bundle $L \to X$ defined over a projective manifold endowed with a continuous metric. In one direction, universality phenomenon indicates that under natural assumptions, asymptotic distribution of (appropriately normalized) zeros of random polynomials is independent of the choice of probability law defined on random polynomials. Another form of universality is asymptotic normality of smooth linear statistics of zero currents. Finally, if time permits, I will also describe some recent results on universality of scaling limits of correlations between simultaneous zeros of random polynomials.

Speaker: **Eric Bedford** (Stony Brook University)
Title: *From the Edge-of-the-wedge theorem to the Scaling method*

Speaker: **Francois Berteloot** (Université de Toulouse)
Title: *Rescaling methods in complex analysis*
Abstract: Rescaling methods are very efficient in complex analysis or geometry because they can be combined with the theory of normal families. We will survey some typical examples of such methods and in particular those introduced by Sergey Pinchuk.

Speaker: **Florian Bertrand** (American University of Beirut)
Title: *Riemann-Hilbert problems with singularities*
Abstract: The study of analytic discs attached to a totally real submanifold M of $\mathbb{C}^n$ leads to the consideration of a regular Riemann-Hilbert problem of a special form. Following this approach, Forstneric, and later on Globevnik, characterized the existence and dimension of a family of deformations of a given analytic disc attached to M in terms of certain indices. However, in case M admits some complex tangencies, the indices mentioned above are no longer well-defined and the Forstneric-Globevnik method falls apart. In this talk, I will focus on a class of such singular Riemann-Hilbert problems. We will see that they can be solved by a factorization technique that reduces them to regular Riemann-Hilbert problems with geometric constraints. In particular, we will deduce the existence of stationary type discs attached to finite type hypersurfaces.

Speaker: **Zbigniew Błocki** (Jagiellonian University)
Title: *Geodesics in the space of Kähler metrics and volume forms*

Abstract: We discuss optimal regularity of geodesics in the space of Kähler metrics of a compact Kähler manifold, as well as the space of volume forms on a compact Riemannian manifold. They are solutions of nonlinear degenerate elliptic equations: homogeneous complex Monge-Ampère equation and Nahm's equation (introduced by Donaldson), respectively. The highest regularity one can expect is $C^{1,1}$.

Speaker: **Alexander Brudnyi** (University of Calgary)
Title: *On the Sundberg approximation theorem*
Abstract: Let H^∞ be the Banach algebra of bounded holomorphic functions on the open unit disk $D \subset \mathbb{C}$. We extend Sundbergs theorem on uniform approximation of functions in BMOA by H^∞ functions to other classes of holomorphic functions on D. In our proofs we use a new characterization of meromorphic functions on D that extend to continuous maps of the maximal ideal space of H^∞ to the Riemann sphere.

Speaker: **Debraj Chakrabarti** (Central Michigan University)
Title: *L^2-cohomology of annuli and Sobolev estimates for the $\overline{\partial}$-problem*
Abstract: We consider the question of L^2-estimates for the $\overline{\partial}$-problem on annuli, a simple but interesting class of non-pseudoconvex domains. We relate this question with W^1-Sobolev estimates on the "hole" of the annulus. We then consider special classes of non-smooth holes for which the questions can be answered. This is joint work with Mei-Chi Shaw and Christine Laurent-Thibaut.

Speaker: **Barbara Drinovec Drnovšek** (University of Ljubljana)
Title: *Minimal hulls and minimally convex domains*
Abstract: Minimal hulls and minimally convex domains were introduced in a series of papers by Harvey and Lawson. They are natural substitutes for polynomial hulls and strictly pseudoconvex domains in the context of minimal surface theory. We present a characterization of the minimal hull of a compact set K in $\mathbb{R}^n$ by sequences of conformal minimal discs whose boundaries converge to K in the measure theoretic sense. We also study some properties of minimally convex domains. This is a report on a joint work with Alarcón, Forstnerič and López.

Speaker: **Peter Ebenfelt** (UCSD)
Title: *Stable umbilical points on perturbations of the sphere in $\mathbb{C}^2$*
Abstract: The standard CR structure on the three dimensional sphere can be deformed in such a way that the deformed structures have no (CR) umbilical points. A 1-parameter family of such deformations was essentially discovered by E. Cartan (and later studied by Cap, Isaev, Jacobowitz). The CR manifolds in this family, however, cannot be embedded in $\mathbb{C}^2$. It is an open question whether the unit sphere can be perturbed in $\mathbb{C}^2$ such that no umbilical points remain on the perturbed CR manifolds. In this talk, we shall discuss an approach to this problem, and describe some recent results. One of the results that will be described guarantees stable (in a sense to be made precise in the talk) umbilical points on generic perturbations of almost circular type. This complements a previous result by the speaker and Son Duong on existence of umbilical points on circular three-dimensional CR manifolds.

Speaker: **Franc Forstnerič** (University of Ljubljana)
Title: *The parametric h-principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$*
Abstract: Let M be an open Riemann surface. It was proved by Alarcón and Forstnerič that every conformal minimal immersion $M \to \mathbb{R}^3$ is isotopic to the real part of a holomorphic null curve $M \to \mathbb{C}^3$. We prove the following substantially stronger result in this direction: for any $n \geq 3$, the inclusion of the space of real parts of nonflat null holomorphic immersions $M \to \mathbb{C}^n$ into the space of nonflat conformal minimal immersions $M \to \mathbb{R}^n$ satisfies the parametric h-principle with approximation; in particular, it is a weak homotopy equivalence. Analogous results hold for several other related maps. For an open Riemann surface M of finite topological type, we obtain optimal results by showing that the above inclusion and several related maps are inclusions of strong deformation retracts; in particular, they are homotopy equivalences. (Joint work with Finnur Lárusson.)

Speaker: **Hervé Gaussier** (Université Grenoble Alpes)
Title: *Prime ends theory in higher dimension*
Abstract: This is a joint work with Filippo Bracci. We try to extend the Carathéodory prime ends theory in higher

dimension, defining the ”horosphere boundary” of complete hyperbolic (in the sense of Kobayashi) manifolds. We prove that a strongly pseudoconvex domain together with its horosphere boundary, endowed with the horosphere topology, is homeomorphic to its Euclidean closure, whereas the horosphere boundary of a polydisc is not even Hausdorff. As an application we study the boundary behaviour of univalent mappings.

Speaker: **Purvi Gupta** (University of Western Ontario)
Title: *Rational density on compact real manifolds*
Abstract: Motivated by the observation that every continuous complex-valued function on the unit circle can be approximated by rational combinations of a single function, we will discuss some conditions under which a manifold M admits N functions whose rational combinations are dense in the space of complex-valued C^k-functions on M. As a result, we will produce an optimal bound on N in terms of the dimension of M. This is joint work with R. Shafikov.

Speaker: **Shulim Kaliman** (University of Miami)
Title: *Algebraic (volume) density property*
Abstract: Let X be a connected affine homogenous space of a linear algebraic group G over $\mathbb{C}$. (1) If X is different from a line or a torus we show that the space of all algebraic vector fields on X coincides with the Lie algebra generated by complete algebraic vector fields on X. (2) Suppose that X has a G-invariant volume form ω. We prove that the space of all divergence-free (with respect to ω) algebraic vector fields on X coincides with the Lie algebra generated by divergence-free complete algebraic vector fields on X (including the cases when X is a line or a torus).

Speaker: **Damir Kinzebulatov** (Indiana University)
Title: *Chern classes of singular metrics on vector bundles*
Abstract: We extend the basic sheaf-theoretic techniques of complex function theory to work within some algebras of holomorphic functions (joint with Alex Brudnyi)

Speaker: **Ilya Kossovskiy** (University of Santa Catharina, Brazil)
Title: *Borel theorem for CR-maps*
Abstract: Following Henri Poincare, numerous results in Dynamics establish the curious phenomenon saying that two smooth objects (e.g., vector fields), which can be transformed into each other by means of a formal power series transformation, can be also transformed into each other by a smooth map. This is a kind of analogue of Borel Theorem on smooth realizations of formal power series. In CR-geometry, similar phenomena hold for real-analytic CR-manifolds, and the usual outcome is that two formally equivalent CR-manifolds are also equivalent holomorphically. However, in our recent work with Shafikov we proved that there exist real-analytic CR-manifolds, which are equivalent formally, but still not holomorphically.

On the other hand, in our more recent work with Lamel and Stolovitch we prove that the following is true: if two 3-dimensional real-analytic CR-manifolds are equivalent formally, then they are C^∞ CR-equivalent. In this talk, I will outline the latter result.

Speaker: **Richard Lärkäng** (Bergische Universität Wuppertal)
Title: *Chern classes of singular metrics on vector bundles*
Abstract: For holomorphic line bundles, it has turned out to be useful to not just consider smooth metrics, but also singular metrics which are not necessarily smooth, and which can degenerate. In relation to vanishing theorems and other properties of the line bundle, one considers plurisubharmonicity properties of the possibly singular metric which correspond to notions of positivity for the line bundle. In particular, having a positive singular metric means that the first Chern form associated to the metric is a closed positive $(1,1)$-current.

More recently, singular metrics on holomorphic vector bundles have been considered, Griffiths positivity of a singular metric on a vector bundle is defined in terms of plurisubharmonicity. For a vector bundle with a Griffiths positive singular metric, there is a naturally defined first Chern class which is a closed positive $(1,1)$-current, but there are examples where the full curvature matrix is not of order 0. I will discuss joint work with Hossein Raufi, Jean Ruppenthal and Martin Sera, where we show that one can give a natural meaning to the k:th Chern form $c_k(h)$

of a singular Griffiths positive metric h as a closed (k,k)-current of order 0, as long as h is non-degenerate outside a subvariety of codimension at least k. The proof builds on pluripotential theory, and in particular, one consider in the spirit of Bedford-Taylor products like $(dd^c\varphi)^q \wedge T$, where φ is plurisubharmonic and T is a closed positive (q,q)-current.

Speaker: **Joël Merker** (Université Paris-Sud)
Title: *Ample Examples*
Abstract: Tuan Huynh, Ph.D. student in Orsay, obtained (IMRN 2015) examples of Kobayashi-hyperbolic hypersurfaces $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of low degree $2n+2$ for $n = 2, 3, 4, 5$, and of degree $\frac{(n+3)^2}{4}$ for $n \geqslant 6$.

Song-Yan Xie, Ph.D. student in Orsay, established in 2015 the ampleness of cotangent bundles (jets of order 1) to generic complete intersections $\mathbb{X}^n \subset \mathbb{P}^{n+c}(\mathbb{C})$ of codimension $c \geqslant n$ with degrees $d_1, \dots, d_c \geqslant (n+c)^{(n+c)^2}$. This result answered fully a conjecture made by Debarre in 2005.

The first part of the talk will present a variation of S. Xie's proof, based on multidimensional resultants, which conducts to an improvement on the degree bound: $d_1, \dots, d_c \geqslant (n+c)^{n+c}$.

In order to reach an effective generic ampleness result about higher order jet bundles, in link with Kobayashi's hyperbolicity conjecture, taking inspiration from Masuda-Noguchi (1996), the second part of the talk will focus on families of hypersurfaces $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ having homogeneous polynomial defining equations of the form:

$$0 = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_{n+1} = \mathsf{fmn}} A_{\alpha_0,\alpha_1,\dots,\alpha_{n+1}}(X) \, \big((X_0)^d\big)^{\alpha_0} \big((X_1)^d\big)^{\alpha_1} \cdots \big((X_{n+1})^d\big)^{\alpha_{n+1}},$$

with *Fermat-Masuda-Noguchi index* $\mathsf{fmn} \geqslant n^2+n$, and with polynomials $A_\bullet\big(X_0 \colon X_1 \colon \cdots \colon X_{n+1}\big)$ homogeneous of relatively low degree $\deg A_\bullet =: \mathsf{a} \geqslant n$, compared with the dominant degree $d \geqslant n^{n^2}$.

Mainly, some appropriately truncated order-n jet bundles will happen to admit (a wealth of) global holomorphic sections, by means of a new process of forming (huge) Macaulay-type matrices, thanks to an application of Hartogs' theorem, a bit similarly as was performed by Siu-Yeung (Invent. 1996) and by Siu (Invent. 2015).

The end of the talk will conclude by presenting a link between the geometry of complex vector bundles and the first complete effective computations of CR curvatures of CR manifolds up to dimension $\leqslant 5$ performed by Samuel Pocchiola (ex-Ph.D. student in Orsay) and Masoud Sabzevari (Shahrekord).

Speaker: **Jasna Prezelj** (University of Ljubljana and University of Primorska)
Title: *Positivity of metrics*
Abstract: Let $p : Z \to Z$ be a submersion from a complex manifold Z to a 1-convex manifold X with an exceptional set X. Let $E \to Z$ Let $a : X \to Z$ be a holomorphic section. Then there exist a conic neighbourhood U of $a(X \setminus S)$ such that U admits a Kähler metric and a metric on E_U with positive Nakano curvature and with at most polynomial poles over $p^{-1}(S)$.

Speaker: **Evgeny Poletsky** (Syracuse University)
Title: *Homotopic properties of holomorphic mappings*
Abstract: Let W be a domain in a complex manifold M. In 2008 B. Jöricke found a way to extend holomorphic functions from W to another manifold and show that it is the envelope of holomorphy of W and in 2013 F. Lárusson and the speaker used a similar approach to subextend plurisubharmonic functions from W to a complex manifold. To define these manifolds the authors considered the space $A(W,M)$ of analytic disks in M whose boundaries lie in M. The new manifolds were defined as the quotients of this space by equivalence relations, where equivalent analytic disks can be connected by a continuous path or a homotopy in $A(W,M)$.

In 1983 L. Rudolph introduced quasipositive elements of braid groups that are fundamental groups of the complements to some set W of planes in Cn. He proved that these elements are boundaries of analytic disks in $A(W,\mathbb{C}^n)$ and form a semi- group. The talk will be divided in two parts. In the first part we will discuss general constructions of extensions of Riemann domains and subextensions of plurisubharmonic functions. In the second part we will address the notion of quasipositive elements in general situation and explain why they form a semigroup.

An important question is whether this semigroup is embeddable into the fundamental group. That is equivalent of asking whether two analytic disks are homotopic as analytic disks when their boundaries are equivalent in the fundamental group. A similar problem was studied by M. Gromov and, recently, by F. Forstnerič and his colleagues for homotopies of submanifolds in elliptic manifolds. In our case the ambient manifold is hyperbolic and the answer is not known. In the special case is when W is an analytic variety in M we will show that this problem can be reduced to the problem involving only real disks.

Speaker: **Nikolay Shcherbina** (University of Wuppertal)
Title: *A domain with non-plurisubharmonic squeezing function*
Abstract: We construct a strictly pseudoconvex domain with smooth boundary whose squeezing function is not plurisubharmonic. This is a joint work with J.E. Fornaess.

Speaker: **Alexander Tumanov** (University of Illinois at Urbana-Champaign)
Title: *Symplectic non-squeezing for the discrete nonlinear Schrödinger equation*
Abstract: The celebrated Gromov's non-squeezing theorem of 1985 says that the unit ball B^n in C^n can be symplectically embedded in the "cylinder" $rB^1 \times C^{n-1}$ of radius r only if $r \geq 1$. Hamiltonian differential equations provide examples of symplectic transformations in infinite dimension. Known results on the non-squeezing property in Hilbert spaces cover compact perturbations of linear symplectic transformations and several specific non-linear PDEs, including the periodic Korteweg - de Vries equation and the periodic cubic Schrödinger equation. We prove a new version of the non-squeezing theorem for Hilbert spaces. We apply the result to the discrete nonlinear Schrödinger equation. This work is joint with Alexander Sukhov.

Speaker: **Elizabeth Wulcan** (Chalmers University of Technology)
Title: *Direct images of semi-meromorphic currents*
Abstract: I will discuss a joint work in progress with Mats Andersson, in which we study and develop a calculus for direct images of semi-meromorphic currents. In my talk I will focus on regularity properties of these and in particular show that the sheaf of such currents is stalkwise injective.

Speaker: **Ahmed Zeriahi** (Université Paul Sabatier, Toulouse)
Title: *Weak solutions to degenerate complex Monge-Ampère flows*
Abstract: Studying the (long-term) behaviour of the Kähler-Ricci flow on mildly singular varieties, one is naturally lead to study weak solutions of "degenerate parabolic complex Monge-Ampère equations". The purpose of this work, is to develop a viscosity theory for degenerate complex Monge-Ampère flows on compact Kähler manifolds. The main ingredient is the "parabolic comparison principle" which allows us to prove uniqueness of the solution to a general complex Monge-Ampère flow starting from a singular metric with bounded potentials in a given Kähler class. Then using our previous results on the "degenerate elliptic side" of the complex Monge-Ampère theory, we are able to construct barriers that allow us to prove the existence of a solution to the Cauchy problem by means of the classical method of Perron. Our general theory allows in particular to define and study the behaviour of the (normalized) Kähler-Ricci flow on projective varieties with canonical singularities, generalizing results of Song and Tian. In the case when the variety is Calabi-Yau or of general type, we prove that the Kähler-Ricci flow converges weakly in the sense of currents (strongly at the level of potentials) to the unique Kähler-Einstein metric on the the variety. The case of intermediate Kodaira dimension is more tricky and will be briefly sketched if time permits. This is a joint work with P. Eyssidieux and V. Guedj which will appear in Advances in Math.

Speaker: **Andrew Zimmer** (University of Chicago)
Title: *Characterizing domains by their automorphism group*
Abstract: It is generally believed that (up to biholomorphism) very few domains have a large automorphism group and a nice boundary. For instance the Wong-Rosay Ball theorem says that a strongly pseudoconvex domain with non-compact automorphism group must be bi-holomorphic to the ball. Later, Bedford and Pinchuk proved that a convex domain of finite type and non-compact automorphism group must be bi-holomorphic to a domain defined by a polynomial. I will discuss a recent result which removes the finite type condition from the Bedford-Pinchuk result but at the cost of assuming that the automorphism group is slightly larger than non-compact. In particular,

a smoothly bounded convex domain is biholomorphic to a domain defined by a polynomial if and only if an orbit of the automorphism group accumulates on at least two different complex faces of the set. The proof of this result combines rescaling arguments and ideas from the theory of metric spaces of non-positively curvature.

Scientific Progress Made

Almost all the talks generated many interesting questions from the audience, related to the results presented in the talks. Some of the questions were about new directions of research, while others pointed to possible connections of the results to other fields of mathematics.

Aside from such questions, there were quite a few longer discussions between groups of participants regarding not only the topics presented in the lectures but also other important open problems. We note here a few such discussions.

A. Rashkovskii gave an informal seminar talk to N. Levenberg and N. Shcherbina on his recent study of local geodesics for plurisubharmonic functions (posted on arXiv). Rather than working in the space of Kähler metrics on compact, complex manifolds, he works on Cegrell classes of plurisubharmonic functions on bounded hyperconvex domains in $\mathbb{C}^n$. Rashkovskii will visit Levenberg in fall 2016 to continue these discussions. Shcherbina also mentioned the possibility of a future visit.

T. Bayraktar, T. Bloom and Levenberg had daily discussions on possible extensions of Bayraktar's results on asymptotic zero distribution of random polynomial mappings in $\mathbb{C}^n$. Bayraktar and Levenberg will organize a special session at an AMS meeting in spring 2017 on randomness in complex geometry.

L. Arosio and F. Lárusson discussed a new research project aimed at relating Oka theory and dynamics. Their discussions resulted in a proof, a few weeks after the workshop, of a new characterisation of ellipticity for Stein manifolds in dynamical terms.

I. Kossovskiy and R. Shafikov discussed connections between singular Levi-flat hypersurfaces and the theory of singular ODEs. This approach may lead to a classification of Levi-flat singularities in the spirit of Burns and Gong [5].

F. Berteloot, H. Gaussier, A. Sukhov and A. Zimmer discussed the relations between hyperbolicity in the sense of Kobayashi and hyperbolicity in the sense of Gromov. It is a striking recent achievement in complex geometry that these fundamental notions are closely related for a wide class of complex manifolds with boundary (such as bounded convex domains of finite type). Many open questions remain. These discussions will lead to the organization of a workshop or summer school concerning the interplay between complex hyperbolic theory, Gromov's theory of hyperbolic spaces, the theory of quasiconformal structures and related topics.

One usually cannot expect major theorems to be proved during a five-day workshop. However, the organizers are confident that the ideas generated by the talks and by the many discussions that took place during the week will lead to important progress in some of the many interrelated topics covered by the conference.

Outcome of the Meeting

Although a single workshop cannot do justice to the breadth and depth of contemporary complex analysis and complex geometry, the organizers believe it was beneficial to bring together a group of experts from diverse subfields to discuss recent results and work in progress and to share ideas on open questions. We chose a coherent collection of interrelated topics for the workshop, representing some of the most vibrant developments in the subject today.

The workshop covered a wide variety of topics of modern research in complex analysis and geometry. There were 39 participants, ranging from leading experts to graduate students (2 in total) and recent PhDs (7 in total). Among the participants were 5 female mathematicians.

The program consisted of 26 talks, each 45 minutes long, with a break of at least 15 minutes in between them. Seven of the talks were by recent PhDs. There were three full days, when the presentations ended by 5pm, while the remaining two days consisted of morning sessions. This allowed ample time for questions and discussions. Many of the talks exposed very recent important results. Quite a few of the talks reported on significant work in progress. Many of the talks gave rise to substantial discussions.

In conclusion, the participants found the workshop exciting and stimulating. The excellent facilities provided at BIRS, together with the inspiring scenery, helped to make the workshop a success.

Participants

Andrist, Rafael (Bergische UniversitŁät Wuppertal)
Arosio, Leandro (Università di Roma 2)
Bayraktar, Turgay (Syracuse University)
Bedford, Eric (Stony Brook)
Berteloot, Francois (Universite de Toulouse)
Bertrand, Florian (American University of Beirut)
Blocki, Zbigniew (Jagiellonian University)
Bloom, Tom (University of Toronto)
Brudnyi, Alex (University of Calgary)
Chakrabarti, Debraj (Central Michigan University)
Drinovec Drnov?ek, Barbara (University of Ljubljana)
Ebenfelt, Peter (University of California at San Diego)
Edigarian, Armen (Jagiellonian University)
Forstneric, Franc (University of Ljubljana)
Gaussier, Herve (Université Grenoble Alpes)
Gupta, Purvi (University of Western Ontario)
KALIMAN, Shulim (University of Miami)
Kinzebulatov, Damir (University of Toronto)
Kossovskiy, Ilya (University of Vienna)
Larkang, Richard (University of Wuppertal)
Larusson, Finnur (University of Adelaide)
Levenberg, Norman (Indiana University)
Magnsson, Benedikt (University of Iceland)
Merker, Joel (Universit Paris-Sud)
Mitrea, Octavian (University of Western Ontario)
Poletsky, Evgeny (Syracuse University)
Prezelj, Jasna (University of Ljubljana and University of Primorska)
Ramos-Peon, Alexandre (University of Bern)
Rashkovskii, Alexander (Stavanger University)
Ritter, Tyson (University of Oslo)
Shafikov, Rasul (University of Western Ontario)
Shcherbina, Nikolay (University of Wuppertal)
Sukhov, Alexandre (Universite des Sciences et Technologies de Lille)
Tumanov, Alexander (University of Illinois at Urbana-Champaign)
Vivas, Liz (Ohio State)
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Wulcan, Elizabeth (Chalmers University of Technology)
Zeriahi, Ahmed (Universite Paul Sabatier (Toulouse))
Zimmer, Andrew (University of Chicago)

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Chapter 10

Variational Models of Fracture (16w5090)

May 8 - 13, 2016

Organizer(s): Blaise Bourdin (Louisiana State University), Corrado Maurini (Université Pierre et Maris Curie), Gilles A. Francfort (Université Paris Nord), Christopher J. Larsen (Worcester Polytechnic Institute)

Variational models of brittle fracture find their origin in Griffith's classical theory, revisited using modern mathematical tools. They preserve the essence of Griffith argument (that brittle fracture can be seen as competition between bulk and surface energies), and recast it as sequences of unilateral minimization principle for a free-discontinuity energy. In doing so, they largely address most of the shortcomings of the classical brittle fracture approach, such as the difficulty to handle changes in cracks topology, the required a priori knowledge of crack path, or the difficulty to handle three-dimensional problems or heterogeneous materials. In contrast, variational models of fracture require no a priori assumption on cracks geometry, can easily handle changes in fracture topology such as new cracks nucleation, branching or merging, and are formulated identically in two or three space dimensions. The numerical implementation of these models also requires special techniques, with the most popular ones being based on some form of regularized or "phase–field" models. In the last few years, the interest of the computing, engineering and industrial communities to this area has been mostly driven by their unprecedented capabilities to tackle problems out of reach of the classical approaches. This has resulted in many extensions of the original variational framework, most of which are currently beyond reach of rigorous mathematical analysis.

Scientific context

Since Francfort and Marigo initiated a revisiting of Griffith's classical theory of fracture of brittle fracture with modern mathematical tools [**?**, **?**], variational models in fracture mechanics have been the subject of intense activity. The interest of the mathematical community for this class of problem can be measured by a large body of work in areas including mathematical analysis, mathematical modeling, numerical analysis and computational science, several workshop and mini-symposia including weeklong meetings in Oberwolfach in 2007 and 2011 and at BIRS in 2011, or G.A. Francfort's plenary lecture at the 2011 ICIAM congress. In recent years, the interest of the engineering community in this class of model has also rapidly increased, mainly motivated by the potential for rigorous, accurate and efficient numerical simulations. This interest can be measured for instance by the size of mini-symposia relating to "phase–field" approach in fracture mechanics congresses, or through M. Ortiz and C. Miehe's plenary lectures at the 2014 WCCM conference. The interest of various industries in this class of problems is also noticeable with ongoing projects at Corning, Chevron, Lafarge, and Airbus, amongst others.

Yet, as the mathematical understanding of this problem is reaching its maturity, we are reaching a tipping point where the mathematics and engineering or computational science communities are becoming increasingly divided, instead of mutually benefiting from each other's progress. The reasons of this growing schism are multiple: the mathematical literature can be very technical and deter engineering students. Conversely, mathematicians may have a hard time translating issues mentioned in the engineering and technical literature into well defined mathematical problems. This is especially true of recent extensions to dynamic or rate dependent problems or modeling of coupled problems in reservoir engineering, fracture of ferromagnetic materials or corrosion cracks in thin coatings, for which a rigorous mathematical understanding is lacking. Finally, graduate students and young researchers often lack a common culture with engineers being unfamiliar with modern mathematical tools such as geometric measure theory or Γ–convergence, and mathematicians lacking awareness of actual problems.

Specific objectives

The goal of this workshop was to bring together a group of mathematicians, mechanicians, engineers and computational scientist sharing an interest in variational models of fracture mechanics in order to achieve a breakthrough in the mathematical understanding of current topics, tools and issues, and in the scope of the numerical applications of the current theories. The specific objectives were:

- O_1: To present the state of the art of the mathematical analysis of problems arising from variational models of fracture. A better understanding of the mathematical issues arising in these problems is essential to reach a deep understanding of the numerical methods. Yet, there is a lack of concise and focussed literature at the graduate level. We will begin this workshop with a few introductory lectures on mathematical modeling and tools so as to give the more applied participants a (possibly critical) overview of the current state of the theory.

- O_2: To gain a better understanding of the challenges facing this class of methods. "Real life" problems can be quite at odds with those favored by mathematicians and are often beyond the reach of rigorous analysis. Engineers and industry partners will be invited to present current or potential applications and algorithms related to the variational models of fracture. The rationale is that a better theoretical understanding of this problem can lead to more efficient numerical tools, while exposure to a broader range of problems will stimulate new theoretical developments.

- O_3: To devise a set of reference problems that can be analyzed rigorously, then used in order to assess the accuracy and efficiency of algorithms. The popularity of benchmark problems in fracture mechanics is highly skewed by the strength and weaknesses of classical methods. The resulting tests are often inappropriate or even non sensical from the standpoint of the variational approach to fracture. Devising proper numerical experiments highlighting specific properties of a model or implementation is difficult and time consuming. In addition to the lack of common reference tests, comparing methods is difficult. We propose to come up with a small set of problems which will be used in the years to come.

Participants

In order to achieve these goals, the organizers seek to build a compact participant list with even representation from the mathematics, mechanics, numerical simulation, experimental, and industrial communities, as well as some experts from adjacent fields.

Presentation highlights

The participants were instructed to focus on the three specific objectives listed above in there presentation and during informal discussion sessions. We summarize some of the presentations that best addressed our objectives, as well as those bringing awareness to new problem, methods and challenges.

The first day was dedicated to four hour-long "keynote" lectures: K. Ravi-Chandar opened the workshop by presenting some and numerical results for mixed-mode fracture. In two dimensions, he proposed a modified compact-tension experiment for which stable mixed-mode I-II crack propagation along a curved crack path can be achieved. In three dimension, he focussed on mixed mode I-III propagation leading to echelon and factory crack propagation. Such problems have received a lot of attention lately including numerical simulations by participant A. Karma and experiment by participant A. Zehnder: "Transition from spiral to factory roof type fracture under tension". Both are excellent benchmarks problems addressing objective O_3.

The following presentations focussed objective O_1 with recent results on variational and phase field models in the mathematics, mechanics and physics communities respectively. G. Dal Maso focussed on recent mathematical results dealing with existence theory for evolutions satisfying a maximum dissipation condition in the dynamic regime. J.-J. Marigo presented recent works on coupling variational models of damage and plasticity. A. Karma focussed on dynamic instabilities of fast moving cracks in brittle solids.

During the next few days several presentations addressed extension of variational and phase field models to plates and shells (B. Roman: "Fracture path in thin sheets", M. Arroyo: "Phase-field modeling of fracture in thin shells", J.-F. Babadjian: "Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination", and B. Schmidt: "On a quantitative piecewise rigidity result and Griffith-Kirchhoff functionals for thin brittle beams"). Applications to hydraulic fracturing also emerged as a critical application that requires a better understanding with presentations from C. Landis: "Phase-Field modeling of Hydraulic Fracture", M.F. Wheeler: "Phase-field modeling of proppant-filled fractures in a poroelastic medium", A. Pandolfi: "A variational model of poro-mechanical damaging material", and K. Yoshioka: "variational fracture modeling applied to hydraulic fracturing (fracking)". Extensions to plasticity and ductile fracture, were also addressed in presentations by G. Lancioni: "A variational approach to gradient plasticity", and M. Ortiz: "Optimal scaling in ductile fracture". These topics clearly emerged as areas of need, as stated in objective O_2.

Objective O_1 was again at the center of presentations focussing on analysis of models by A. Chambolle: "Some remarks on the energy release rates in planar linearized elasticity" and K. Pham: "Stability analysis of homogeneous states in gradient damage models". Alternate numerical methods and extension were the focus of by M. Negri': "Convergence in time of discrete evolutions generated by alternate minimizing schemes", and M. Kimura: "Unidirectional gradient flow and its application to a crack propagation model".

Finally multiple participants presented alternative models of damage and fracture, and innovative numerical methods, including R. Haber: "Spacetime interfacial damage model for dynamic fracture in brittle materials", G. Allaire "Damage and fracture evolution in brittle materials by shape optimization methods", R. Lehoucq: "recent developments in peridynamic mechanics", M. Sarkis: "Finite element methods on non-aligned meshes for interface problems", A. Lew: "high-fidelity simulation of brittle fracture problems with Universal Meshes", and M. Luskin: "Lattice stability of hybrid atomistic to continuum coupling methods for fracture modeling".

It is also to be noted that feedback on challenges facing the industry was provided by K. Yoshioka (Chevron ETC) and V. Subramanian (Corning inc.).

Outline of presentations

Grégoire Allaire: Damage and fracture evolution in brittle materials by shape optimization method. This joint work with François Jouve and Nicolas Van Goethem is devoted to a numerical implementation of the Francfort-Marigo model of damage evolution in brittle materials. This quasi-static model is based, at each time step, on the minimization of a total energy which is the sum of an elastic energy and a Griffith-type dissipated energy. Such a minimization is carried over all geometric mixtures of the two, healthy and damaged, elastic phases, respecting an irreversibility constraint. Numerically, we consider a situation where two well-separated phases coexist, and model their interface by a level set function that is transported according to the shape derivative of the minimized total energy. In the context of interface variations (Hadamard method) and using a steepest descent algorithm, we compute local minimizers of this quasi-static damage model. Initially, the damaged zone is nucleated by using the so-called topological derivative. We show that, when the damaged phase is very weak, our numerical method is able to predict crack propagation, including kinking and branching. Several numerical examples in $2d$ and $3d$ are discussed.

Jean-François Babadjian: Reduced models for linearly elastic thin films allowing for fracture, debonding or delamination. This talk is devoted to highlighting the interplay between fracture and delamination in thin films. The usual scaling law on the elasticity parameters and the toughness of the medium with respect to its thickness gives rise to traditional cracks which are invariant in the transverse direction. We will show that, upon playing on this scaling law, it is also possible to observe debonding effects (delamination as well as decohesion) through the appearance of cracks which are orthogonal to the thin direction. Starting from a three-dimensional brittle elastic thin film, we will first present how both phenomena can be recovered independently through a Gamma-convergence analysis as the thickness tends to zero. Then, working on a ?toy model" for scalar anti-plane displacements, we will show how both phenomena can be obtained at the same time. Some partial results in the full three-dimensional case will be presented. These are joint works with Blaise Bourdin, Duvan Henao, Andrs Leon Baldelli. and Corrado Maurini.

Antonin Chambolle: Some remarks on the energy release rates in planar linearized elasticity The talks reviewed a series of results obtained in the past ten years with J.-F. Babadjian, G. Francfort, A. Lemenant and J.-J. Marigo on the singularities at the tip of cracks in planar elasticity with minimal regularity assumptions, and the appropriate definition of a generalised Energy Release Rate in such a setting.

In [**?**, **?**], we have shown that in the variational approach to brittle fracture (which generalises the classical Griffith theory, see [**?**, **?**]), if one considers that a crack may grow in an arbitrary way which is not necessarily smooth at infinitesimal scales, then one should arrive to the conclusion that a straight fracture under a mode II loading should become unstable for lower loads than classically predicted (by criteria such as the "principle of local symmetry" or "maximum ERR") and kink "brutally" rather than in a continuous way. Obviously, the "correct" scale cannot be prescribed by Griffith's model and the statement, as such, is unphysical: however it should give an idea of how the macroscopic behaviour of brittle materials should appear in such situations

In [**?**] (antiplane case) and [**?**] (planar elasticity), we have partially extended these results to situations where the initial crack itself is not even smooth, showing that also in such situations, the singularity at the tip of an arbitrary crack which looks like a half-line at infinitesimal scales is (as expected) the same as for a smooth crack ending in a point, and similar results should be expected. A natural conclusion (which is mathematically out of hand by now) should be that quasi-static fractures whose length is growing in a continuous way (as a function of time) should be at least C^1 curves.

Gianni Dal Maso: A maximal dissipation condition for dynamic fracture: an existence result in a constrained case. The study of dynamic fracture is based on the dynamic energy-dissipation balance. It is easy to see that this condition is always satisfied by a stationary crack together with a displacement satisfying the system of elastodynamics. Therefore to predict crack growth a further principle is needed. In this talk we introduce a maximal dissipation condition that, together with elastodynamics and energy balance, provides a model for dynamic fracture, at least within a certain class of possible crack evolutions. In particular, we prove the existence of dynamic fracture evolutions satisfying this condition, subject to smoothness constraints, and exhibit an example to show that maximal dissipation can indeed rule out stationary cracks.

Robert Haber: Spacetime Interfacial Damage Model for Dynamic Fracture in Brittle Materials. While most of fundamental physical questions in dynamic fracture mechanics are settled science, there remains a significant gap between this fundamental understanding and our ability to apply it in computational models of failure in the complex systems and materials that arise in geophysics, biology, and contemporary engineering design. This talk describes recent progress at the University of Illinois and the University of Tennessee Space Institute in developing new numerical methods and models intended to close at least some aspects of this gap. A spacetime discontinuous Galerkin (SDG) method is the numerical foundation for our fracture model. This particular SDG method [**?**] is tailored to the requirements of hyperbolic systems, and differs from most others in that it is asynchronous, locally implicit, embarrassingly parallel, and supports fine-grain adaptive meshing. It enforces jump conditions with respect to Riemann solutions on element boundaries to preserve the characteristic structure of the underlying system. As with other DG methods, conservation fields balance to within machine precision on every (spacetime) element. We model crack opening and closure with specialized Riemann solutions for the various modes of frictional dynamic contact [**?**]. We weakly enforce these Riemann solutions using the same framework that enforces jump conditions and boundary conditions at inter-element and domain boundaries elsewhere in the SDG formulation. This approach produces contact conditions that are distinct from those that arise from simple constraints against inter-element penetration. We can implement cohesive fracture models in this SDG framework by incorporating traction? separation laws in the Riemann solutions [**?**]. However, in this presentation we focus on interfacial damage as an alternative means to model fracture along sharp interfaces. Time-delay evolution equations determine the damage rate as functions of the tractions and velocity jumps across fracture surfaces. A probabilistic model for microscopic flaws provides a mechanism for nucleating new fracture surfaces and is sufficient to captures crack branching. Adaptive refinement ensures that the solution fields are well resolved at multiple crack tips and along wave fronts. The same adaptive procedures continuously reconfigure the mesh so that it follows the crack paths determined by our physical model. We discuss some of the open challenges in modeling fracture with interfacial damage and present several numerical examples to demonstrate existing capabilities.

Work in collaboration with Reza Abedi, Mechanical, Aerospace & Biomedical Engineering, University of Tennessee Space Institute (UTSI) / Knoxville (UTK), 411 B. H. Goethert Parkway, Tullahoma, TN 37388

Alain Karma: Phase-field modeling of rapid fracture in linear and nonlinear elastic solids. This talk will discuss phase-field modeling of dynamic instabilities of fast moving cracks in brittle solids. Experiments in thin gels have shown that cracks can attain extreme speeds approaching the shear wave speed when micro branching, which limits propagation to smaller speeds in thick samples, is suppressed. Furthermore, they have revealed the existence of an oscillatory instability with an intrinsic system-size-independent wavelength above a threshold speed. In apparent contradiction with experimental observations, the commonly used phase-field formulation of dynamic fracture yields crack that branch by tip splitting at roughly half the shear wave speed. A phenomenologically based phase-field formulation is proposed that can model crack propagation at extreme speeds by maintaining the wave speed constant inside the microscopic process zone. Simulations of this model for linear elasticity outside the process zone produce crack that tip split above a high threshold speed but no oscillations. In contrast, simulations for nonlinear neo-Hookean elasticity yield crack oscillations above a ultra-high speed threshold. Those oscillations have an intrinsic wavelength that scale with the size of the nonlinear zone surrounding the crack tip, which can be much larger than the process zone scale, and bear striking similarity with observed oscillations in thin gels.

This work was carried out in collaboration with Chih-Hung Chen and Eran Bouchbinder and his supported by a grant from the US-Israel Binational Science Foundation.

Masato Kimura: Unidirectional gradient flow and its application to a crack propagation model. We consider a nonlinear diffusion equation with irreversible property and construct a unique strong solution by using implicit time discretization. A new regularity estimate for the classical obstacle problem is established and is used in the construction of the strong solution.

As an application, we consider a quasi-static fracture model of brittle material using the idea of the phase field model. The Francfort-Marigo energy which is based on the classical Griffith theory is introduced, where the sharp crack profile is approximated by a smooth damage function using the idea of the Ambrosio-Tortorelli regularization. The crack propagation model is derived as a gradient flow of the energy of the damage variable with an irreversible constraint. Some numerical examples in various settings computed by finite element method are also presented in the talk.

The contents is based on the joint works with Goro Akagi (Kobe Univ.) and with Takeshi Takaishi (Hiroshima Kokusai Gakuin Univ.).

Giovanni Lancioni: A variational approach to gradient plasticity. In this talk, a variational model for gradient plasticity is proposed, which is based on an energy functional sum of a stored elastic bulk energy, a non-convex dissipative plastic energy, and a quadratic non-local term, depending on the gradient of the plastic strain. The basic modelling ingredients are presented in a simple one-dimensional setting, where the key physical aspects of the phenomena can easily be extracted. The evolution laws are deduced by using the mathematical tool of incremental energy minimization, and they are commented, highlighting the main differences and similarities with variational damage models. The typical assumptions of classical plasticity, such as yield condition, hardening rule, consistency condition, and elastic unloading, are obtained as necessary conditions for a minimum. Then, analytical solutions are determined, and attention is focused on the correlations between the convex-concave properties of the plastic energy and the distribution of the deformation field. The issue of solution stability is also addressed. Finally, some numerical results are discussed. First, tensile tests on steel bars and concrete samples are reproduced, and, then, a more complex two-dimensional crystal plasticity is proposed, and the process of microstructures evolution in metals is described by assuming a double-well plastic potential.

Chad Landis: Phase-field Modeling of Hydraulic Fracture. In this talk a theoretical framework implementing the phase-field approach to fracture is used to couple the physics of flow through porous media and cracks with the mechanics of fracture. The main modeling challenge addressed in this work, which is a challenge for all diffuse crack representations, is on how to allow for the flow of fluid and the action of fluid pressure on the aggregate within the diffuse damage zone of the cracks. The theory is constructed by presenting the general physical balance laws, postulating a kinematic ansatz for an effective porosity, and conducting a consistent thermodynamic analysis to constrain the constitutive relationships. Constitutive equations that reproduce the desired responses at the various limits of the effective porosity are proposed in order to capture Darcy-type flow in the intact porous medium and Stokes-type flow within open cracks. A finite element formulation for the solution of the governing model equations is presented and discussed. Finally, the theoretical and numerical model is shown to compare favorably to several important analytical solutions. More complex and interesting calculations are also presented to illustrate some of the advantageous features of the approach.

Richard Lehoucq: Recent developments in peridynamic mechanics. My presentation introduces the peridynamic model for predicting the initiation and evolution of complex fracture patterns. The model, a continuum variant of Newton's second law, uses integral rather than partial differential operators where the region of integration is over a domain. The force interaction is derived from a novel nonconvex strain energy density function, resulting in a nonmonotonic material model. The resulting equation of motion is proved to be mathematically well-posed. The model has the capacity to simulate nucleation and growth of multiple, mutually interacting dynamic fractures. In the limit of zero region of integration, the model reproduces the classic Griffith model of brittle fracture. The simplicity of the formulation avoids the need for supplemental kinetic relations that dictate crack growth or the need for an explicit damage evolution law.

Adrian Lew: High-fidelity Simulation Of Brittle Fracture Problems With Universal Meshes We describe our approach to simulating curvilinear brittle fractures in two-dimensions based on the use of Universal Meshes [**?**, **?**, **?**, **?**, **?**]. A Universal Mesh is one that can be used to mesh a class of geometries by slightly perturbing some nodes in the mesh, and hence the name universal. In this way, as the crack evolves, the Universal Mesh is always deformed so as to exactly mesh the crack surface. The advantages of such an approach are: (a) no elements are cut by the crack, (b) new meshes are automatically obtained as the crack evolves, (c) the crack faces are exactly meshed with a conforming mesh at all times, and the quality of the surface mesh is guaranteed to be good, and (d) apart from duplicating degrees of freedom when the crack grows, the connectivity of the mesh and the sparsity of the associated stiffness matrix remains unaltered. In addition to the mesh, we are now able to compute stress intensity factors with any order of convergence, which gives us unprecedented accuracy in computing the crack evolution. As a result, we observe first order convergence of the crack path as well as the tangent to the crack path in a number of different examples. In the presentation I will introduce the notion of a Universal Mesh, illustrate the progress we have made so far with some examples, and then focus on the simulation of curvilinear fractures, and on the tools we created to compute stress intensity factors. In particular, show examples in which the computed crack path converge to the exact crack path, regardless of the mesh. If time permits, simulation of thermally induced fracture and hydraulic fractures will be discussed.

Mitchell Luskin: Lattice Stability of Hybrid Atomistic to Continuum Coupling Methods for Fracture Modeling. Hybrid atomistic to continuum coupling methods have been developed to obtain the accuracy of atomistic modeling in the neighborhood of crack tips, while using continuum modeling to include long range elastic effects. We will present a survey of recent work to analyze the lattice stability error introduced by atomistic to continuum coupling methods. These error estimates are then utilized to develop accurate blended coupling methods with controllable error.

Joint work with Christoph Ortner, Mathew Dobson, Xingjie Helen Li, Derek Olson, and Brian Van Koten.

Matteo Negri: Convergence in time of discrete evolutions generated by alternate minimizing schemes We consider two quasi-static evolutions, of BV-type, for the Ambrosio-Tortorelli energy. Both are obtained by time discretization and by alternate minimization schemes, thus taking full advantage of the separate quadratic structure of the energy.

For the first we will employ a constrained alternate minimization scheme in which the time-update configuration is found by an iterative procedure, either finite or infinite. In the latter case the updated configuration is an equilibrium point for the energy. This algorithm can be recast both as a separate gradient flow, with respect to a suitable family of intrinsic norms, and also by a "quasi-Newton" method. Both the representations highlight the underlying family of intrinsic norms for the evolution. After re-parametrizing the evolution by means of an arc-length parameter we can conveniently pass to the limit, characterized in terms of a (parametrized) BV-evolution. In particular we show that in the regime of stable (or steady state) propagation the limit evolution satisfies equilibrium for the displacement variable and a suitable form of Griffith's criterion for the phase-field variable, written in terms of a phase-field energy release rate. We further show that the irreversibility constraint, given by the monotonicity of the phase-field variable, is thermodynamically consistent, since the associated dissipated energy is non-decreasing in time. Further we characterize the unstable regime of propagation in terms of gradient flows (in the parametrization variable) with respect to the intrinsic norms. The fact that the limit evolution is "simultaneous" in the two variable, even if the algorithm is not, is justified by continuous dependence.

In the second case we consider an irreversible L^2-gradient flow for the phase field variable combined with the equilibrium equation for the displacement field. We obtain again the time-continuous evolution as the limit of a time-discrete approximation, in which the incremental problem is a one-step alternate minimization. We can characterize the limit both in terms of energy balance and PDEs without relying on chain rule and thus on compactness in time-Sobolev spaces. Moreover, we study the limit as the viscosity vanishes. To this end, as in the previous case, we first parametrize the evolutions by arc-length and prove that their lengths are uniformly finite. This delicate technical step is obtained by means of a suitable discrete Gronwall argument, which in turns provides also the local regularity in time-Sobolev spaces. Then we can pass to the limit, which is characterized again in terms of a quasi-static BV-evolution. In this case we first show the energy balance and then deduce, by means of the chain rule, the corresponding PDEs. We can show that in the regime of stable propagation the limit evolution is again in equilibrium, in both the variables, and that Griffith's criterion holds, again for a phase-field energy release. Finally, unstable regimes of propagation are characterized by a system of PDEs which is nothing but the original time-continuous system re-written in the parametrization variable.

We briefly discuss the fact that different underlying norms can lead to qualitatively similar evolutions, in particular as far as steady state propagation is concerned. Such evolutions do not necessarily coincide since solutions to this systems are in general non-unique. We finally suggest some alternative 'norm' for the evolution in terms of the phase-field length.

Michael Ortiz: Optimal scaling in ductile fracture. Abstract: This work is concerned with the derivation of optimal scaling laws, in the sense of matching lower and upper bounds on the energy, for a solid undergoing ductile fracture. The specific problem considered concerns a material sample in the form of an infinite slab of finite thickness subjected to prescribed opening displacements on its two surfaces. The solid is assumed to obey deformation-theory of plasticity, in the case of metals, and classical rubber elasticity for polymers. When hardening exponents for metals are given values consistent with observation, or when chain failure is accounted for in polymers, the energy is found to exhibit sublinear growth. We regularize the energy through the addition of nonlocal energy terms of the strain-gradient plasticity type for metals and fractional strain-gradient elasticity for polymers. This nonlocal regularization has the effect of introducing an intrinsic length scale into the energy. Under these assumptions, ductile fracture emerges as the net result of two competing effects: whereas the sublinear growth of the local energy promotes localization of deformation to failure planes, the nonlocal regularization stabilizes this process, thus resulting in an orderly progression towards failure and a well-defined specific fracture energy. The optimal scaling laws derived here show that ductile fracture results from localization of deformations to void sheets in metals and to crazes in polymers, and that it requires a well-defined energy per unit fracture area. The optimal scaling laws show that ductile fracture is cohesive in nature, i.e., it obeys a well-defined relation between tractions and opening displacements. Finally, the scaling laws supply a link between micromechanical properties and macroscopic fracture properties. In particular, they reveal the relative roles that microplasticity and surface energy play as contributors to the specific fracture energy of the material.

Anna Pandolfi: A variational model of poro-mechanical damaging material. Deterioration of mechanical and hydraulic properties of rock masses and subsequent problems are closely related to changes in the stress state, formation of new cracks, and increase of permeability in porous media saturated with freely moving fluids. In fully saturated rocks, fluid and solid phases are interconnected and the interaction between fluid and rock is characterized by coupled diffusion-deformation mechanisms that convey an apparent time-dependent character to the mechanical properties of the rock. The two governing equations of the coupled problem are the linear momentum balance and the continuity equation (mass conservation). The kinematic quantities that characterize this picture are the porous solid displacement and the rate of fluid volume per unit area. Hydro-mechanical coupling arises from the influence of the mechanical variables (stress, strain and displacement) on the continuity equation, where the primary variable is the fluid pressure, and from the influence of the hydraulic variables (pore pressure and seepage velocity) on the equilibrium equations, where the primary variables are the displacements. We describe a coupled approach to model damage induced by hydro-mechanical processes in low permeability solids. We describe the solid as an anisotropic brittle continuum where the damage is characterized by the formation of nested micro-structures in the form of equi-distant parallel faults, characterized by distinct orientation and spacing. The particular geometry of the faults allows for the analytical derivation of the porosity and of the anisotropic permeability of the solid. The fractured medium can be regarded as an anisotropic porous material. Classic methods can be applied to describe the porous-mechanical behavior of the solid to estimate the flow of fluids across the medium according to the presence of a fluid pressure gradient. The approach can be used for a wide range of engineering problems, ranging from the prevention of water or gas outburst into underground mines to the prediction of the integrity of reservoirs for underground CO2 sequestration or hazardous waste storage. The work is done in collaboration with M.L. De Bellis, G. Della Vecchia and M. Ortiz.

Kim Pham: Stability analysis of homogeneous states in gradient damage models. In this talk I will talk about the stability of homogeneous states for gradient damage models. We will show how to exploit the second order derivative of the total energy to discriminate stable and unstable homogeneous states depending on the hardening properties, the loading and the size of the sample.

This is based on joint work with J-J Marigo and Corrado Maurini.

Benoit Roman: Tearing : fracture path in brittle thin sheets. In this presentation i gave a short review of several tearing experiments, and tried to explain the reasons why i think that the determination of fracture path in thin sheets is a good problem for the variational approach to fracture,

A first reason is that experiments show that in many configurations fracture in thin sheets follow very robust, reproducible trajectories. This gives clear experimental features that can be used to test a variational theory.

A second reason is that the description of these tearing experiments fall out of the classical linear elastic approach to fracture. A first difficulty is that plate equations are based on averaging mechanical load through the thickness. This is incompatible with fracture mechanics, which studies the stress field at the tip of the crack, requiring a detailed description in the thickness. One approach would therefore be to generalize the notion of stress intensity factor to linear plate equations, but to our knowledge this line of work is not very much advanced, and many tools need to be defined before the crack path can be defined. A second problem is that because large out of plane bending is involved, the elastic problem is most likely to be strongly non-linear.

If linear fracture mechanics fails, a variational approach to fracture including the complete non-linear plate description is completely possible. We have used a first simplified model, which consists in assuming that the sheet is inextensible (conserves its length) but infinitely bendable. In this framework, the elastic energy of the plate is zero, so that the work of the operator is entirely dissipated into fracture energy during propagation. The usual notions of modes (corresponding to stress intensity decomposition) has no meaning in this framework since we have neglected the elasticity of the system. However this approach leads to simple prediction for the crack path which found to be often close to the experimental observations.

The first successes of this simplified approach are encouraging, but some features of the crack path are not correctly described in the inextensible model. We are also lacking estimates of the error in the path due to these strong assumptions. A real description of the system should include both bending and stretching energies within a complete variational model based on non-linear plate mechanics.

Marcus Sarkis: Finite Elements Methods on Non-Aligned Meshes for Interface Problems. We define two finite element methods for elliptic problems with possibly discontinuous diffusion coefficients and divergence constraints where the meshes are not aligned with the interface. The first method is based on Immersed Interface Methods while the second one on Nitsche's methods. In order to obtain apriori error estimates totally independent of the contrast between diffusion coefficients and independent on how the interface crosses the mesh, we consider stabilizations based on the jump of the flux as well as the jump of solution across elements.

Bernd Schmidt: On a quantitative piecewise rigidity result and Griffith-Euler-Bernoulli functionals for thin brittle beams. We study a planar thin brittle beam subject to elastic deformations and cracks described in terms of a nonlinear Griffith energy functional acting on SBV deformations of the beam. In particular we consider the case in which elastic bulk contributions due to finite bending of the beam are comparable to the surface energy which is necessary to completely break the beam into several large pieces.

In the limit of vanishing aspect ratio we rigorously derive an effective Griffith-Euler-Bernoulli functional which acts on piecewise $W^{2,2}$ regular curves representing the midline of the beam. The elastic part of this functional is the classical Euler-Bernoulli functional for thin beams in the bending dominated regime in terms of the curve's curvature. In addition there also emerges a fracture term proportional to the number of discontinuities of the curve and its first derivative [**?**].

A key ingredient in the Γ-convergence proof is a novel quantitative geometric rigidity estimate for special functions of bounded deformation in a planar setting, recently obtained in joint work with M. Friedrich. It generalizes a result by Friesecke, James and Müller for Sobolev functions obtained in nonlinear elasticity theory and a qualitative piecewise rigidity result by Chambolle, Giacomini and Ponsiglione for brittle materials which do not store elastic energy. We show that for each deformation there is an associated triple consisting of a partition of the domain, a corresponding piecewise rigid motion being constant on each connected component of the cracked body and a displacement field measuring the distance of the deformation from the piecewise rigid motion [**?**].

Vijay Subramanian: Variational fracture modeling at Corning Incorporated. The variational approach to fracture has been in development for the past two decades. Over two years ago Corning began a collaboration with Prof. Blaise Bourdin at Louisiana State University. The presentation summarizes some of the developments that have occurred as part of the collaboration as well as summarize some of the internal efforts within Corning Incorporated.

Some of the businesses at Corning Incorporated for which fracture and damage plays an important role on performance are: Specialty Materials, Display Technologies, Environmental Technologies, and Pharmaceutical Technologies to name a few. The corresponding applications in these businesses would be strong glass for portable handheld devices (Gorilla Glass), Display glass substrates for televisions and hand held devices, catalytic converters and diesel particular filters, and pharmaceutical tubes respectively. The typical sources of field failures for these glasses and ceramics may come from sharp contact damage (impact, scratch, indentation), thermal shock, and/or mechanical loading. Some of the key areas of interest at Corning where the field of variational fracture can provide differentiation is in the following: fracture and strength analysis of thin-glass sheets (strengthened and non-strengthened), crack nucleation and growth in ceramic substrates due to thermo-mechanical loading, and finite elasto-plastic fracture. Many of these areas would also require large deformation capabilities as well as the ability to handle crack growth evolution under crack-face contact. A suitable reference problem for crack-face contact could be the wing-crack problem which involves an inclined crack subject to compression. Solving the wing-crack problem using the variational fracture approach may require additional development and it is hoped that future efforts will focus on devising suitable unilateral contact laws that would allow solution of wing-cracks in a robust manner.

Mary F. Wheeler: Phase-field modeling of proppant-filled fractures in a poroelastic medium This work presents proppant and fluid-filled fracture with quasi-Newtonian fluid in a poroelastic medium [**?**, **?**]. Lower-dimensional fracture surface is approximated by using the phase field function. The two-field displacement phase-field system solves fully-coupled constrained minimization problem due to the crack irreversibility. This constrained optimization problem is handled by using active set strategy. The pressure is obtained by using a diffraction equation where the phase-field variable serves as an indicator function that distinguishes between the fracture and the reservoir. Then the above system is coupled via a fixed-stress iteration. The transport of the proppant in the fracture is modeled by using a power-law fluid system. The numerical discretization in space is based on Galerkin finite elements for displacements and phase-field, and an Enriched Galerkin method is applied for the pressure equation in order to obtain local mass conservation. The concentration is solved with cell-centered finite elements. Nonlinear equations are treated with Newton's method. Predictor-corrector dynamic mesh refinement allows to capture more accurate interface of the fractures with reasonable number for degree of freedoms.

Keita Yoshioka: Variational Fracture Modeling Applied to Hydraulic Fracturing (Fracking). Despite many applications of hydraulic fracturing, coupling of reservoir fluid flow, heat transfer, and fracture(s) propagation, especially in the presence of other pre-existing fractures, remains a major challenge in predicting the well stimulation and the evolution of well injectivity in the petroleum industry. To date simulation has focused upon the problem of a single planar in mode-I driven by a pressurized fluid using classical fracture mechanics models, or superposition of single planar fractures whilst neglecting the interaction between fractures. In contrast, realistic applications involve multiple fractures propagating along complex and unknown paths. Recently, the variational approach to fracture, which was originally proposed by Francfort and Marigo [**?**] and numerically implemented by Bourdin et al. [**?**, **?**], has been increasingly applied to simulation of hydraulic fracturing because of its ability to track any number of arbitrary fractures in an efficient manner. In this talk, we present a variational fracture model extended to hydraulic fracturing by accounting for pressure force within the fracture and in-situ stresses. We then show illustrative examples to demonstrate that the model responses are closely matched with existing analytical solutions of fluid-driven fracture propagation. Finally we will present several applications to practical problems.

Alan Zehnder: Fracture Surface Transition for Notched Bars in Torsion. Keywords: mixed-mode, factory roof, spiral fracture, PMMA, brittle fracture It is well known that a brittle rod loaded in torsion will fail on a spiral surface angled 45 degrees to the axis of the rod so that local Mode-I conditions are maintained. It is also known that under Mode-III loading, pre-cracked brittle materials will fail with a surface that starts out planar to the original fracture surface but with a linked set of fractures at 45 degrees to the surface, forming what prior authors have deemed the ?factory roof? fracture pattern. We explore the transition from the globally flat, but locally faceted, factory roof fracture to the spiral fracture surface by performing torsion experiments on circumferentially notched rods of Polymethylmethacrylate (PMMA). Varying the notch depth we observe a transition from the factory roof to spiral fracture. Micro-CT scanning is used to image the fracture surfaces and to reveal the pattern of internal cracks that form during failure. Results of this work are posed as a challenge to the computational simulation of crack surface evolution.

Joint work with Natasha Zella.

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Chapter 11

Recent Advances in Hydrodynamics (16w5102)

June 5 - 10, 2016

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Recent Developments and Open Problems[1]

Recent developments in the theory, numerics, and experimental aspects of fluid mechanics have centered on a more quantitative understanding and modeling of fluid phenomena—understanding that goes beyond universal laws and dimensional scaling, even though these still retain an important role.

In particular, the fundamental role that boundaries play in modeling incompressible fluid flows has been underscored by a renewed impetus in studying boundary layers, from both an analytical and modeling point of view. On the analytical side, significant progress has been made in understanding boundary layer thickness and the importance of boundaries, even for addressing regularity. Conditions that guarantee well-posedness of the Prandtl equations have been determined, in tandem with studies of the resulting instabilities and ill-posedness where these conditions are violated [8, 18, 21, 29, 30]. Progress has also been made in justifying the Prandtl approximation under non-classical slip boundary conditions, such as Navier-slip boundary conditions [17] or for special classes of flows, such as flow with symmetry [35, 34] and linear flows [16], and in seeking physically motivated criteria for the vanishing viscosity limit to hold [10, 27, 28, 32]. By the vanishing viscosity limit we mean convergence strongly in the energy norm of solutions of the Navier-Stokes equations to solutions of the Euler equations.

On the computational side, new methods, such as wavelet and penalization methods [36, 37], have emerged that allow for a more accurate and stable resolution of the small scales associated with boundary layers and the modeling of vorticity near walls. Yet, modeling the behavior of viscous fluids near rigid walls remains one of the most outstanding open problems in hydrodynamics today with an impact on many real-life problems, such as skin friction and drag reduction. More specifically, from a mathematical point of view, it is still unknown whether the vanishing viscosity limit holds for regular data and domains for short time, and if, in this case, the Prandtl approximation to the Navier-Stokes solution is valid; i.e., if it converges to the actual solution as the viscosity vanishes. Furthermore, while it is clear that such an approximation is no longer valid when boundary layer separation occurs, the precise relation between layer separation, the vanishing viscosity limit, and the Prandtl approximation is still not understood, except in special cases.

[1] In view of the large literature on the subject, most of the references cited in this and other sections pertain to relevant results by participants in the workshop and their co-authors.

There are still several open question related to the well-posedness of the Prandtl equations. It is known that they are ill-posed in Sobolev spaces and well-posed for analytic and nearly analytic (Gevrey class) data [19], but the precise Gevrey regularity needed is still open. Boundary layers arise in several other contexts in hydrodynamics. For example, for the Euler-α model, which is a reduced model for an inviscid second-grade fluid, convergence to the Euler solution can be shown under Navier boundary conditions and used for an improved existence result of strong Euler solutions in 3D [4]. Boundary layers are of fundamental importance in modeling flow through porous media, as they impact the resulting filtration law at the macroscale. Modeling of multi-phase flow has improved significantly recently, especially in non-standard geometries, such as karstic geometry [25]. However, a rigorous justification of the filtration laws is still lacking in many regimes.

A related area that has seen a resurgence of interest is flow stability and controllability, from the perspective of both partial differential equations (PDEs) and dynamical systems (including the existence of attractors, such as inertial manifolds). Significant work has been done recently on determining modes and forms for various hydrodynamics equations, also in the context of uncertainty quantification ([15] and references therein). Instabilities around suitably chosen flow profiles, in particular shear flows, have been key to establishing ill-posedness of the Prandtl equations. Fast rotation is recognized as a stabilizing mechanism in three-dimensional flows, by making the flow nearly two-dimensional, leading in this case to global well-posedness of the Navier-Stokes equation. However other stabilizing mechanisms have been recognized, both for inviscid and viscous flows, involving oscillations in time [11] as well as space. In the latter case, vorticity may be transferred to small scales and lead to asymptotic stability even in inviscid flows. This phenomenon is called inviscid damping, akin to Landau damping in kinetic equations [3].

Recent progress has been made on controllability of the Navier-Stokes equation, which has practical applications in engineering problems, but many questions remain unanswered. Controllability is especially relevant in fluid-structure interaction problems, which have been the focus of significant work recently, especially regarding the motion of solid objects in an incompressible fluid [18]. A main open question in this area is the existence of solutions past collision in the inviscid case and when slip is present in the viscous case.

Flow instabilities connect naturally to perhaps the most fundamental open question in mathematical hydrodynamics; that is, the question of finite-time blowup for solutions to the Navier-Stokes and Euler equations. Recent careful numerical simulations seems to suggest that regular Euler solutions may blow up in finite time [23]. In this context, inviscid regularizations of the Euler equations, such as the Euler-α and Euler-Voigt models may provide criteria for blow up that are more easily monitored numerically [31]. In Ref. [23], the possible blow up occurs at the boundary of the domain. In fact, the arguments in Ref. [23] have informed analytical work on the growth of vorticity in two-dimensional Euler flows, for which global existence and uniqueness is known. Examples have been constructed of solutions that saturate the double exponential growth of gradients of vorticity, the theoretical upper bound, again at points on the boundary [26]. The question of whether such growth can occur at interior points remains open. For viscous flows, global well-posedness has been established for reduced, but physically relevant models, in particular for the primitive equations of the oceans and atmosphere, for which it holds under no additional smallness assumptions on the initial data [22]. At the same time, finite-time blow up can occur in this model in the inviscid case [7].

At the level of weak solutions, significant progress has been made in the inviscid case. Using convex integration, weak solutions of the Euler equations known as wild solutions, with interesting properties, have been constructed in both two and three dimensions [14]. Strong non-uniqueness holds for these solutions, in part because the behavior of the associated energy can be arbitrarily prescribed. Wild solutions have provided examples of energy-dissipative Euler solutions, which considerably narrow the existing gap in proving Onsager's conjecture. Onsager's conjecture states that flows with velocities that are Hölder continuous of exponent $1/3$ or higher are energy conservative, while there exists energy-dissipative solutions if the exponent is less than $1/3$. The conservative part of the conjecture has been established, with the minimal regularity needed measured in certain Besov spaces (see [9] and references therein). The dissipative part is still open, though the theory of wild solutions allows one to construct energy-dissipative Euler solutions with regularity close to Hölder $1/3$; these solutions are integrable in time but have space regularity less than $1/3$ [13].

Onsager's conjecture is utilized to justify employing irregular, energy-dissipative Euler solutions to model turbulence, as numerically and experimentally it is observed that the rate of energy dissipation does not vanish as the viscosity approaches zero. A major open problem in this context is whether it is possible to construct such dissipative solutions as limits of Navier-Stokes solutions as the viscosity vanishes (at least in the absence of boundaries). While the understanding of turbulence from a rigorous and quantitative point of view is still lacking in many respects, important progress has been made in obtaining rigorous bounds on quantities of interest, such as structure functions, as well as in modeling and experimental observation, even for inhomogeneous and anisotropic turbulence, such as wall-bounded turbulence. On the computational side, progress has been made in the efficiency of pseudospectral methods, such as the method of implicit dealiasing [5]. Penalty methods have been developed to exploit the efficiency and high spectral accuracy of the fast Fourier transform in the presence of arbitrary, nonperiodic boundaries (e.g. [36]). For homogeneous Dirichlet boundary conditions, a convergence proof of the penalty method for the Navier-Stokes equations was given by Angot [2]. Advances have also been made in Lagrangian methods, including numerical techniques [33] and experimental procedures for bridging from Eulerian to Lagrangian statistics [24]. Experimentally, improved particle velocimetry allows accurate tracking of particles in turbulent flows, even close to walls. There has also been a renewed interest in Lagrangian methods on the analytical side. One advantage of using a Lagrangian formulation for the Euler equations is that Lagrangian paths are analytic even for rough data.

Lastly, the interplay between geometry, analysis, and mechanics is becoming ever more relevant. On the computational side, improved pseudospectral and wavelet methods, based on tessellation, such as Voronoi cells, can naturally handle curved geometries, as in global circulation and climate models (see [1] for the sphere). On the analytical side, recent work on the fluid equations in hyperbolic geometries and in the relativistic setting [6, 12] has elucidated some of the obstructions created by the Euclidean structure, for example in resolving non-uniqueness of weak Navier-Stokes solutions.

Presentation Highlights

The first two talks set the stage for much of what was to come, introducing two of the main themes of the workshop: stability/instability of fluid flow and the role that boundaries play in turbulence.

The opening talk, by Edriss Titi, explained how dispersion can act to stabilize or destabilize the solution to PDEs. He started with the example of rotation, acting as a dispersive mechanism, regularizing the solution to Burger's equation and the Navier-Stokes equations. He contrasted this with the example of the Kuramoto-Sivashinsky equation, in which dispersion acts as a destabilizing mechanism.

In the second talk, Kai Schneider presented, in a very graphical manner, recent, detailed numerical calculations of a (2D) vortex dipole interacting with a (flat) wall with no-slip boundary conditions. The numerics suggest that the Prandtl expansion holds in such a situation up to a critical time, after which the vorticity detaches from the boundary, the Prandtl equations become singular, and the expansion breaks down. The calculations also suggest that beyond the critical time, a new asymptotic description of the flow appears possible. The calculations also indicate energy dissipation following detachment of the boundary layer.

Here are some highlights from a few other notable talks at the workshop:

Nicholas Kevlahan explained his method of using dynamically adaptive wavelets for the solution of the shallow water equations. The wavelets provide both adaptivity of discretization and a way to interpolate between scales. This allows the finest level of resolution to be determined dynamically by local error estimates, and decouples the method of solution from the manner in which the discretization is adapted. This results in a robust, flexible framework for computation that can be efficiently parallelized.

Franck Sueur gave a talk on the controllability of the Navier slip-with-friction boundary condition. This is a recent work where the speaker together with his collaborators showed that given a square integrable and divergence-free vector field, there is a time and a boundary control that makes a weak solution trivial at that time. The strategy of the proof is elegant and seeks solutions of the rescaled problem with vanishing viscosity, and small data. It consists of the following three steps:

- High Reynolds number control on a time scale of order 1.
- Well-prepared dissipation of the boundary layer on a time scale $O(T/\varepsilon)$.
- Low Reynolds number control on a time scale $O(T/\varepsilon)$.

Yasunori Maekawa gave a talk on the Prandtl expansion in the case of the half-space with periodic horizontal boundary conditions. In this work, he is building toward the result that the vanishing viscosity limit holds when the initial data is analytic in a strip near the boundary, improving upon results from the 1990s for data analytic in the whole domain.

Dragos Iftimie talked about the zero-α limit for the α-Euler model. Adam Larios discussed the related Euler-Voigt equations, a regularization of the Euler equations, and a new blowup criteria for the Euler equations based upon the behavior of these solutions (which are globally well-posed) as the parameter α vanishes.

Matthias Hieber's described the extension of global strong well-posedness results for the 3D viscous primitive equations from the L^2 setting (pioneered by Cao and Titi) to the L^p setting.

Jean-Christophe Nave described a very clever and efficient numerical method for calculating the flow map for transport equations as it evolves over time as an iteration of diffeomorphisms. He presented a number of numerical examples based upon passive transport, though the approach is equally applicable to active transport equations like the 2D Euler equations.

Sylvie Monniaux showed how to apply abstract functional analytic techniques to a very practical problem: the solution of the Navier-Stokes equations with time-dependent Navier-type boundary conditions, thereby extending existing (relatively recent) results for the time-independent case.

Sina Ghaemi described particle image velocimetry in near-wall turbulence. Planar and volumetric particle image velocimetry and particle tracking velocimetry was applied to a turbulent channel flow at a Taylor Reynolds number of 190. These state-of-the-art experimental techniques yield accurate measurement of turbulence statistics in the inner layer of wall flows, which is of fundamental importance for the development of passive and active flow control systems. Sina illustrated how passive drag reduction can be achieved using superhydrophobic surfaces or polymeric drag reducers.

Scientific Progress Made

The workshop served to promote the development of junior talents in the field and to facilitate the integration of young scientists into the community. The participants exchanged recent research results and discussed challenges and interesting new topics.

In addition, the workshop has enabled several participants (who so far worked on equations without boundaries) to get a state-of-the-art view of boundary effects and their importance to fluid flow. We believe the workshop has provided them with concrete open problems to work on in this context. A similar statement can be made regarding the ever-growing importance of accurate, faithful numerical simulations in modern fluid mechanics.

Moreover, several new collaborations developed between the participants, existing collaborations were advanced, and new directions of research have opened up. In particular:

- Kai Schneider reported that he and Jean-Christophe Nave made significant progress on their paper, "A characteristic mapping method for two-dimensional incompressible Euler equations," while at the workshop.
- Hantaek Bae worked with Jim Kelliher, and Edriss S. Titi and Slim Ibrahim worked on a continuing common project on primitive equations.
- For Marcelo Disconzi it was an opportunity to finish a manuscript he had been writing with Magdalena Czubak and Chi Hin Chan. He also appreciated the access it gave him to recent developments in areas, such as boundary layers and numerics, that come in contact with his own research, although not specifically topics that he has worked on himself. It also allowed him to exchange ideas about his work with many people who have a different view/perspective on the topic and to interact (and sometimes meet for the first time) with leading researchers in the field (something particularly important to younger researchers).

Outcome of the Meeting

On Thursday evening, we held an informal discussion, in which the great majority of the attendees discussed both their experiences at the workshop and where they see the field headed. Some themes that emerged were the following:

- Over the past 10 or 20 years questions related to the regularity of solutions, especially of the Navier-Stokes and Euler equations, have occupied much of the attention and resources of researchers from the pure end of hydrodynamics. There was a trend among the talks at this workshop, reflected in the larger community, toward questions related to the stability of solutions. This was generally applauded for having greater meaning and impact among numerical and experimental researchers, and indeed among engineers and physicists. In addition, instability studies provide a possible means to probe the existence of finite-time singularities in hydrodynamics.
- Another trend observed both in this conference and in the field at large was an increasing focus on the role that boundaries play in affecting the behavior of fluids. This was clearly evident in pure, numerical, and experimental talks at this conference.
- The numerical talks provided what could be viewed as a survey of the state of the art in the computation of inviscid and high Reynolds number fluid flow. These talks were extremely useful for the more pure researchers attending.
- The role of dissipation of energy—Onsager's conjecture—had a prominent place in the conference, including both positive and negative results.
- The α and α-Voigt models, originally conceived as mathematical artifices for regularizing solutions to fluid equations, were of interest to numerical researchers.
- The pure researchers gained useful insight into what issues the numericists find important and what their limitations are.
- The effective use of wavelets in the numerical solution of incompressible fluids equations was seen as real advance (and somewhat surprising since the wavelets cannot be divergence-free if compactly supported).
- Talks also covered control theory (the talk of Franck Sueur) and filtration laws (the talk by Xiaoming Wang).

Participants

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Chapter 12

New Directions in Iwasawa Theory (16w5043)

June 26 - July 1, 2016

Organizer(s): Frauke Bleher (University of Iowa), Ted Chinburg (University of Pennsylvania), Ralph Greenberg (University of Washington), Georgios Pappas (Michigan State University), Romyar Sharifi (University of Arizona)

Objectives A main goal of the BIRS workshop "New Directions in Iwasawa Theory" was to prepare the ground for anticipated significant scientific developments in number theory, and more specifically in Iwasawa theory. Iwasawa theory is undergoing rapid change in several directions, so one of the aims of the workshop was to map out some primary new objectives for Iwasawa theory.

The workshop brought together researchers and advanced Ph.D. students working on Iwasawa theory and related topics in K-theory, commutative algebra and modular forms.

Two specific objectives of the workshop were to

(i) discuss recent advances in Iwasawa theory, and

(ii) develop deeper ties between Iwasawa theory and algebraic K-theory.

The workshop included researchers with a wide range of expertise in number theory, algebraic K-theory, homological algebra and arithmetic geometry. It also introduced young researchers to previous work, as well as to new directions for research in these areas.

The workshop was timely in view of the recent proofs of a number of Main Conjectures about first Chern classes. These include the non-commutative Main Conjecture over totally real fields in 2010, the final form of the commutative Main Conjecture over imaginary quadratic fields in 2011, and recent progress on Main Conjectures relating to modular forms.

Besides the generous support from the Banff International Research Station, this workshop was also supported by a Focused Research Group award from the National Science Foundation concerning Iwasawa theory and Chern classes. The P.I.'s on this NSF FRG award were the organizers (Bleher, Chinburg, Greenberg, Pappas and Sharifi) and the consultants on the FRG grant were Mahesh Kakde and Martin J. Taylor.

The workshop had 41 participants and eighteen scheduled 50-minute talks.

Talks The 50-minute talks of the workshop gave insights and results related to recent advances in Iwasawa theory. In the following we give a description of each talk, in the order in which the talks were given.

Ted Chinburg: Iwasawa theory in higher codimension. In this talk, Chinburg gave a survey of recent work on higher codimension Iwasawa theory, which is joint work with Frauke Bleher, Ralph Greenberg, Mahesh Kakde, George Pappas, Romyar Sharifi and Martin Taylor [2]. This work has to do with relating p-adic L-functions to the behavior of Iwasawa modules that are supported in codimension larger than one as modules for an Iwasawa algebra. One idea, which was discussed in this talk, is that the natural analytic invariants arising from Katz p-adic L-functions pertain to the derived top exterior powers of Iwasawa modules. For first Chern classes, passing to the derived top exterior power makes no difference, but for higher Chern classes it does. This is analogous to the fact that Stark's conjectures pertain to regulators rather than to the individual entries of matrices whose determinants are regulators.
For the video of the talk, see [10].

Cristian Popescu: Towards module structure in classical Iwasawa theory. In this talk, Popescu discussed aspects of his recent joint work with Corey Stone on higher Fitting ideals of various Iwasawa modules. Given a commutative Noetherian ring R and a finitely generated R-module M, one looks at a presentation $R^m \xrightarrow{A} R^n \to M \to 0$ and defines the i^{th} Fitting ideal $\mathrm{Fit}^i_R(M)$ to be the ideal generated by the determinants of all $(n-i) \times (n-i)$ minors of A. If R is a PID then the Fitting ideals $\mathrm{Fit}^i_R(M)$, $0 \le i \le n$, determine M up to isomorphism. However, if $R = \Lambda = \mathcal{O}[[t]]$ contains $\mathbb{Z}_p$, then this is not true in general. Popescu discussed a conjecture of Kurihara [21, 22] about higher Fitting ideals in this context.
For the video of the talk, see [26].

Samit Dasgupta: On the Gross-Stark Conjecture.

In this talk, Dasgupta discussed joint work with Mahesh Kakde and Kevin Ventullo [14]. In the 1980's, Gross conjectured [17] a formula for the expected leading term at $s = 0$ of the Deligne–Ribet p-adic L-function associated to a totally even character ψ of a totally real field F. The conjecture states that after scaling by $L(\psi\omega^{-1}, 0)$, this value is equal to a p-adic regulator of units in the abelian extension of F cut out by $\psi\omega^{-1}$. In this talk, Dasgupta described a proof of Gross's conjecture.
For the video of the talk, see [13].

Francesc Castella: Λ-adic Gross-Zagier formula for elliptic curves at supersingular primes.

In this talk, Castella discussed joint work with Xin Wan [9]. In 2013, Kobayashi proved an analogue of Perrin-Riou's p-adic Gross-Zagier formula for elliptic curves at supersingular primes [20]. Castella explained an extension of Kobayashi's result to the Λ-adic setting. The main formula he presented is in terms of plus/minus Heegner points up the anticyclotomic tower. He also discussed the proof of this formula, which is via Iwasawa theory, based on the connection between Heegner points, Beilinson-Flach elements and their explicit reciprocity laws, rather than on calculations inspired by the original work of Gross-Zagier.
For the video of the talk, see [8].

Bharathwaj Palvannan: On Selmer groups and factoring p-adic L-functions.

In this talk, Palvannan first discussed the construction by Haruzo Hida of a 3-variable Rankin Selberg p-adic L-function. Two of the variables of this L-function are "weight" variables and one of its variables is the "cyclotomic" variable. He then pointed out that Samit Dasgupta [12] has factored a certain restriction of this 3-variable p-adic L-function (when the two weight variables are set equal to each other) into a product of a 2-variable p-adic L-function (related to the adjoint representation of a Hida family) and the Kubota-Leopoldt p-adic L-function. Palvannan then discussed his proof of the corresponding result involving Selmer groups that is predicted by the main conjectures [24]. A key technical input in this proof is the study of the (height one) specialization of Selmer groups.
For the video of the talk, see [25].

Haruzo Hida: Ring theoretic properties of Hecke algebras and cyclicity in Iwasawa theory.

In this talk, Hida showed that one can formulate a certain Gorenstein property of subrings of the universal deformation ring (i.e., the corresponding Hecke algebra) as a condition almost equivalent to the cyclicity of the Iwasawa module over $\mathbb{Z}_p$-extensions of an imaginary quadratic field if the starting residual representation is induced from the imaginary quadratic field. He discussed this connection in some details.
For the video of the talk, see [18].

Masato Kurihara: Iwasawa theory and Rubin-Stark elements.
In this talk, Kurihara talked about joint work with David Burns and Takamichi Sano [3]. More precisely, he presented an Iwasawa theoretic discussion of Rubin-Stark elements and zeta elements. Moreover, he discussed equivariant main conjectures and their consequences.
For the video of the talk, see [23].

Malte Witte: On zeta-isomorphisms and main conjectures.
In this talk, Witte explained first the setup of the zeta-isomorphism conjecture of Fukaya and Kato, which is a generalization of the equivariant Tamagawa number conjecture. He then turned to the noncommutative main conjecture for totally real fields and discussed a unicity result for the noncommutative zeta functions constructed by Kakde. Finally, Witte explained how this unicity result can be used to construct zeta-isomorphisms in the sense of Fukaya and Kato. See [34].
For the video of the talk, see [35].

Kazım Büyükboduk: On the anticyclotomic main conjectures for modular forms.
In this talk, Büyükboduk reported on recent joint work with Antonio Lei [5, 6] on the anticyclotomic Iwasawa theory of the base change of an elliptic modular form to an imaginary quadratic field K in which the prime p splits. Büyükboduk and Lei treated both the definite and indefinite cases in both p-ordinary and non-p-ordinary situations. One of their main results is an equality (up to powers of p) that is predicted by the main conjectures in the definite p-ordinary set up and a Λ-adic Birch and Swinnerton-Dyer formula in the indefinite case.
For the video of the talk, see [7].

Florian Sprung: The main conjecture for elliptic curves at non-ordinary primes.
In this talk, Sprung explained the proof of the main conjecture for elliptic curves at non-ordinary primes. More precisely, suppose E is an elliptic curve over $\mathbb{Q}$ and that $p > 2$ is a prime of good reduction. In the ordinary main conjecture one assumes that p does not divide a_p, whereas in the non-ordinary main conjecture the assumption is that p divides a_p. Sprung's results generalize work of Wan [33], who worked under the assumption that $a_p = 0$.
For the video of the talk, see [29].

Karl Rubin: Heuristics for the growth of Mordell-Weil ranks in big extensions of number fields.
In this talk, Rubin reported on joint work with Barry Mazur. He discussed some heuristics for modular symbols, and consequences of those heuristics for Mordell-Weil ranks. For example, these heuristics predict that every elliptic curve over $\mathbb{Q}$ has finite Mordell-Weil rank over the $\hat{\mathbb{Z}}$-extension of $\mathbb{Q}$.
For the video of the talk, see [27].

Otmar Venjakob: Wach modules, regulator maps and ε-isomorphisms in families.
In this talk, Venjakob discussed joint work with Rebecca Bellovin on the "local ε-isomorphism" conjecture of Fukaya and Kato for (crystalline) families of $G_{\mathbb{Q}_p}$-representations. This can be regarded as a local analogue of the global Iwasawa main conjecture for families, extending earlier work of Kato for rank one modules, of Benois and Berger for crystalline representations with respect to the cyclotomic extension, as well as of Loeffler, Venjakob and Zerbes for crystalline representations with respect to abelian p-adic Lie extensions of $\mathbb{Q}_p$. Nakamura has shown Kato's conjecture for (φ, Γ)-modules over the Robba ring, which means in particular only after inverting p, for rank one and trianguline families. The main ingredient of (the integrality part of) the proof consists of the construction of families of Wach modules generalizing work of Wach and Berger and following Kisin's approach via a corresponding moduli space.
For the video of the talk, see [30].

Olivier Fouquet: Congruences between motives and congruences between values of L-functions.

In this talk, Fouquet discussed the following question: If two motives are congruent, is it the case that the special values of their respective L-functions are congruent? Or, more precisely: Can the formula predicting special values of motivic L-functions be interpolated in p-adic families of motives? Fouquet explained how the formalism of the Weight-Monodromy filtration for p-adic families of Galois representations sheds light on this question (and suggests a perhaps surprising answer). See [15].
For the video of the talk, see [16].

Peter Schneider: Rigid character groups, Lubin-Tate theory, and (φ, Γ)-modules.

In this talk, Schneider described joint work with Laurent Berger and Bingyong Xie [1]. Namely, for a finite extension L of $\mathbb{Q}_p$, they formulate a new theory of (φ, Γ)-modules whose coefficient ring is the ring of holomorphic functions on the rigid character variety of the additive group o_L, resp. a "Robba" version of it.
For the video of the talk, see [28].

Mirela Çiperiani: Local points of supersingular elliptic curves on $\mathbb{Z}_p$-extensions.

In this talk, Çiperiani first discussed work of Kobayashi and Iovita-Pollack, which describes how local points of supersingular elliptic curves on ramified $\mathbb{Z}_p$-extensions of $\mathbb{Q}_p$ split into two strands of even and odd points. She then discussed a generalization of this result to $\mathbb{Z}_p$-extensions that are localizations of anticyclotomic $\mathbb{Z}_p$-extensions over which the elliptic curve has non-trivial CM points. See [11].

Ming-Lun Hsieh: Hida families and triple product p-adic L-functions.

In this talk, Hsieh presented a construction of the three-variable p-adic L-function attached to the triple product of three Hida families. This p-adic L-function is a three-variable power series with p-integral coefficients interpolating central L-values of triple product L-functions in the balanced case. He gave the explicit interpolation formula at all critical specializations and discussed some problems on this p-adic L-function.
For the video of the talk, see [19].

Ashay Burungale: On $\mathfrak{p}$-anticyclotomic Iwasawa theory.

In this talk, Burungale presented joint work with Haruzo Hida. Let F be a totally real field, and et p be an odd prime unramified in F and $\mathfrak{p}$ a prime above p. Let K/F be a p-ordinary CM quadratic extension and $K_{\mathfrak{p}}^{-}$ the maximal p-anticyclotomic extension of K unramified outside $\mathfrak{p}$. Burungale discussed results on the μ-invariant of certain p-adic L-functions over K along the $\mathfrak{p}$-anticyclotomic tower. He also described relevant questions regarding the $\mathfrak{p}$-anticyclotomic Selmer groups.
For the video of the talk, see [4].

Preston Wake: Ordinary pseudorepresentations, modular forms and Iwasawa theory.

In this talk, Wake discussed joint work with Carl Wang Erickson [31]. Pseudorepresentations appear naturally in the context of modular forms that are congruent to Eisenstein series. Wake talked about the difficulties that arise when defining "ordinary pseudorepresentation", and how to resolve these difficulties. He explained how the deformation theory of pseudorepresentations is related to cyclotomic Iwasawa theory and the geometry of the ordinary eigencurve.
For the video of the talk, see [32].

Schedule

The schedule of the workshop allowed for ample discussions among the participants. This was appreciated by everyone. On the last day of the conference, there were two scheduled one-hour discussion sessions, which were used by many participants to work in groups on old and new projects.

Sunday, June 26	16:00	Check-in begins (Front Desk - Professional Development Center - open 24 hours)
	17:30 - 19:30	Buffet Dinner, Sally Borden Building
	19:30	Informal gathering (Corbett Hall, 2nd floor lounge)

Monday, June 27	7:00 - 8:45	Breakfast
	8:45 - 9:00	Introduction and welcome by BIRS station manager
	9:00 - 9:50	Chinburg: *Iwasawa theory in higher codimension.*
	10:00 - 10:30	Coffee break
	10:30 - 11:20	Popescu: *Towards module structure in classical Iwasawa theory.*
	11:30 - 13:00	Lunch
	13:00 - 13:50	Guided Tour of The Banff Centre (meet in the 2nd floor lounge, Corbett Hall)
	13:50 - 14:00	Group Photo (meet in foyer of TCPL)
	14:00 - 14:50	Dasgupta: *On the Gross-Stark Conjecture.*
	15:00 - 15:30	Coffee break
	15:30 - 16:20	Castella: Λ*-adic Gross-Zagier formula for elliptic curves at supersingular primes.*
	16:40 - 17:30	Palvannan: *On Selmer groups and factoring p-adic L-functions.*
	17:45 - 19:30	Dinner

Tuesday, June 28	7:00 - 9:00	Breakfast
	9:00 - 9:50	Hida: *Ring theoretic properties of Hecke algebras and cyclicity in Iwasawa theory.*
	10:00 - 10:30	Coffee break
	10:30 - 11:20	Kurihara: *Iwasawa theory and Rubin-Stark elements.*
	11:30 - 13:30	Lunch
	14:00 - 14:50	Witte: *On zeta-isomorphisms and main conjectures.*
	15:00 - 15:30	Coffee break
	15:30 - 16:20	Büyükboduk: *On the anticyclotomic main conjectures for modular forms.*
	16:40 - 17:30	Sprung: *The main conjecture for elliptic curves at non-ordinary primes.*
	17:45 - 19:30	Dinner

Wednesday, June 29	7:00 - 9:00	Breakfast
	9:00 - 9:50	Rubin: *Heuristics for the growth of Mordell-Weil ranks in big extensions.*
	10:00 - 10:20	Coffee break
	10:20 - 11:10	Venjakob: *Wach modules, regulator maps and ε-isomorphisms in families.*
	11:30 - 12:20	Fouquet: *Congruences between motives and values of L-functions.*
	12:30 - 13:30	Lunch
		Free Afternoon
	17:30 - 19:30	Dinner

Thursday, June 30	7:00 - 9:00	Breakfast
	9:00 - 9:50	Schneider: *Rigid character groups, Lubin-Tate theory, and (φ, Γ)-modules.*
	10:00 - 10:30	Coffee break
	10:30 - 11:20	Çiperiani: *Local points of supersingular elliptic curves on $\mathbb{Z}_p$-extensions.*
	11:30 - 13:30	Lunch
	14:00 - 14:50	Hsieh: *Hida families and triple product p-adic L-functions.*
	15:00 - 15:30	Coffee break
	15:30 - 16:20	Burungale: *On $\mathfrak{p}$-anticyclotomic Iwasawa theory.*
	16:40 - 17:30	Wake: *Ordinary pseudorepresentations, modular forms & Iwasawa theory.*
	17:45 - 19:30	Dinner

Friday, July 1	7:00 - 9:00	Breakfast
	9:00 - 10:00	Discussions
	10:00 - 10:30	Coffee break
	10:30 - 11:30	Discussions
	11:30 - 12:00	Checkout by Noon (Front Desk - Professional Development Centre)
	12:00 - 13:30	Lunch

Participants

The participants of the workshop and their affiliations at the time of the workshop were as follows.

Agboola, Adebisi (University of California Santa Barbara)
Bleher, Frauke (University of Iowa)
Bouganis, Athanasios (Durham University)
Burungale, Ashay (Univeristy of Arizona)
Bykboduk, Kazm (Koc University of Istanbul)
Castella, Francesc (University of California, Los Angeles)
Chinburg, Ted (University of Pennsylvania)
Ciperiani, Mirela (The University of Texas at Austin)
Dasgupta, Samit (University of California, Santa Cruz)
Fouquet, Olivier (Universite Paris-Sud)
Greenberg, Ralph (University of Washington)
Harron, Robert (University of Hawaii at Manoa)
Hida, Haruzo (University of California, Los Angeles)
Hsieh, Ming-Lun (National Taiwan University)
Johnson-Leung, Jennifer (University of Idaho)
Kakde, Mahesh (King's College London)
Kim, Chan-Ho (Korea Institute for Advanced Study)
Kolster, Manfred (McMaster University)
Kurihara, Masato (Keio University)
Lamplugh, Jack (University College London)
Lei, Antonio (Universit Laval)
Palvannan, Bharathwaj (University of Washington - Seattle)
Ponsinet, Gautier (Universit Laval)
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Ramdorai, Sujatha (University of British Columbia)
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Sano, Takamichi (Osaka University)
Schneider, Peter (Mathematisches Institut Muenster)
Sharifi, Romyar (University of California, Los Angeles)
Sprung, Florian (Princeton University)
Taylor, Martin (Merton College - Oxford University)
Tomaskovic-Moore, Sebastian (University of Pennsylvania)
Trihan, Fabien (Sophia University in Tokyo)
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Wang Erickson, Carl (Brandeis University)
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[28] P. Schneider. Rigid character groups, Lubin-Tate theory, and (φ, Γ)-modules. http://videos.birs.ca/2016/16w5043/201606300859-Schneider.mp4

[29] F. Sprung. The main conjecture for elliptic curves at non-ordinary primes. http://videos.birs.ca/2016/16w5043/201606281641-Sprung.html

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Chapter 13

Developing a Comprehensive, Integrated Framework for Advanced Statistical Analyses of Observational Studies (16w5091)

July 3 - 8, 2016

Organizer(s): Michal Abrahamowicz (McGill University, Montréal), James Carpenter (London School of Hygiene & Tropical Medicine), Willi Sauerbrei (Medical Center, University of Freiburg, Freiburg

The STRATOS initiative — motivation, mission, structure and main aims

The overall objective of the 16w5091 Workshop was to jump-start the research activities of the international initiative for STRengthening Analytical Thinking for Observational Studies (STRATOS). This was achieved by providing the first opportunity for a large group of 38 STRATOS members, and seven research trainees, from 12 countries on 4 continents, to meet in-person, exchange ideas, continue with article writing and together develop an operational plan for further development and internal homogenization of the scientific and knowledge translation endeavors of the nine topic groups and several cross-cutting panels. As the attached Workship Scientific Program shows, the majority of sessions were organized following the structure of STRATOS Topic Groups (TGs) or Panels. Therefore, to facilitate understanding of the rather unique character and goals of the workshop, we begin this Report with a short overview of the STRATOS initiative, including its *raison d'être*, overall mission, internal structure and main objectives.

Lack of statistical expertise is now recognized as a significant brake on scientific progress across a wide range of empirical sciences. For instance, a recent article in The Economist (19/10/2013), 'Unreliable research: Trouble at the lab.' commented: "Scientists' grasp of statistics has not kept pace with the development of complex mathematical techniques for crunching data. Some scientists use inappropriate techniques because those are the ones they feel comfortable with; others latch on to new ones without understanding their subtleties. Some just rely on the methods built into their software, even if they don't understand them."

According to the general paradigm of modern sciences, statistical analysis methods are key to translate raw empirical data into new insights in, and deeper understanding of, complex processes affecting human health, the economy, environment, and many other phenomena studied in different branches of science. Yet, the complexity of such processes, and of the observable data they generate, create numerous analytical challenges. In the 21st century, parallel progress in the theory of mathematical statistics and computational resources and technology led to dynamic developments in statistical methodology, resulting in a large number of increasingly complex, ever more flexible, statistical techniques and models that allow researchers to account for several complexities frequently encountered in analyzing real-life data. Unfortunately, many of these important developments are ignored in everyday practice of data analysis, including analyses reported in influential publications in high-impact medical or social sciences journals. Consequently, the design and analysis of recent, often complex and costly, observational studies of human health and welfare often exhibit serious weaknesses. This leads to misleading inferences, which may, in turn, adversely affect the effectiveness or safety of different treatments, social or economic policy programs etc.

Formulating and overcoming these formidable challenges requires a well structured, highly interactive collaboration between a large, international group of statistical experts, whose research combines development of new methodology with collaborative research on real-life applications and whose joint, complementary expertise covers different sub-areas of statistical research. Indeed, such need arises due to a combination of (i) ever increasing complexity and variety of analytical challenges encountered in the majority of important observational studies, together with (ii) the increasing trend for narrow specialization, necessary to achieve cutting-edge novel developments in modern statistics. Together, these two trends imply that no single university-based (bio-)statistics department and no team of (bio-)statisticians (working in even the largest private or public research institutes or, e.g., pharmaceutical companies) is able to ensure state-of-the-art expertise regarding even a reasonable fraction of analytical challenges encountered in real-life applications. At present, particular challenges are being addressed by leading authorities in different areas of statistical research, but little effort is invested in combining the results of these separate developments and ensuring their material impact on the practice of data analysis. This situation provided the motivation for, and driving vision behind, the STRengthening Analytical Thinking for Observational Studies (STRATOS) initiative. The STRATOS initiative was launched in 2013 at the 34th conference of the International Society of Clinical Biostatistics (ISCB). It is connected to this society and had dedicated invited sessions or mini-symposia at all ISCB annual meetings in 2013–16. The initiative brings together leading, internationally recognized, methodological experts in several areas of statistics essential for the analyses of observational studies, who—at present—are grouped in nine Topic Groups (TGs), each focusing on a different, highly relevant area of statistical research. The experts' joint, largely complementary, knowledge allows the initiative to address complex analytical challenges in the design and analysis of observational studies, by developing, validating and comparing state-of-the-art methods for various topics. To increase the impact of our endeavours on empirical research, individual topic groups are working to summarize our findings by developing practical guidance regarding 'best practice' to address a particular set of analytical challenges (e.g. handling of missing data or dealing with right-censored time-to-event data). The guidance covers such practical issues as e.g. the awareness of potential pitfalls due to inappropriate use of 'conventional' methods, the choice of appropriate, validated analytical methods able to overcome specific challenges, and software that can be used to implement these advanced methods.

We are entering the era of 'big data' with automated collection of very large amounts of data and the paradigm of empirical sciences shifting toward 'data science'. In a recent Editorial, David Hand (J Roy Statist Soc (JRSS) A (2016), 179: 629–631) stresses the importance of distinguishing between two types of activity relating to big data. The first involves primarily data manipulation: sorting, searching, matching, and so on. Examples include online route finders and apps for updated status of bus and train traffic, with the associated issues addressed mostly by computer scientists and mathematicians. The second type of big data activity seeks to go beyond the data at hand, with the ultimate goals being either prediction of future data, or understanding of the mechanisms and processes that have generated the collected data. Achieving these goals will rely primarily on state-of-the-art statistical and machine learning methods.

However, 'big data' will not help answering such essential prognostic or etiology questions if researchers use designs and statistical methods which are unsuitable, e.g. by being unable to account for the complexity of the underlying dynamic multi-factorial processes. Therefore, it is of central importance to gain knowledge about advantages and disadvantages of alternative statistical approaches, and their dependence on the data structure. Equally important is to develop, validate and explain to end-users state-of-the-art methods that can address frequent limitations (e.g. missing data, measurement errors, unmeasured confounding) and complexities of the data (e.g. non-linear relationships of continuous variable with the outcome, time-varying effects, mediation). These challenges are relevant for small, intermediate and big data sets, and the STRATOS initiative aims to address all

	STRATOS Topic Groups	Chairs
1	Missing data	James Carpenter, Kate Lee
2	Selection of variables and functional forms in multivariable analysis	Michal Abrahamowicz, Aris Perperoglou, Willi Sauerbrei
3	Initial data analysis	Marianne Huebner, Saskia le Cessie, Werner Vach
4	Measurement error and misclassification	Laurence Freedman, Victor Kipnis
5	Study design	Mitchell Gail
6	Evaluating diagnostic tests and prediction models	Gary Collins, Carl Moons, Ewout Steyerberg
7	Causal inference	Els Goetghebeur
8	Survival analysis	Michal Abrahamowicz, Per Kragh Andersen, Terry Therneau
9	High-dimensional data	Lisa McShane, Joerg Rahnenfuehrer

Table 13.1: Topic groups and their chairs

of them. In the health sciences, studies with a relatively small sample size and a very large number of potential (genetic) predictors ($p >> n$) have become popular. Obviously, the analysis of such studies raises many methodological issues. To cope with such problems we have recently created 'High dimensional data' TG9. The work of this group is summarized in Section 13, and other interactions at Banff relating to 'Big Data' are described in Section 13.

One of the fundamental objectives of the STRATOS approach is to develop guidance documents for data analysts and researchers with different levels of statistical training, skills and experience. Specifically, we have indentified three levels of statistical knowledge, each of which would require a somewhat different type of targeted guidance document, and we have outlined the main criteria to be used when developing the guidance documents aimed at the analysts at each level. Initially, we are working to derive guidance documents for experienced statisticians ('level 2'), which requires work on state-of-the-art documents for each specific topic group. For each topic considered (see next section) several analytical strategies have been proposed in the statistical literature, but knowledge about their properties and relative strengths and weaknesses is often insufficient, as meaningful comparisons are rare and evidence based guidance documents are lacking. For more details see Sauerbrei et al, (Statistics in Medicine, 2014, **33**, 5413–5432) and the STRATOS website http://www.stratos-initiative.org/

Several members of STRATOS have recently worked on various guidelines for reporting in the health sciences, research now coordinated by the equator network (http://www.equator-network.org/). Positive experiences in reporting guideline initiatives have been an important driving force and a source of inspiration for the discussions undertaken during the BIRS workshop.

STRATOS topic groups

August 2013, when the STRATOS initiative was launched, it had seven Topic Groups (TGs) and 45 members. By July 2016, these numbers have increased to 9 TGs (see Table 13.1), with a total of 82 members, from 15 countries.

Cross-cutting panels

To co-ordinate the activities of different topic groups (TGs) , share best research practices, and disseminate research tools and results across TGs, several cross-cutting panels have been created in 2015–16. These panels aim to address the 'generic' issues, common to all or most of the TGs, by developing recommendations (sometimes rather loose as for simulation studies, sometimes more strict as for STRATOS publications) and coordinating the efforts of the individual TGs. Recommendations aim to support, integrate and harmonize work within and across the TGs. They will also help increase transparency in deriving final guidance documents for the entire STRATOS initiative.

The BIRS Workshop

Participants.

Between the official launching of the STRATOS initiative in August 2013 and the Spring of 2016, some of the members met during the annual ISCB conferences, and each of the topic groups (TGs), as well as the steering group conducted more or less regular teleconferences, in addition to (separate) small face-to-face, 1–2 days long, meetings of subgroups of 3–6 members of some of the TGs. However, the BIRS workshop offered the first opportunity for the members of all 9 TGs and the panels to meet together, and discuss both TG-specific and general STRATOS-wide projects over the extended period of five working days, often followed by less formal evening meetings of subgroups interested in particular methodology, or organizational issues, related e.g. to strategic planning or STRATOS modus operandi. Together these varied, highly interactive activities were essential to start the complex, long-term process whose ultimate objective is to develop a comprehensive, integrated framework for the statistical analysis of observational studies.

The popularity, and interest, of the international statistical research community in the workshop were reflected in both (1) the fact that we have filled all the 42 available places at BIRS, and in addition had three non-resident observers; and (2) the meeting's truly global character. Indeed, the 45 participants/observers came from 12 countries on 4 continents (most participants came from Canada, USA and Europe but Asia (J. Freedman from Israel) and Australia (K. Lee) were also represented). The participants represented the full spectrum of career stages, including (i) leading international authorities in different fields of statistics and biostatistics; (ii) several mid-career and junior faculty members; as well as 7 research trainees (2 post-doctoral fellows and 5 PhD students). About 40% of participants were female.

Overview of the presentations and discussions

On Monday morning, we started with a general introduction to the initiative, overview of its long-term objectives and mode of operation, and the main aims of the BIRS meeting. This was followed by 4 keynote talks by internationally recognized authorities in selected areas covered by the STRATOS initiative: Per Kragh Andersen (survival analysis, TG8), James Carpenter (missing data, TG1), Frank Harrell (selection of variables and functional forms in multivariable analysis, TG2), and Els Goetghebeur (causal inference, TG7). To jump-start the discussions over the next 4 days, the four keynote speakers provided a broad overview of the state of research in their respective areas and identified some outstanding methodological challenges, as well as pointed out some of the most frequent mis-conceptions and errors in the current practice of data analysis.

The key component of the meeting involved presentations and highly interactive discussions about the activities of nine TGs. TG-specific presentations, usually given by 2 or 3 members of a given TG focused on: (i) overview of methods currently used in everyday practice of data analysis, and their potential limitations, and/or (if applicable) of the existing guidelines, tutorials or methodological review papers covering the methods relevant to a particular TG, (ii) main analytical issues that should be clarified in future guidance documents, (iii) progress already achieved by a given TG, (iv) outstanding challenges, some of which may require either primary research on developing new methods or secondary simulation-based research on validation and comparison of alternative existing methods, (v) short- and mid-term plans, including outlines for future papers, as well as (vi) identification of future collaborative links to some of the other STRATOS TGs, in order to tackle more complex issues that will require combining cutting-edge methods developed by individual TGs (e.g. handling of missing data on unmeasured confounding in survival analysis (TG1, TG7, TG8), or selection of variables and functional forms in high-dimension analyses (TG2 and TG9). All TGs worked on a roadmap for guidance documents, which included intended publications for the next 12 months. Key members were identified for most scientific projects. More details on the presentations given, and conclusions reached by members of individual TGs, are found in Sections 13–13 below.

In addition, the STRATOS Steering Group has more recently decided to create various cross-cutting 'Panels', each of which was given a mandate to cover specific strategic issues that are pertinent to many TGs, and/or to the operation of the STRATOS initiative as a whole. Given the cross-cutting character of the panels, the BIRS workshop offered an unique opportunity for direct in-person discussions of the panels missions and their modus operandi. Therefore, the panels were given an important role at the meeting. Accordingly, during the 5 days of the BIRS meeting, the necessity and role of each of the 10 panels were discussed, usually after a detailed presentation from one of the chairs. General discussions were often followed by in-person meeting of members of different TGs interested in a given panel, and discussions in smaller groups. Several panels recruited new members (Sections 13–13 of the Report provide a summary of the activities of the respective panels during the BIRS workshop).

To allow more detailed discussions and time for outlining the content of future manuscripts within TGs, as well as discussions between members of different TGs and/or panels, we also decided dedicate a substantial amount of time to separate meetings in smaller groups. Often discussions were continued in the evenings. Results of small groups discussions were presented and discussed on Friday morning, in a general session that summarized the conclusions of the BIRS meeting and provided an outlook for main activities planned for the next two years.

Summary of the BIRS workshop achievements and plans for future

There was a general consensus that the BIRS workshop was very successful: (i) each of the nine topic groups made substantial progress in shaping and concretizing their research plans, and was able to outline or even draft the manuscript(s) (more than 15 TG-specific manuscripts are currently in progress, at various stages of completion, across the STRATOS initiative), (ii) first multi-disciplinary projects, tackling complex analytical challenges and issues relevant for more than one TG were outlined, (iii) members of the initiative refined the purpose of, and agreed members for, the cross-cutting panels, and (iv) progress was made on an effective dissemination strategy including an enhanced, interactive, website and social media presence, as well as concrete steps to build long-term partnerships with various societies and organization including those that involve mostly (a) statisticians or (b) end-users of statistical methods.

All participants expressed the strong wish having more regular meetings in smaller groups (TG meetings, joint meetings of members from two or more TGs and/or one or two panels with somewhat overlapping mandates (e.g. Knowledge Translation vs Publication panels). There was also an overwhelming consensus that, given the achievements of the BIRS meeting, it will be essential to organize the next general meeting of the STRATOS team in about two years. Several colleagues suggested that an ideal location for the next meeting, in 2018, will be Oberwolfach (Germany), whose research station has a mission similar to BIRS. Accordingly, four of the Steering Group members started working on the new application in the days after the BIRS meeting and, on July 31, 2016 have submitted an application for a workshop in 2018 in Oberwolfach, entitled 'Next stages in Developing an Integrated Framework for Advanced Statistical Analyses of Observational Studies'.

Summary of Progress and Plans by 9 Topic Groups

All topic groups benefitted immensely from face-to-face meetings, with many STRATOS members meeting for the first time. This was vital for both cementing interpersonal relationships and establishing new long-term research collaborations, the continuation of which will be almost exclusively by email and teleconferences.

Project management and plans for the first topic group-specific STRATOS manuscripts, were discussed within each TG; it was agreed that authorship will broadly follow the ICMJE guidelines. In general, all TG members who expressed an interest are invited to actively contribute to a given project, and input from other TG members should be sought at various stages, e.g. before presentations and for revisions of manuscript drafts. More specific details of the protocol for STRATOS publications are given below, in a sub-section 13 on the Publication Panel.

TG-led papers were finalized, revised and developed, as summarized below:

TG1: Missing data

Discussions and interactions

Five members of the group were present (including the co-chairs); it was the first time some of us had met, and good collaborative relations were established. We reviewed progress so far, and decided to re-focus our efforts as outlined below.

Research: current

Our first paper, led by Rod Little, will be aimed at a 'level 1' audience, and will detail when analysis of records with no missing data alone is likely to be appropriate. Preliminary discussions with the New England Journal of Medicine suggested they would be open to considering a short 'research methods' article on this topic.

The second paper, led by Kate Lee, with aspects relevant for both level 1 and 2 researchers will outline the proposed framework for the conduct and reporting of the analysis of observational studies with missing data. This framework was presented by James Carpenter in the plenary session, and refined following the ensuing stimulating discussions.

These papers should be drafted by January 2017.

Research: future

Beyond this, we also planned a paper led by James Carpenter aimed at 'level 2' researchers, presenting and comparing a wide range of the different approaches to analyzing data that are missing at random, using an extract from publicly available UK youth cohort study.

TG2: Selection of variables and functional forms in multivariate analysis

Discussions and interactions

The plenary talk by Frank Harrell stimulated an ongoing discussion about the utility of variable selection vis-a-vis different approaches to variables selection.

Research: current

The meeting allowed the plans of the group to be consolidated as follows:

- Michal Abrahamowicz and Gary Collins (TG6) will lead on a literature review of the relevant methods currently used in applied research, with a draft circulated to the group by December 2016;
- Aris Perperoglou and Matthias Schmid will lead the review and assessment of the splines packages available in R, circulating a draft to the group by November 2016;
- Pre-modelling data analysis was identified as an important, and somewhat neglected, area and Matthias Schmid and Werner Vach will co-ordinate a draft of a level 1 paper to be circulated in December 2016.

Research: future

- The group agreed that the conflicting, and often missing, information about the pros and cons of the many spline approaches, in the current literature, is confusing to researchers and calls for a systematic assessment and clarification. Aris Perperoglou, Matthias Schmid and Willi Sauerbrei will draft a level 2–3 article for the group by mid 2017.
- Spikes at zero are a challenge for many methods, and Heiko Becher and Frank Harrell will draft a level 2 manuscript evaluating the suitability of methods in this setting for Summer 2017.
- Willi Sauerbrei, Matthias Schmid, Doug Altman and Aris Perperoglou will work on a level 1 manuscript assessing how analysts handle modeling of the functional forms for continuous prognostic factors (whether they categorize them, etc.)

TG3: Initial data analysis

Discussions and interactions

Three members of the TG, Heike Hoffmann, Marianne Huebner, Werner Vach, were present. The group agreed to drop 'Descriptive' from the name, and focus on initial data analysis of observational health care studies. Future aims for TG3 are data visualization, skewed covariates, literature review, IDA reporting guidance, IDA for longitudinal data, and electronic health records.

The group took full advantage of the opportunity for face-to-face discussions with other topic groups, in particular with:

- TG9 (High dimensional data analysis), to exploit approaches of common interest. As a result, Lara Lusa (TG9) has joined TG3;

- TG1 (Missing data), to discuss the visualization of missing data patterns, and how this can be used to show predictors of data being missing.
- TG2 (Variable selection), leading to planning a paper 'What you should think about before starting a regression analysis', led by Werner Vach and Matthias Schmid.

Research: current

The following research presentations were given by group members at Banff:

- Marianne Huebner: 'Overview of framework and reporting of initial data analysis';
- Heike Hoffmann: 'Data visualization for initial data analysis'

These informed discussions leading to finalization of a level 3 manuscript, entitled 'A contemporary framework of initial data analysis', which has been submitted;

As mentioned above, plans were developed for a paper 'What you should think about before starting a regression analysis', led by Werner Vach and Matthias Schmid.

A paper on data visualization for initial data analysis led by Heike Hoffmann and Dianne Cook is in progress.

Research: future

The following projects are being developed:

- a literature review
- initial data analysis for longitudinal data, especially from electronic health records
- issues with skewed data

TG4: Measurement error and misclassification

Discussions and interactions

Five members were present in Banff: Veronika Deffner, Larry Freedman, Ruth Keogh, Victor Kipnis, and Pamela Shaw. It was agreed to:

- nominate TG members for the panels as follows: Simulation Panel—Victor Kipnis (secondary representative Pamela Shaw); Website—Ruth Keogh; Publications—Pamela Shaw;
- use, and advocate to the Datasets Panel, the NHANES, OPEN, IDATA and Whitehall datasets, and
- Suggest to the Organizations Committee linking with the ISEE, Farr Institute, ICDAM, and EpiCongress (SER).

Research: current

Previous plans to write three papers by Spring 2017 were consolidated as follows:

- a level 1–2 report of our literature survey on the use of methods to adjust for measurement errors in four areas of epidemiology;
- a level 2 guidance paper on measurement error and misclassification of variables in epidemiology, and
- a guidance paper on dietary measurement errors for nutritional epidemiologists.

Research: future

- Discussions with Mitchell Gail (TG5, Design) elaborated several areas where improved design could address measurement error issues. The two groups plan to collaborate on this in 2017.
- Discussions with Lisa McShane (TG9, High-dimensional data), and James Carpenter (TG1, Missing data) suggested fruitful areas of collaboration, and these will be developed further in 2017.

TG5: Study design

Discussions and interactions

Suzanne Cadarette joined TG5, and discussions on TG5 representatives on various STRATOS panels were initiated. Working groups were established with the following topic groups where improved designs need to be developed and adopted:

- TG4 (Measurement error): led by Laurence Freedman and Mitch Gail, and
- TG9 (High dimensional data): led by Lisa McShane and Mitch Gail, focusing on use of stored bio-specimens for biomarker studies for prognosis or treatment prediction.

Research: current

The TG has an advanced draft of an overview of design issues in observational studies; it was agreed to strengthen this with three examples:

- Australian weighted cohort with nested case-control design (Elizabeth Williamson);
- GWAS studies (Mitch Gail), and
- Administrative database example to illustrate confounding by indication (Elizabeth Williamson).

The manuscript should be submitted by the end of 2016.

Research: future

The following projects are in the planning stage:

- paper on design aspects of routinely collected data (Elizabeth Williamson, Suzanne Cadarette, and Andrea Rotnitsky)
- design for the phases of (i) development and (ii) validation of risk prediction models in new prognostic factor(s) (Gary Collins, Peggy Sekula).

TG6 Evaluating diagnostic tests and prediction models

Discussions and interactions

During the meeting, a Letter to the Editor highlighting methodological flaws of a recently published prediction model in the journal Gynecological Oncology Reports was written. The letter has subsequently been published (on behalf of TG6):

Van Calster B, Steyerberg EW, Bourne T, Timmerman D, Collins GS, on behalf of TG6 of the STRATOS Initiative. Flawed external validation study of the ADNEX model to diagnose ovarian cancer. Gynecological Oncology Reports 2016; DOI: 10.1016/j.gore.2016.09.003

Research: current

- Design for the phases of development and validation of a prognostic model, with TG5 (Design), led by Gary Collins and Peggy Sekula.

Research: future

- Perform a number of systematic reviews evaluating current practice in the medical literature to identify areas where guidance is most needed.
- Write an overview/review paper describing the various approaches for assessing model calibration; provide some guidance for non-statisticians.
- An article on how to compare the performance of alternative prediction models in an external validation.

TG7: Causal inference

Discussions and interactions

Causal inference is clearly related to a number of the TGs, and this lead to a number of fruitful interactions, in particular with TG1 (missing data), TG4 (Measurement error), TG8 (Survival analysis), and TG9 (High dimensional data)

Research: current

The aim is to provide insight and guidance in this very complex landscape by identifying and publishing important studies, where causal effects have been correctly (or incorrectly) handled:

- a case study on the effects of breastfeeding on the baby's weight was identified,
- discussions with TG8 (Survival) focused on the challenges of phrasing the causal question in the competing risks setting.

Research: future

- Use the breast feeding example to construct a realistic simulation framework.
- Develop the sketch paper on missing data issues in causal inference in collaboration with TG1.

TG8: Survival Analysis.

Discussions and interactions

All but two TG8 members were present. Members took full advantage of the opportunity for fruitful discussions, laying the foundation for progress that would be been substantially more difficult and time consuming without the BIRS meeting.

Research: current

Discussions focused on the content and structure of the two first TG8 papers, both aimed at the 'level 2' readership (applied statisticians who are not experts in survival analysis). A writing plan was agreed, which should see these two articles submitted to Statistics in Medicine in the first half of 2017.

The first paper will concentrate on the relatively common setting where individual members of a (retrospective or prospective) cohort are followed until they are right-censored or developed a single end-point (e.g. 'all-cause mortality'). Following a brief introduction to survival analysis, this paper will provide a detailed discussion of three specific issues, about which there has been considerable research, which has yet to fully permeate practice. These are: (i) modeling of time-varying covariates (i.e. variables which change their value(s) over time); (ii) accounting for time-dependent effects (e.g. non-proportional hazards), and (iii) 'immortal time bias' and related biases.

The second paper will deal with the situation where several end-points are analyzed simultaneously, in a multi-state model, that involves transitions between different health states. These states may be mutually exclusive and/or may follow each other. In the former case a competing risks approach will be described and promoted. Examples and illustrations will build on Therneau's vignette to his 'survival' package for the R software.

Research: future

Papers for 'level 1' (non statisticians) will be planned at a later stage and will build on the two above-mentioned 'level 2 papers'. These papers for level 1 must be carefully calibrated with the excellent series of four papers from British Journal of Cancer, published in 2003 by Altman and others, who provide a broad, reader-friendly overview of (i) many analytical challenges and (ii) most popular statistical methods that may be used to address most of these challenges.

TG9: High dimensional data

Discussions and interactions

Three members of this, the most recently created, TG were present. Discussions and interactions were very important. Discussions focused on:

- what the notion 'high-dimensional' should encompass, deciding that as a working definition we would use the term to describe the situation where a large number of variables is available for the analyses.
- deciding that data from molecular medicine (i.e. "-omics" data, like genomics, transcriptomics, proteomics, metabolomics), as well as data from other sources such as electronic health records and comparative effectiveness studies, would be included within the scope of TG9.
- agreeing that the following sub-topics would be used to structure the group's work: Data Pre-processing, Exploratory Data Analysis, Data Reduction, Multiple Testing, Prediction Modeling/Algorithms, Comparative Effectiveness and Causal Inference, Design Considerations, Data Simulation Methods, Resources for Publicly Available High-dimensional Data Sets.

Research: current

An initial project will collate and review issues arising from the analyses of gene expression data from cancer studies that TG members have been involved with and/or have access to, leading to identification of typical, widely used methodologies, their strengths and limitations, and hence guidance for practitioners.

Research: future

Tentative plans include preprocessing methods and data analysis pipelines for specific types of omics data TG members are working with.

Cross-cutting Research Panels

Underpinning the initiative are a number of panels, formed from members of the various Topic Groups and some additional STRATOS members whose main contributions to the initiative involve work on specific panels. Panels provide a common scientific framework for the initiative's activities, determine some aspects of its modus operandi and provide guidelines regarding issues relevant to all TGs.

As this was the first meeting of the initiative, a part of the Scientific Program was set aside to consolidate the mandates and membership of these panels, which have a key role in the success of the initiative.

Below we present brief reports of the panels' activities during the BIRS workshop.

Publication panel (PP)

Before the meeting, the PP panel prepared and circulated a draft of the publication policies and procedures for the initiative. This covers issues relevant for peer-reviewed papers, letters, conference presentations and web materials. The panel presented the draft policy in a plenary session, with the focus on rules for authorship of STRATOS publications, as well as responsibilities and timelines of the PP. Following a constructive discussion, the consensus was reached regarding the operational definition of a 'STRATOS publication', and the corresponding standards for internal STRATOS review process.

The full version will be published on the website and initiative members alerted by email. Key requirements agreed upon are:

- Papers should be open access, if at all possible;
- Results need to be reproducible, with both (i) the code for implementation of the analyses, or simulation studies; and (ii) the data used in the analyses, being available to research community at large.

Glossary Panel (GP)

This panel aims to standardize language, terminology and notation used within the STRATOS initiative's publications and guidance documents. To this end, the GP supports all members helping them to use consistent, clear and unambiguous language in all outputs. In turn, it is hoped this homogenization will positively impact the broader statistical research community.

Prior to the meeting, the panel had prepared a 300 page document of about 800 terms, drawing closely on Simon Day's Dictionary for Clinical Trials (2nd edition, Chichester: John Wiley and Sons, 2007. ISBN 047031916X, 9780470319161). The GP will use this as a starting document for the STRATOS glossary. The current version was presented at a plenary session, and suggestions for (i) the structure and searchability, (ii) specific terms for inclusion, and (iii) cross-referencing, were discussed. Members were invited to provide extensions and improvements. The panel will develop plans for incorporating the glossary into the STRATOS website, alongside with user-friendly tools that will allow some highly qualified researchers, to add new terms.

Review Panel (RP)

This panel is responsible for establishing a common framework for conducting systematic reviews, of both (i) methodological and (ii) applied literature, across all STRATOS topic groups. The focus of these reviews will be, respectively, to (i) identify the un-resolved methodological challenges and issues that will require systematic investigation by analytical means and/or simulations (e.g. validation or comparisons of alternative methods to address specific challenge(s) relevant for a given TG), and (ii) establish which of the relevant analytical methods are applied in substantive studies and identify common pitfalls in the analyses of real-life data. A guidance document was circulated and the issues were considered in a plenary session led by Gary Collins. This panel will now incorporate the feedback into a revised document, which will be considered by the Steering Committee.

Simulation Panel (SP)

Simulations are key to the work of all TGs. They will be essential to (i) validate some of the recently developed methods, (ii) compare how the relative advantages and disadvantages of alternative methods depend on the true underlying data structure, and (iii) at time, illustrate the pitfalls and potentially serious errors induced by conventional methods (frequently used in the applied research), especially when the underlying assumptions are violated. This panel develops and promotes principles for, and provides examples of, best practice for simulation studies. Discussions during the BIRS workshop focused on drawing on the literature to identify the key principles, and find useful published examples, which will help TG members with the design, conduct, analysis and reporting of targeted simulation studies. This will help ensure the conduct of simulation studies is as consistent as possible across TGs (especially when they touch on similar issues), and support the accessibility, transparence, and reproducibility principles, that are fundamental to all STRATOS research activities.

Data Sets Panel (DP)

This panel coordinates the search for, and criteria for use of, publicly available real-life datasets that will be used by different TGs to illustrate applications of the state-of-the-art methods and/or specific methodological issues or pitfalls. For reproducibility, it was agreed all data sets used in STRATOS publications will be publicly available via the STRATOS website. Meta-data will be used to provide an easy overview of datasets. The DP is developing common protocols for these activities.

Website Panel (WP)

The STRATOS website is central for both (a) communication within the initiative, and (b) for general research community, as the primary access point for (future) guidance documents, software packages and codes, data, as well as information about individual TGs and the initiative as a whole. During the BIRS workshop, a plenary session reviewed the current website and discussed the strategy for future development and enhancements of the website. Key characteristics for a good website, such as appearance, content, functionality, website usability, and search engine optimization, were discussed. Many important properties are already met but several further extensions are required.

It was also generally agreed to make a start toward integrating social media activities into the STRATOS website. The hashtag #STRATOSi was registered using the healthcare hashtags project (http://www.symplur.com/search/stratosi). Members agreed to link to the STRATOS web pages to their individual home and research pages, which will increase the visibility of the STRATOS website.

Further panels

The work of the remaining panels, that focus mostly on organizational issues, progressed, and is summarized below:

- Membership Panel: the protocol for processing requests to join the initiative has been agreed by the Steering Group and published on the website. This is essential given increasing visibility of the initiative within the international research community results in a increasing number of researchers who would like to join STRATOS.
- Contact with other Organizations: STRATOS is closely linked with the International Society of Clinical Biostatistics; links are being developed with the International Biometric Society (see section 13 on "post-BIRS activities). A list of other relevant scientific Societies that will be approached was drawn up.
- Knowledge Translation: In collaboration with the Publication Panel (PP), the panel will spearhead publicizing the work of the initiative to the target audiences. A number of imaginative approaches for this emerged during discussions at the BIRS workshop.
- Bibliography Panel: it was agreed this panel will oversee creation of a comprehensive bibliography for the initiative, which will be available on the STRATOS website.

STRATOS post-BIRS Activities

Informed and stimulated by the discussions at the BIRS meeting, the following activities have taken place since:

- Invited, videorecorded, STRATOS session at the International Biometric Conference (July 12, 2016, Victoria, Canada; https://biometricconference.org/)
- STRATOS Mini-symposium at ISCB2016 (25 August, 2016, Birmingham, UK; http://www.iscb.info/ISCB2016.html).
- Special STRATOS session at the HEC2016 conference (Health—Exploring Complexity: An Interdisciplinary Systems Approach; September 1, 2016, Munich, Germany; http://www.hec2016.eu/).

Slides of all 19 talks presented at these sessions and mini-symposia are available on the STRATOS website.

Future Challenges and Plans

Big data The emergence of 'Big Data', and the teams of researchers who focus on this complex topic, is a key driver for STRATOS, as this poses particular challenges and opportunities across the spectrum of statistical research and applications. However, the term 'Big Data' encompasses many diverse areas and many types of data sources: e.g. electronic health records, large administrative and insurance-oriented databases, related registry data, -omics, and imaging data. Reflecting this diversity, STRATOS deliberately decided not to have a 'Big Data' topic group, but instead to encourage all TGs to consider how their work relates to, can be motivated by, and/or adapted in order to respond to specific challenges induced by the type of 'Big Data' relevant to their focus. As the TG reports (summarized in Section 13 above) illustrate, this is already happening. For example, the missing data topic group (TG1) has been involved in development and dissemination of techniques for large electronic health record datasets; the design TG5 is including large routinely collected data examples in their first publication. Furthermore, challenges related to variable selection and modeling of functional forms for continuous variables (that are focus of TG2) are of paramount importance in the analyses of datasets. Finally, the High Dimensional DataTG9 is concerned with the specific analytical problems that arise with -omics data, where the number of variables is typically far greater than the number of study subjects.

Future funding It was agreed that TGs should be on the alert for finding external funds to support future, smaller workshops, which could facilitate pushing forward their research agendas. Such workshops could constitute attractive 'satellite meetings' to established conferences, thereby minimizing travel costs. For example, the UK MRC Methodology Panel funds such workshops.

Steering Group members will also reach out to national funding agencies to explore opportunities for receiving financial support for the vital, but less glamorous, work of the underpinning research panels.

Complementing this, the TGs are encouraged to identify topics for primary research, and—as the BIRS workshop convincingly demonstrated—STRATOS provides a natural forum for the emergence of exciting new primary research collaborations.

Future Meeting of the entire STRATOS Initiative team The Steering Group believes that the STRATOS activities will greatly benefit from a second international meeting of all TGs and panels, in 2018, prior to releasing the first of the STRATOS guidance documents on our interactive website.

To this end, immediately after the BIRS meeting, in July 2016 an application was submitted for a research week at the Mathematisches Forschungsinstitut, in Oberwolfach (MFO), Germany, in 2018. Several members of the initiative have participated in previous MFO workshops and know well the stimulating, inspiring and interactive atmosphere at this special place.

Prior to the 2018 meeting, STRATOS topic groups and panels will circulate their draft guidance documents, focusing on the issues fundamental for their respective areas. At the meeting, we will use a structured format to discuss, revise and finalize these documents.

The second key focus of the planned 2018 meeting will be on novel research projects requiring development and validation of new multi-stage, methodology, to be developed jointly by several Topic Groups, to simultaneously address analytical challenges frequently encountered in real-life observational studies, and as well as the emerging challenges induced by Big Data. Specifically, the following challenges are envisaged:

- identification of efficient and flexible multivariable methods for variable selection and modeling functional dose-response relationships, both for low and high dimensional data;

- robust, accessible methodology for analyzing complex survival and multi-state data, and

- dynamic approaches to causal inference.

Alongside, and related to this, an important, highly innovative aspect of the meeting will focus on further steps for combining cutting-edge methods from different fields, in order to match the complexity of analyses of real-life observational studies (where many of these analytical challenges have to be simultaneously addressed). For example, a running thread will be addressing coherently, across the three topics, the challenges of missing data, unmeasured confounding, and measurement errors, common to most observational studies. Given the complexity of the data and methodologies, while analytic work is key to developing new methods of estimation and statistical inference, this must be complemented by validation and comparison of alternative methods in comprehensive, contextually relevant simulation studies.

Participants

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Chapter 14

Whittaker Functions: Number Theory, Geometry and Physics (16w5039)

July 24 - 29, 2016

Organizer(s): Ben Brubaker (Univ. of Minnesota), Daniel Bump (Stanford University), Gautam Chinta (City University - New York), Solomon Friedberg (Boston College), Paul E. Gunnells (UMass - Amherst)

The Unreasonable Ubiquity of Whittaker Functions

Classical Whittaker functions were initially defined by Whittaker as solutions to the confluent hypergeometric differential equation. Like many special functions, they have a group theoretic meaning, and the classical Whittaker functions are an instance of a more general class of special functions on a reductive group G over a local field F. Gradually it has been realized that Whittaker functions functions play a critical role in such varied areas as automorphic forms, algebraic combinatorics, geometry of flag varieties and mathematical physics.

Let G be a split, reductive group over a local field F with maximal unipotent subgroup U. Consider functions f on $G(F)$ satisfying the transformation property:

$$f(ug) = \psi(u)f(g)$$

where ψ is a non-degenerate character of U. Then $G(F)$ acts on such functions by right translation. Any function appearing in an irreducible subspace of this regular representation is called a *Whittaker function*. An irreducible representation (π, V) of $G(F)$ has at most one space of Whittaker functions isomorphic to π as a representation of G. They are intimately related with the representation theory of another group, the Langlands dual group. Whittaker functions exhibit what one might call an "unreasonable ubiquity" as they arise crucially in many different contexts in mathematics and physics.

Two developments in the 1970's led to the current landscape. On the one hand, it was realized that a group-theoretic uniqueness principle explains many of the remarkable properties of Fourier coefficients of automorphic forms; over time this has led to the pervasive role of Whittaker functions in the theory of L-functions in modern number theory. On the other hand, a seemingly separate development of Whittaker functions occurred in mathematical physics, particularly around the classical and quantum Toda lattice.

Eventually these threads of separate origin merged into a common tapestry. After Givental studied stationary phase integrals for Whittaker functions coming from the quantum Toda lattice, the geometric role of Whittaker functions led to important developments in the quantum cohomology of flag varieties and mirror symmetry. The geometry of flag varieties also proved important in the theory of automorphic forms, and since the 1990's the geometric Langlands theory has developed with equal inputs from number theory and physics. Here Whittaker functions appear as sheaves on affine flag varieties. Meanwhile the theory of quantum groups, itself having separate origins in the physics of solvable lattice models and quantum mechanics on the one hand, and geometry on the other, also sheds light on Whittaker functions. In the current picture, the separate origins of Whittaker functions in number theory and physics can be understood as part of a larger picture.

Yet the current picture is in great flux. *Crystal bases* are combinatorial analogs of Lie group representations that appeared mysteriously useful in the theory of p-adic Whittaker functions, particularly on the metaplectic group. Pursuing this phenomenon leads to surprising connections with quantum groups and the Yang-Baxter equation. Also in the p-adic theory, Iwahori Whittaker functions are connected with representations of Hecke algebras, Demazure operators, Kazhdan-Lusztig theory and the geometry of flag varieties. Pursuing these connections leads into deep algebraic combinatorics such as the theory of non-symmetric Macdonald polynomials and the double affine Hecke algebra. *Geometric crystals* are algebraic varieties with extra structure whose tropicalizations are crystal bases; these have appeared in the theory of archimedean Whittaker functions, particularly in connection with the Toda lattice and its geometric connections, and other connections in mathematical physics such as Brownian motion. Many of the same geometric connections appear in both the archimedean and non-archimedean theories and it appears possible to interpolate between the two theories. Both theories relate to the geometric Langlands theory. Recent progress on Eisenstein series on Kac-Moody groups and their Whittaker functions has been motivated by connections to both mathematical physics (string theory, quantum gravity) and number theory, leading to rich interactions between the fields. Meanwhile Zeta functions of prehomogenous vector spaces, which have applications in number theory, are related to Whittaker functions via the theory of multiple Dirichlet series.

The previous BIRS workshops "Whittaker Functions, Crystal Bases, and Quantum Groups" (10w5096, June 2010) and "Whittaker Functions: Number Theory, Geometry and Physics" (13w5154, October 2013) brought together a diverse group of researchers in these different fields. Due in part to collaborations initiated or furthered during these meetings, rapid developments in several interdisciplinary research areas are occurring. This current workshop served to highlight the rapid developments in interdisciplinary research areas due, in part, to collaborations initiated or furthered during the past meetings.

In the 2016 workshop "Whittaker Functions: Number Theory, Geometry and Physics" (16w5039) we continued studies in these directions by bringing together researchers in number theory, mathematical physics, combinatorics and representation theory to explore these different manifestations of Whittaker functions.

The 2016 Meeting in Banff

The conference talks highlighed recent developments and open problems concerning Whittaker functions in number theory, geometry, combinatorics, and physics. Most talks were videotaped. In the following we summarize the highlights of each talk. Junior speakers are indicated by a bullet (•). Speakers from under-represented groups are indicated with a star ($\star$).

Benjamin Brubaker (University of Minnesota) spoke on *Hamiltonian interpretation of p-adic Whittaker functions*. Whittaker functions on p-adic groups are expressible as partition functions of six-vertex models on a rectangular lattice; at least, this is known for Cartan types A and B and expected more generally. This lecture explained that in type A, this may alternately be viewed as the discrete time evolution of a one-dimensional system of free fermions. The Hamiltonian dictating the evolution arises from the Lie superalgebra $\mathfrak{gl}(1 \mid 1)$ and the Whittaker function may thus be viewed as a kind of generalized "tau function" in the terminology of the Kyoto school. The work reported on is joint with A. Schultz based on `arXiv:1606.00020`.

Daniel Bump (Stanford University) spoke on *Metaplectic Whittaker functions and the Yang-Baxter equation*. In 2012, Brubaker, Bump, Friedberg, Chinta and Gunnells proposed statistical-mechanical models for p-adic Whittaker functions on the degree n metaplectic cover of $GL(r)$. This lecture concerned recent work of Brubaker, Buciumas and Bump, in which the corresponding Yang-Baxter equations have been found. The corresponding quantum group is identified as a Drinfeld twist of $U_q(\hat{\mathfrak{gl}}(n))$. The effect of the Drinfeld twisting is to introduce Gauss sums into the R-matrix. The scattering matrix of the intertwining operators on the (nonunique) Whittaker models, previously studied by Kazhdan-Patterson and Chinta-Gunnells, is thus reinterpreted as the R-matrix of this quantum group. Moreover, the internal states of these generalized ice-type models (which are not visible to the intertwining operator) are built up by tensoring from $(n+1)$-dimensional supersymmetric modules for the quantum affine Lie superalgebra $U_q(\hat{\mathfrak{gl}}(n \mid 1))$.

Yuanqing Cai (•) (Boston College) spoke on *Fourier coefficients of theta functions on metaplectic groups*.

Kazhdan and Patterson constructed generalized theta functions on covers of general linear groups as multi-residues of the Borel Eisenstein series. These representations and their unique models were used by Bump and Ginzburg in the Rankin-Selberg constructions of the symmetric square L-functions for $GL(r)$. This lecture discussed the two other types of models that the theta representations may support. The first are semi-Whittaker models, which generalize the models used in the work of Bump and Ginzburg. Secondly, the lecturer discussed unipotent orbits, how to determine the unipotent orbits attached to theta functions, and how to determined the covers when these models are unique. He concluded by outlining some applications in Rankin-Selberg constructions.

Reda Chhaibi (•) (Institut de Mathématiques de Toulouse) spoke on *A probabilistic approach to the Shintani-Casselman-Shalika formula*. Jacquet's Whittaker function for a group G, in the non-Archimedean case, is essentially proportional to a character of an irreducible representation of the Langlands dual group – a Schur function in the case of GL_n. This statement is known as the Shintani-Casselman-Shalika formula. This talk presented a probabilistic proof of this formula that is the natural generalization of Shintani's. In it, the appearance of the Weyl character formula is explained from a reflection principle for random walks.

YoungJu Choie ($\star$) (POSTECH) spoke on *Periods of modular forms on* $\Gamma_0(N)$ *and Jacobi theta functions*. Generalizing a result of Zagier for modular forms of level one, she gave a closed formula for the sum of all Hecke eigenforms on $\Gamma_0(N)$ multiplied by their odd period polynomials in two variables, as a single product of Jacobi theta series for any squarefree level N. She also showed that for $N = 2, 3$ and 5 this formula completely determines the Fourier expansions all Hecke eigenforms of all weights on $\Gamma_0(N)$.

Holley Friedlander (•, $\star$) (Dickinson College) spoke on *Twisted Weyl group multiple Dirichlet series over the rational function field*. Similar to zeta functions associated to algebraic function fields, Weyl group multiple Dirichlet series associated to algebraic function fields are rational functions in several variables. The denominators of these rational functions are known, but the numerators are not well understood. Like zeta functions, we expect the coefficients of the numerators to encode information about the arithmetic of the defining curve. As a step toward understanding this relationship, this talk described the support of Weyl group multiple Dirichlet series defined over the rational function field $\mathbb{F}_q(t)$. In particular, the speaker showed that up to a variable change, all such series can be expressed as a finite sum of simpler local series, which act analogue to Euler factors in the construction of the global object.

Nadya Gurevich ($\star$) (Ben Gurion University of the Negev) spoke on *The twisted Satake map and the Casselman Shalika formula*. In this lecture, working with an arbitrary split group, she explained how to identify the unramifed Whittaker space with the space of skew-invariant functions on the lattice of coweights and deduce the Casselman-Shalika formula from it.

Angele Hamel ($\star$) (Wilfrid Laurier University) spoke on *Factorial characters and Tokuyama's identity for classical groups*. In this talk she introduced factorial characters for the classical groups and derived a number of central results. Classically, the factorial Schur function plays a fundamental role in traditional symmetric function theory and also in Schubert polynomial theory. She developed a parallel theory for the classical groups, offering combinatorial definitions of the factorial characters for the symplectic and orthogonal groups, and further proving flagged factorial Jacob-Trudi identities and factorial Tokuyama identities. These identities are established by manipulating determinants through the use of certain recurrence relations and by using lattice paths. This work is joint with Ron King.

Axel Kleinschmidt (Max Planck Institute for Gravitational Physics) spoke on *Automorphic forms and lattice sums in exceptional field theory*. This talk concerned the connection between automorphic forms and string theory. Automorphic forms appear in string theory considerations for example in scattering amplitudes. The simplest cases where their form is understood is related to processes that are (partially) controlled by an additional symmetry, called supersymmetry. The lecture explained how one may use the modern language of exceptional field theory to determine parts of these scattering amplitudes that give alternative expressions for Eisenstein series on exceptional groups (attached to small representations). This approach also gives some insight into certain non-Eisenstein series that are expected to arise in string theory.

Kyu-Hwan Lee (University of Connecticut) spoke on *Convergence and holomorphy of Kac-Moody Eisenstein series*. Let G be a Kac-Moody group associated to a nonsingular, symmetrizable generalized Cartan matrix. First, this lecture considered Eisenstein series on G induced from quasi-characters, and explained how to prove the almost-everywhere convergence of Kac-Moody Eisenstein series inside the Tits cone for spectral parameters in the Godement range. For a certain class of Kac-Moody groups satisfying an additional combinatorial property, the lecturer then showed the absolute convergence everywhere in the Tits cone for spectral parameters in the Godement range. Next, he considered Eisenstein series on G induced from unramified cusp forms on finite-dimensional Levi subgroups of maximal parabolic subgroups. Under some 'ample' conditions on a maximal parabolic subgroup, he proved that the Eisenstein series are entire on the whole complex plane. This is joint work with L. Carbone, H. Garland, D. Liu and S.D. Miller.

Cristian Lenart (SUNY Albany) spoke on *New results on Kirillov-Reshetikhin modules and Macdonald polynomials*. In a series of papers with S. Naito, D. Sagaki, A. Schilling, and M. Shimozono, the lecturer has developed two uniform combinatorial models for (tensor products of one-column) Kirillov-Reshetikhin (KR) modules of affine Lie algebras; he also showed that their graded characters coincide with the specialization of symmetric Macdonald polynomials at $t = 0$. This lecture presented the latest work in this direction: the extension of the above results corresponding to the non-symmetric Macdonald polynomials. It also explained the connection of this work to the q-Whittaker functions of Braverman-Finkelberg and their results, which extend to quantum K-theory Schubert calculus.

Anthony Licata (Australian National University) spoke on *Partial orders on the Weyl group, monoids in the braid group, and homological algebra*. Let W be a finite Weyl group. Associated to W there are two important partial orders – (weak) Bruhat order, and absolute order. These two partial orders are related to a pair of combinatorial lattices, and these lattices are in some (not completely understood) senses 'dual' to one another. As a result many other structures related to W come in dual pairs; for example, the braid group of W has a pair of 'dual' Garside structures, and a pair of 'dual' positive monoids. This talk explained these themes, and showed how these dual structures appear in the higher representation theory of the braid group. This work is joint with Hoel Queffelec.

Stephen Miller (Rutgers University) spoke on *Do growing Whittaker functions actually occur in automorphic forms?* Whittaker's original work involves both exponentially decaying and growing functions, generalizing the K- and I-Bessel functions that occur in the Fourier expansions of cusp forms on rank-1 groups. Decaying Whittaker functions are the bedrock of Fourier expansions of L^2 cusp forms for general groups. There is a corresponding beautiful theory of non-decaying Whittaker functions, dating back to work of Kostant and Casselman-Zuckerman in the 1970s. However, there are no known examples of automorphic functions which have non-decaying Whittaker functions in their Fourier expansions outside of rank-1 examples. This is consistent with a tantalizing conjecture of Miatello-Wallach, which asserts that all automorphic eigenfunctions on higher rank groups automatically have moderate growth (and hence decaying Whittaker functions). This lecture, based on joint work with Tien Trinh, presented some cases of that conjecture for $SL(3, \mathbb{Z})$, which rule out analogs of the classical modular j-function. (Note: Due to flight cancelations, Prof. Miller was unable to attend the conference, but he prepared a video presentation as a substitute and was available by phone for questions and comments as it was shown.)

Dinakar Muthiah ($\bullet$) (University of Alberta) spoke on *Double affine Bruhat order and Iwahori-Hecke algebras for p-adic loop groups*. Recently, Braverman, Kazhdan, and Patnaik have constructed Iwahori-Hecke algebras for p-adic loop groups. Unsurprisingly, the resulting algebra is a variation on Cherednik's DAHA. The p-adic construction also comes with a basis (the double-coset basis) consisting of indicator functions of Iwahori double cosets. Braverman, Kazhdan, and Patnaik also proposed a (double affine) Bruhat preorder on the set of double cosets, which they conjectured to be a poset. This lecture described a combinatorial presentation of the double-coset basis and also an alternative way to develop the double affine Bruhat order that is closely related to this combinatorics; from this perspective the order is manifestly a poset. One new feature is a length function that is compatible with the order. It also discussed joint work with Daniel Orr proving that the length function can be specialized to take values in the integers. This proves finiteness of chains in the double-affine Bruhat order, and it gives an expected dimension formula for (yet to be defined) transversal slices in the double affine flag variety. The lecture conclued by discussing how these results are stepping stones for developing Kazhdan-Lusztig theory in this setting and offered a number of open questions.

Maki Nakasuji ($\star$) (Sophia) spoke on *Casselman's basis, Yang-Baxter basis, and Kostant-Kumar's twisted group algebra*. Casselman's basis is the basis of Iwahori fixed vectors of a spherical representation of a connected reductive p-adic group over a non-archimedean local field, which is dual to the intertwining operators at the identity indexed by elements of the Weyl group. The problem of Casselman is to express Casselman's basis in terms of another natural basis, and vice versa. This talk provided one solution to Casselman's problem, using the Yang-Baxter basis of the Hecke algebra and Kostant-Kumar's twisted group algebra. This is joint work with H. Naruse.

Omer Offen (Technion) spoke on *Integrability of matrix coefficients and periods of automorphic forms*. Let G be a p-adic reductive group and H a symmetric subgroup. This lecture presented joint work with Max Gurevich giving a criterion for H-integrability of matrix coefficients of representations of G. This is a generalization of Casselman's criteria for square integrability. Chong Zhang has applied these results to show that for some symmetric subgroups all H-invariant linear forms of square integrable representations emerge as H-integrals of matrix coefficients. In particular, in a global setting, this provides information on the local components of factorizable period integrals of automorphic forms.

Manish Patnaik (•) (University of Alberta) spoke on *Kac-Moody Eisenstein series*. Eisenstein series play an important role in the theory of automorphic forms on reductive groups. They are essential in Langlands program, in particular, spectral decomposition, Arthur trace formula, and Langlands-Shahidi method of studying automorphic L-functions. On the other hand, Kac-Moody groups are infinite dimensional groups, and only recently there have been attempts to define Eisenstein series and automorphic forms on Kac-Moody groups which found an application in string theory in physics. This lecture explained obstacles to doing so and recent positive developments.

Daniel Persson (•) (Chalmers University of Technology) spoke on *Automorphic representations, Whittaker vectors, and black holes*. Automorphic forms on exceptional Lie groups appear naturally in string theory compactifications. They manifest themselves as couplings in higher derivative corrections and in terms of generating functions of black hole microstates. This lecture explained how certain Fourier coefficients attached to the minimal automorphic representations of E_6, E_7, and E_8 are determined by maximally degenerate Whittaker vectors. This fact allows for a simple method for calculating explicit Fourier coefficients which are relevant in string theory. Various recent results, conjectures and open problems were then outlined.

Anne Puskás (•, $\star$) (University of Alberta) spoke on *Metaplectic Iwahori-Whittaker functions and Demazure-Lusztig operators*. Metaplectic Demazure-Lusztig operators are built on the Chinta-Gunnells action, and (analogously to their non-metaplectic counterparts) are useful in the study of p-adic (metaplectic) Whittaker functions. This talk presented work, joint with Manish Patnaik (University of Alberta), that relates metaplectic Iwahori-Whittaker functions to these operators directly. This process gives a metaplectic analogue of earlier work of Brubaker-Bump-Licata in the non-metaplectic setting. She also discussed aspects of the extension of relevant formulae to the affine Kac-Moody groups.

Arun Ram (University of Melbourne) spoke on *Alcove walks and the Peterson isomorphism*. He described a labeling of the points of the moduli space of genus 0 curves in the complete complex flag variety using the combinatorial machinery of alcove walks. Following Peterson, this geometric labeling explains the 'quantum equals affine' phenomenon which relates the quantum cohomology of this flag variety to the homology of the affine Grassmannian. This is joint work with Elizabeth Milicevic.

Anne Schilling ($\star$) (University of California, Davis) spoke on *Cohomology of affine Grassmannians, combinatorics, and crystals*. In it, she explained how the theory of crystal bases can be used to understand some of the affine combinatorics that arises in the (co)homology of the affine Grassmannian. In particular, this approach leads to a combinatorial interpretation of the coefficients of a Schur function in the expansion of a Stanley symmetric function (based on joint work with Jennifer Morse). She also presented some new results and conjectures regarding the Schur expansion of the k-Schur functions, which are dual to affine Stanley symmetric functions.

Takashi Taniguchi (Kobe University) spoke on *Orbital exponential sums for prehomogeneous vector spaces*. Exponential sums arise various context in algebraic and analytic number theory. This talk explained how they may appear in the study prehomogeneous vector spaces, and outlined a new method for evaluating them explicitly. As an application, he showed that there are 'many' quartic field discriminants with at most eight prime factors. This is joint work with Frank Thorne.

Nicolas Templier (Cornell University) spoke on *Kloosterman families, quantum cohomology, and geometric Langlands*. In it, he explained his proof, joint with Thomas Lam, of cases of the Rietsch mirror conjecture that the Dubrovin quantum connection for projective homogeneous varieties is isomorphic to the pushforward D-module attached to Berenstein-Kazhdan geometric crystals. The idea is to recognize the quantum connection as Galois and the geometric crystal as automorphic. He explained surprising relations with the works of Frenkel-Gross, Heinloth-Ngo-Yun and Zhu on Kloosterman sheaves. The isomorphism comes from global rigidity results where Hecke eigensheaves are determined by their local ramification. It implies combinatorial identities for the counts of rational curves, the Peterson variety presentation and other consequences.

Ian Whitehead (•) (University of Minnesota) spoke on *The Chinta-Gunnells construction for affine groups*.

Let W be the Weyl group of a simply-laced affine Kac-Moody Lie group, excepting type A affine root systems of even rank. In this talk, it was explained how to construct a multiple Dirichlet series that is meromorphic in a half-space, satisfying a group W of functional equations, using generalizations of the Chinta-Gunnells construction. This series is analogous to the multiple Dirichlet series for classical Weyl groups constructed by Brubaker-Bump-Friedberg, Chinta-Gunnells, and others. It is completely characterized by four natural axioms concerning its coefficients, axioms which come from the geometry of parameter spaces of hyperelliptic curves. Evidence was presented to suggest that this series appears as a first Fourier-Whittaker coefficient in an Eisenstein series on the twofold metaplectic cover of the relevant Kac-Moody group. The construction is limited to the rational function field, but it also describes the p-part of the multiple Dirichlet series over an arbitrary global field

Outcome of the Meeting

The workshop was well-attended by researchers at all points in their careers, and from a wide range of countries: Canada, USA, Australia, China, France, Germany, Japan, Korea, Israel, and Singapore. After the completion of the workshop, the organizers received many positive comments from both junior and senior participants, and in the year since the workshop ended there have been a number of important papers by participants that are directly related to the workshop themes. Moreover, the broad themes, including interaction between mathematicians and physicists related to Whittaker models, have continued in other settings, notably last fall's program at the Simons Center for Geometry and Physics entitled "Automorphic forms, mock modular forms and string theory."

The workshop itself generated a great deal of energy among the participants. In terms of immediate feedback, one participant wrote

> "The workshop was very fruitful for my research."

and another wrote

> "One week at BIRS gave me a year's worth of new ideas and projects to tackle. BIRS workshops are some of some of the most fruitful conferences I've ever attended, thanks to their camaraderie, beautiful atmosphere, and intense mathematical focus.

This workshop is the third we have organized on the broad theme of Whittaker functions and their appearance in different fields. (The prior two were 10w5096 and 13w5154.) This continuity has led to a strong group of researchers in diverse areas who are aware of and build on these cross-field and even cross-disciplinary connections. We are very appreciative of the opportunity to organize such a workshop periodically, and we hope to continue this string of successes by organizing a week-long workshop at Banff in 2019 or 2020 emphasizing the latest developments. We look forward to continuing this high-impact workshop series.

Participants

Brubaker, Benjamin (University of Minnesota)
Bump, Daniel (Stanford University)
Cai, Yuanqing (Boston College)
Chhaibi, Reda (Institut de Mathmatiques de Toulouse)
Chinta, Gautam (City University of New York)
Choie, YoungJu (POSTECH)
Friedberg, Solomon (Boston College)
Friedlander, Holley (Dickinson College)
Grodzicki, Will (University of Minnesota)
Gunnells, Paul (University of Massachusetts)
Gurevich, Nadya (Ben-Gurion University of the Negev)
Hamel, Angele (Wilfrid Laurier University)
Hoffstein, Jeffrey (Brown)
Karasiewicz, Edmund (Rutgers)

Kleinschmidt, Axel (Max Planck Institute for Gravitational Physics)
Lee, Kyu-Hwan (University of Connecticut)
Lenart, Cristian (SUNY Albany)
Leslie, Spencer (Boston College)
Licata, Anthony (Australian National University)
Liu, Dongwen (Zhejiang University)
McNamara, Peter (University of Queensland)
Muthiah, Dinakar (University of Alberta)
Nakasuji, Maki (Sophia)
Offen, Omer (Technion)
Patnaik, Manish (University of Alberta)
Patterson, Samuel (Universitaet Goettingen)
Persson, Daniel (Chalmers University of Technology)
Pusks, Anna (University of Alberta)
Ram, Arun (University of Melbourne)
Schilling, Anne (University of California Davis)
Scrimshaw, Travis (Univ. of Minnesota)
Strasser, Ben (University of Minnesota)
Takeda, Shuichiro (University of Missouri)
Taniguchi, Takashi (Kobe University)
Templier, Nicolas (Cornell University)
Tingley, Peter (Loyola University Chicago)
Wen, Jun (Univ of Massachusetts)
Whitehead, Ian (University of Minnesota)
Zhang, Lei (National University of Singapore)

Chapter 15

Bridges Between Noncommutative Algebra and Algebraic Geometry (16w5088)

September 11 - 16, 2016

Organizer(s): Michael Artin (Massachusetts Institute of Technology), Jason Bell (University of Waterloo), Colin Ingalls (University of New Brunswick), Lance Small (University of California, San Diego), James Zhang (University of Washington)

Overview of the Field

Noncommutative algebra is a rich and diverse field that has influences rooted in algebraic and differential geometry, representation theory, algebraic combinatorics, mathematical physics, and other areas. Rapid developments in noncommutative algebra, especially in noncommutative algebraic geometry, influence many other mathematical disciplines. We briefly give an overview of some of the main trends that shape our field at this time, with an emphasis on the areas represented during the workshop.

1. **Noncommutative surfaces.** Since curves, having one parameter, are close to being commutative, (or finitely generated modules over their centers), as shown in [5, 6], it is natural to study noncommutative surfaces next. In algebraic geometry there is a classification of surfaces with the most crucial results being Castelnouvo's and Enriques' which characterize rational and ruled surfaces respectively. Artin's conjecture predicts that noncommutative surfaces are either rational, ruled or module finite over their centers [2]. The study of noncommutative rational surfaces produced seminal work in this area [7, 8], as well as new techniques and objects such as twisted homogeneous coordinate rings. Recently, there have been remarkable developments in the theory of noncommutative surfaces (and 3-folds) that are finite modules over their centers and these results extend several commutative programs to the noncommutative setting.

2. **Birationally commutative surfaces and birationally Sklyanin surfaces** In [31, 48, 49, 50] Keeler, Rogalski, Sierra and Stafford have defined and studied a family of noncommutative surfaces that are birationally commutative, some of which are naïve blowups of commutative surfaces. These surfaces behave pathologically in some ways; for example, while they are often noetherian, they do not retain the noetherian property when one tensors with certain commutative noetherian rings. They have shown that all birationally commutative surfaces that are generated in degree 1 are either part of this new family of algebras or are twisted homogeneous coordinate rings. Recently, Rogalski-Sierra-Stafford [46, 47] have started studying noncommutative surfaces that are birationally equivalent to Sklyanin algebras of dimension three. By using noncommutative blowups, Rogalski-Sierra-Stafford have classified all connected graded subalgebras of

Sklyanin algebras of dimension three which are birationally equivalent to these Sklyanin algebras and are maximal with respect to inclusion. These results are crucial to understanding general noncommutative surfaces. Rogalski's talk was based on his joint work with Sierra and Stafford.

3. **Noncommutative Crepant Resolutions.** The study of noncommutative crepant resolutions represents a fundamental connection between resolutions of (commutative) singularities and noncommutative algebra. Noncommutative algebras can be used to build resolutions as moduli spaces and exhibit derived equivalences between them. They are also important in applications of algebraic geometry to string theory. The area began with Van den Bergh's definition [52, 53] and his new approach to Bridgeland's proof of Orlov's conjecture that crepant resolutions of terminal Gorenstein singularities are derived equivalent. Recent results in this area include work of Iyama-Wemyss [29, 30] and Buchweitz-Leuschke-Van den Bergh [12, 13] which build resolutions of singularities from noncommutative algebras. This area was touched upon in some of the talks, including the talk of Faber, which we describe in detail below.

4. **(Twisted or skew) Calabi-Yau algebras.** Calabi-Yau triangulated categories were introduced by Kontsevich in 1998 and Calabi-Yau algebras were introduced by Ginzburg in 2006 as a noncommutative version of the coordinate rings of Calabi-Yau varieties. Since, the study of Calabi-Yau algebras/categories has been related to a large number of research areas, including quivers with superpotentials, differential graded algebras, cluster algebras/categories, string theory and conformal field theory, noncommutative crepant resolutions, and so forth. Twisted or skew Calabi-Yau algebras are a generalization and a companion of Calabi-Yau algebras. It is well-known that all connected graded Artin-Schelter regular algebras are twisted Calabi-Yau and that only a subclass of Artin-Schelter regular algebras are Calabi-Yau. One invariant attached to a twisted Calabi-Yau algebra is its Nakayama automorphism. Several homological identities that relate the Nakayama automorphism with other invariants have been proved. These identities helps to understand several crucial questions concerning group/Hopf actions and smash products, quantum groups, noncommutative resolutions, twisted superpotentials, and noncommutative McKay correspondence. Calabi-Yau algebras and their twisted counterparts were discussed in the talks of Mori, Reyes, Sierra and others.

5. **Hopf Algebra Actions.** The invariant theory of finite group/finite dimensional Hopf algebra actions on noetherian Artin-Schelter regular algebras has been developed, see [33, 34, 35] and a nice survey by Kirkman [32], which includes noncommutative versions of several classical results such as the Watanabe theorem (a criterion for the fixed subrings being Gorenstein), the Shephard-Todd-Chevalley theorem (a criterion for the fixed subring being regular) and the Kac-Watanabe theorem (a criterion for the fixed subrings being a complete intersection). The classification of finite quantum subgroups of quantum $\mathrm{SL}_2(\mathbb{C})$, namely, so-called quantum binary polyhedral groups, has been completed in [17], leading to the study of the Kleinian or Du Val singularities of Hopf actions on noncommutative surfaces and noncommutative McKay correspondence [18, 19]. Hopf actions on the Weyl algebras and commutative domains have recently been studied by Cuadra, Etingof, Walton and others [21, 22, 24, 25, 26] with several surprising results. These studies are closely connected to the representation theory of finite groups/Hopf algebras, homological aspects of noncommutative algebra and noncommutative algebraic geometry. These topics were discussed in the talks of Walton and Kirkman.

Recent Developments and Open Problems

We summarize a few of the main open problems in noncommutative algebra, and in particular, noncommutative algebraic geometry, today. We place emphasis on the problems that were viewed as most important by speakers and participants at the conference and we do not claim to give a complete list of all of the most important open problems in the area.

1. <u>Artin's Conjecture</u>

   Artin [2] conjectured that every division algebra of transcendence degree two that arises from a noncommutative projective surface is either finite-dimensional over its center or is isomorphic to a Sklyanin division ring or to a division ring of the form $K(x;\sigma,\delta)$, where K is the function field of a smooth projective curve, σ is an automorphism of K, and δ is a σ-derivation of K. This has been one of the major open problems in noncommutative algebraic geometry for more than twenty years. By looking at rank one discrete valuations of the division algebras in Artin's list, it can be shown that they are non-isomorphic. This conjecture is very natural after the landmark work of Artin-Stafford [5, 6] that classified all noncommutative projective curves (equivalently, noncommutative graded domains of Gelfand-Kirillov dimension two). It is generally believed that the conjecture is currently beyond available methods, but there has been a large amount of work devoted to understanding this conjecture. As some examples, we note the work of Chan, Hacking, Ingalls, Kulkarni [14, 15, 16], the manuscript by Artin and de Jong, and the series of papers by Keeler, Rogalski, Sierra and Stafford [31, 46, 47, 48, 49, 50], some of which look at classifying projective surfaces in a given birational class on Artin's list.

2. Noncommutative McKay Correspondence

   One influential program in noncommutative algebra is the noncommutative McKay correspondence. This program is highly interdisciplinary, involving the study of (1) Group/Hopf algebra actions on noncommutative algebras/schemes, (2) quotient singularities of noncommutative schemes, (3) representation theory of Hopf algebras and fixed subrings, (4) homological algebraic aspects (such that various derived categories are related), (5) combinatorial information coming from algebra, geometry and representation theory (such as McKay quivers and ADE quivers), and most importantly, the connections between these fields. Some work in dimension 2 has been finished [17, 18, 19]. As the classification of Artin-Schelter regular algebras of dimension 3 is now known, thanks to the work of Artin, Schelter, Tate and Van den Bergh [3, 7, 8], this question could be tractable in dimension 3, and would lead to significant new mathematics. In the commutative case, the Auslander theorem is a key step in establishing the McKay Correspondence. One immediate open question in this program is the noncommutative version of the Auslander theorem [18, Conjecture 0.2]: Let H be a semisimple Hopf algebra and suppose that A is a noetherian Artin-Schelter regular graded H-module algebra such that the homological determinant of the H-action is trivial. Then is there a graded algebra isomorphism $A\#H \cong \mathrm{End}(A_{A^H})$?

3. Noncommutative discriminants

   The noncommutative discriminant is a powerful invariant that has long been used to study orders and lattices in central simple algebras [45]. Recently, the noncommutative discriminant has effectively been used to tackle (1) the automorphism group of noncommutative algebras and the Tits alternative, (2) the isomorphism problem, (3) the Azumaya locus of algebras that are finite modules over their center, and (4) the Zariski cancellation problem in the noncommutative setting. Nguyen-Trampel-Yakimov introduced a new method of using Poisson primes to calculate successfully the discriminant for quantum matrices, the negative part of the quantized universal enveloping algebras, and quantum Schubert cell algebras at roots of unity [44]. Levitt and Yakimov then computed the discriminant for quantized Weyl algebras at roots of unity [37]. Gaddis-Kirkman-Moore applied other methods to deal with discriminants for Ore extensions and for skew group rings [27]. Despite these successes, it is still extremely difficult to compute the discriminant in the general noncommutative case, partly due to the fact that it is not easy to understand the Poisson structure in a general situation. One challenging question is to understand and calculate more effectively the discriminant from different viewpoints. One should relate this question to the classification of group actions on noncommutative algebras. Another interesting question is how to use discriminants to study singularities of the center of noetherian PI Artin-Schelter regular algebras. Further, it is also essential to understand how the discriminant is related to other significant invariants such as superpotential and Hochschild cohomology.

Presentation Highlights

A common theme throughout the workshop was the high quality of the lectures at this conference; indeed, there were many excellent lectures delivered over the course of the week. We include here some of the highlights.

Walton spoke on her work with Chirvasitu and Wang [20], in which they defined a universal quantum group that preserves a pair of Hopf comodule maps whose underlying vector space maps are preregular forms defined on dual vector spaces. This construction generalizes a construction of Bichon-Dubois-Violette [10], where the target of these comodule maps is the base field. Using this new construction, she showed how one can recover various earlier constructions of quantum groups, including those introduced by Dubois-Violette-Launer [23], Takeuchi [51], Artin-Schelter-Tate [4], and Mrozinski [41]. Particularly their work gives an explicit presentation of a universal quantum group that coacts simultaneously on a pair of N-Koszul Artin-Schelter regular algebras with arbitrary quantum determinant.

Kirkman gave a wonderful talk on extending a classical result of Auslander, who proved that when a finite linear group contains no reflections and acts naturally on a polynomial ring with fixed subring B, then the skew group algebra generated by the polynomial ring and the group is isomorphic to the ring of B-linear endomorphisms of the polynomial ring. A common theme in the field of Hopf algebra actions is to extend results of this flavor to the setting of non(co)commutative Hopf algebra actions on Artin-Schelter regular algebras, where one must define what one means by containing no "reflections". Bao-He-Zhang [9] developed the notion of pertinency, and used this notion to prove the Auslander Theorem for certain group/Hopf actions on various Artin-Schelter regular algebras. Kirkman touched upon work in progress with Gaddis, Moore and Won that proves the Auslander Theorem for the permutation action of S_n on a class of noetherian PI Artin-Schelter regular algebras and her work with Chan, Walton and Zhang [17] that proves the Auslander theorem for Artin-Schelter regular algebras of dimension 2 with the action of a semisimple Hopf algebra H so that A is a graded H-module algebra under an action that is inner faithful and has trivial homological determinant.

Coinciding somewhat with Kirkman's talk, Faber spoke on joint work with Buchweitz and Ingalls [11] also regarding the Auslander theorem, albeit from a completely different point of view. Here we have G, a finite subgroup of $\mathrm{GL}(n, K)$ for a field K whose characteristic does not divide the order of G and which acts naturally on the polynomial ring in n variables. Unlike Kirkman's talk, which dealt with extending the result of Auslander to Hopf algebra actions in which there are no "reflections", this took an orthogonal point of view and looked at what happens when G has reflections. By studying the geometry of the discriminant of the group action and using the theory of noncommutative resolutions of singularities, the authors were able to extend the Auslander theorem on the algebraic version of the McKay correspondence to reflection groups.

For many, one of the highlights of the workshop was the work of Yakimov [37, 44] on noncommutative discriminants. Discriminants play a key role in the study of PI algebras, orders in central simple algebras, Azumaya loci of PI algebras, and the study of the isomorphism and automorphism problems for PI algebras. Previously, they were computed only for very few PI algebras and with great difficulty. Yakimov presented a general method for computing discriminants of PI algebras coming from Poisson geometry. From a different perspective the technique builds a bridge to the theory of discriminants of number fields, where factorizations into primes are replaced by factorizations into Poisson primes.

There were several interesting talks concerning the structure and classification of Artin-Schelter regular algebras. Mori's talk was based on his joint work with Smith and Ueyama on Artin-Schelter regular algebras of global dimension three [38, 39, 40]. It is well-known that every N-Koszul Artin-Schelter regular algebra is determined by a twisted superpotential. They proved that, in dimension three, the correspondence twisted superpotential is uniquely determined by the algebra. Using the twisted superpotential associated to an Artin-Schelter regular algebra A, they computed (1) the Nakayama automorphism of A, (2) the group of graded algebra automorphisms of A, and (3) the homological determinant of a graded algebra automorphism of A.

Another Artin-Schelter regular algebra talk was given by Reyes. As mentioned before, a connected graded algebra is Artin-Schelter regular if and only if it is twisted Calabi-Yau. In joint work with Rogalski, Reyes showed that when A is graded but not connected graded, the twisted Calabi-Yau property is also equivalent to a suitable analogue of the Artin-Schelter regularity in the sense of Martinez-Villa and Minamoto-Mori. Reyes also described the structure and properties of graded twisted Calabi-Yau algebras in dimension at most 2.

Sierra gave an overview of her interesting joint work with Lecoutre [36]. In this work, they generalize a construction of Pym and construct a family of Calabi-Yau Artin-Schelter algebras, $R(n)$, which gives a Poisson deformation of a polynomial ring in n variables for each n. She then went on to give an overview of this new Poisson bracket and showed that the Poisson spectra of the associated Poisson algebras are homeomorphic to the prime spectra of Zhang twists of her original algebras $R(n)$.

Hochschild cohomology is another topic that underwent extensive discussion. Negron gave a survey on his joint work with Schedler and Witherspoon on the Hochschild cohomology of global quotient orbifolds (and other work in [42, 43]). A global quotient orbifold is the stack quotient of a quasi-projective scheme by a finite group in characteristic 0. The vector space structure on the cohomology of such an object was given in a recent paper of Arinkin-Căldăraru-Hablicsek [1]. One of Negron's main results is to describe the algebraic structure of the Hochschild cohomology in terms of the geometries of the fixed spaces under the actions of individual group elements.

A complete list of talks can be found at the BIRS website.

Scientific Progress Made

As a result of this meeting, many people working in different areas of noncommutative algebra were able to engage in fruitful discussions and start projects with people working on related areas (see the section on testimonials of participants for examples). In fact, there are many examples of projects that were begun at this meeting. We note that this meeting had many younger researchers, postdocs, and graduate students and this meeting was particularly useful in allowing them to network and create joint projects with more senior researchers. One of the often repeated remarks made by different participants was an interest in the new work of Yakimov, which brought together many new and exciting ideas.

Outcome of the Meeting

This workshop brought together 41 researchers, which included a mix of Ph.D. students, postdocs, and senior researchers from Europe, China, and North America, working in different areas of noncommutative algebra and geometry, including ring theory, noncommutative algebraic geometry, representation theory, the study of Hopf algebra actions, and quantum groups.

Many of the participants noted that this was a great conference with many interesting talks. The program was full of lectures highlighting recent progress on difficult problems and we include here some testimonials from participants about research programs that they started at Banff and thoughts on results disseminated during the lectures.

Chelsea Walton said: "I enjoyed Richard Ng's talk on "Traces of Powers of Antipodes," joint with Cris Negron. He gave a nice introduction to invariants of representation categories of Hopf algebras ("gauge invariants") including various examples. Much as (generalized) Frobenius-Schur indicators are used to tell apart representation categories, knowing that the sequence of traces of powers of antipodes (under the Chevalley property condition) is a gauge invariant is a super useful result. It was only a 30 minute talk, but I could have listened for another 60 minutes—I like this material and the way in which Ng gives talks. He's super clear and I always learn a lot from him. I'm currently thinking about the dual problem, invariants of co-representation categories of Hopf algebras, in a joint project with Adriana Mejia, Susan Montgomery, Sonia Natale, and Maria Vega (which also began at BIRS this past March). So, Negron and Ng's work is definitely on my radar—it's a great result."

Ragnar-Olaf Buchweitz said: "Concerning mathematical content, I list the four talks I liked best (refraining from any comments on Eleonore, as that is joint work). For me, the best talk was the one by Yakimov. This should be a game changer, as they say. Beautiful mathematics delivered impeccably. I greatly appreciated the talks by Sarah Witherspoon, Ellen Kirkman, and Susan Sierra. Nice surveys with interesting results in the first two cases, beautiful results nicely presented in Sue's case.

I commend you for giving so many young researchers an opportunity to present their work. I had heard of some of them, but hadn't met several of them before (e.g. Manuel Reyes, Matt Satriano who also gave nice talks.) Colin, Eleonore and I appreciated the opportunity to continue working together, we used every lunch break to do so."

Alexandru Chirvasitu said: "This was my first time visiting BIRS, and it has been every bit as impressive as conversations with my colleagues had led me to believe. I'd like to echo one of them in saying that I feel very lucky to have been able to be there.

The conference was very conducive to collaboration (there seem to be a couple brewing as far as I am concerned) and very intense, and the Station and its program seem tailor made to foster this type of atmosphere (no distractions, beautiful place, availability of any amenity one might think of, research facilities / tools of all types, etc. etc.).

All in all it was a terrific week, and I hope to be able to attend one of these meetings again sometime."

Paul Smith said: "It was good to see some fresh faces, and hear some fresh voices, like Negron, Woods, Belmans, Chirvasitu, Satriano, Faber, Reyes.

It was good to have some experts in algebraic geometry there. I learned a bit more about deformation theory and low-dimensional Hochschild cohomology (for rings and varieties and stacks) from Cris Negron, Max Lieblich, Ragnar Buchweitz, and Pieter Belmans. I now feel more confident about the precision and worth of some "deformation-theoretic" questions I have been thinking about recently and, related to that, how to formalize questions about the behavior in families of algebras and abelian categories that are important to noncommutative algebraic geometry. Belmans, Chirvasitu, and I, and Mori, had some interesting conversations about Poisson structures and deformation quantization (we also had some interesting email exchanges with Brent Pym about these things)."

Louis Rowen wrote: "I think it was an excellent meeting. It had an excellent balance between geometry and algebra, showing how algebra still impacts meaningfully in related areas. I was very interested to see the diverse work in AS regular algebras, which has motivated me to take a closer look at them.

Also there was an impressive mix of generations, which gave me the opportunity to meet some sharp younger mathematicians."

Lutz Hille wrote: "One highlight have been the talks of Ellen Kirkman and Eleonore Faber, which I had preferred to see more closely. Both deal with fundamental open questions: the action of a group, respectively a Hopf algebra, on a commutative (or even noncommutative) ring. Whereas group actions are studied since a long time, the action of a Hopf algebra is more recent research. Both are, in my opinion, fundamental questions in noncommutative algebraic geometry. The case Faber considered is certainly one of the main open questions, the case Kirkman has considered seem to be new and hopefully the beginning of a new subject with many applications.

The talk of Ken Goodearl was the one most closest to my own research, I work myself on problems about irreducible components. It was nice to see somebody else working on this and the case (he considered mainly) of all relations of a fixed length is certainly a nice first step. Even if I know, the general problem is hopeless, since any algebraic variety essentially occurs as a representation space, even for a very small dimension vector, there might be good classes where one can solve the problem. Thus, one has to concentrate on particular relations.

The most exciting talk was certainly Cris Negron, I was deeply impressed about his power and the deepness of his results, I certainly keep in contact with him and I got ideas how this is related to my own work."

Martin Lorenz said the following: "This meeting gave me the opportunity to catch up on current developments, especially in the area of Hopf algebras and their actions. I found the talks by Walton, Witherspoon, Ng, Kirkman and Chirvasitu to be very useful in that regard, but I enjoyed the lectures by Yakimov, Goodearl and Launois very much as well."

Milen Yakimov said: "I have learned a great deal of math both in terms of new directions of research and methods that I can incorporate in my own research. I was particularly excited that the workshop represented a very wide range of topics and not a single concrete direction: from Hopf symmetries of regular algebras, structure of Jacobian algebras for superpotentials and twisted CY algebras, noncommutative blow ups, to structure of division algebras, the McKay correspondence to reflection groups, and birational automorphisms of projective varieties. The breadth of the workshop represents the dynamics of the area and the numerous connections with other fields that emerged in the last years. During the workshop I had the opportunity to initiate 2 research projects with Tim Hodges. The first is on establishing a Levi type decomposition theorem for all Belavin-Drinfeld quantum groups. The second is on describing a large class of noncommutative projective spaces via quantum Hamiltonian reduction."

Following this workshop, there will be a closely related workshop at ICMS, Edinburgh in the summer of 2017, entitled “Linking noncommutative rings and algebraic geometry”. This BIRS conference will inevitably play a major role in shaping the research done from now until this conference.

Participants

Ardakov, Konstantin (University of Oxford)
Bell, Jason (University of Waterloo)
Belmans, Pieter (University of Antwerp)
Brown, Ken (University of Glasgow)
Buchweitz, Ragnar-Olaf (University of Toronto Scarborough)
Chirvasitu, Alexandru (University of Washington)
De Laet, Kevin (University of Antwerp)
Faber, Eleonore (University of Michigan)
Goodearl, Kenneth (University of California - Santa Barbara)
Hille, Lutz (University of Mnster)
Hodges, Timothy (NSF)
Ingalls, Colin (Carleton University)
Kirkman, Ellen (Wake Forest University)
Krashen, Daniel (Rutgers University)
Kulkarni, Rajesh (Michigan State University)
Launois, Stphane (University of Kent)
Lenagan, Tom (University of Edinburgh)
Lieblich, Max (University of Washington)
Lorenz, Martin (Temple University)
Lu, Diming (Zhejiang University)
McKinnie, Kelly (University of Montana)
Mori, Izuru (Shizuoka University)
Nasr, Amir (University of New Brunswick)
Negron, Cris (Massachusetts Institute of Technology)
Ng, Siu-Hung (Louisiana State University)
Reyes, Manuel (Bowdoin College)
Rogalski, Daniel (University of California at San Diego)
Rowen, Louis (Bar Ilan University)
Saltman, David J (Center for Communications Research - Princeton)
Satriano, Matt (University of Waterloo)
Sierra, Susan (University of Edinburgh)
Small, Lance (University of California - San Diego)
Smith, Paul (University of Washington)
Stafford, J. Toby (University of Manchester)
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Chapter 16

Integrodifference Equations in Ecology: 30 years and counting (16w5121)

September 18 - 23, 2016

Organizer(s): Christina Cobbold (Glasgow), Mark Kot (Washington), Mark Lewis (Alberta), Frithjof Lutscher (Ottawa)

Overview of the Field

Many ecosystems experience drastic changes: species go extinct, non-native species invade, climatic conditions change, human activities disrupt habitat and dispersal pathways. Some ecosystems are closely managed for the benefit of humans: forestry provides essential material, and fisheries are indispensable for food security. Ecologists focus on understanding causes and mechanisms for current change to understand future impact of development and climate change. Ecosystem managers need solid scientific bases to reach decisions about land use and conservation. The spatial and temporal scales involved in these processes are typically too large to conduct detailed empirical work. Mathematical modeling of individual and population processes is essential to understand ecosystem function and predict ecosystem response to change. Ecosystem complexity, in turn, poses considerable challenges to mathematical theory and analysis of the resulting models. Novel quantitative tools are needed to meet these challenges.

Integrodifference equations (IDEs) are a class of spatially explicit, dynamical systems models that closely reflect the strongly seasonally synchronized life stages of many ecological species. In the simplest case, a generation consists of a growth phase and a temporally distinct dispersal phase. Particular examples include some invasive insects causing great damage in North America and elsewhere (e.g., Emerald Ash borer), and native species on the verge of extinction worldwide (e.g., Fender's blue butterfly).

IDEs project the density of a population forward from one generation to the next by considering the stationary growth phase and the dispersal phase in sequence. In the simplest case, denoting $N_t(x)$ as the density of a population in (discrete) generation t at (continuous) spatial location x, we write the IDE to obtain N_{t+1} as

$$N_{t+1}(x) = \int K(x-y)F(N_t(y))dy. \tag{16.0.1}$$

Function F represents the growth phase of the life cycle, and dispersal kernel K is the probability density function of offspring location. Thus, IDEs constitute discrete, non-local, infinite-dimensional dynamical systems. They are a discrete-time analogue of reaction-diffusion equations (RDEs), and some of the mathematical theory for IDEs parallels that for RDEs. The non-local nature of the equations and the discrete-time setting makes some of this work considerably more challenging than for RDEs.

In 1986, Kot and Schaffer introduced IDEs as models for spatial ecological processes and laid the foundation for their analytical investigation and application [12]. Building also on previous analysis of IDEs in genetics [24], several pioneering papers then explored the qualitative behavior of IDEs, in particular stability properties, stochastic and chaotic dynamics, spreading speeds, traveling wave theory and pattern formation [1, 2, 7, 9, 19]. In 1996, the discovery that IDEs can support accelerating waves of species invasions (a property that classical RDEs do not have) caught the attention of theoretical and empirical ecologists and immediately publicized IDEs in the ecological community [10].

Since then, a growing number of empiricists and theoreticians have worked to understand the qualitative behavior of these equations, and to connect the theory with observations [15]. IDEs were extended to include structured and interacting populations, stochasticity, spatial heterogeneity, genetic aspects and evolutionary processes. The purpose of the BIRS workshop was to review of the tremendous work being done worldwide and to bring together researchers in all related fields (analysts, modellers, theoretical and empirical ecologists) for a preview of developments and challenges to come. Specifically, the mathematical theory of IDEs has much to learn from and to offer to other areas of dynamical systems with non-local operators, such as integrodifferential equations, the emerging fields of integral projection models and impulsive reaction-diffusion equations, as well as spatial stochastic processes. Close interaction between analysts, modellers, and ecologist at the workshop ensured that model development remained relevant to empiricists and that empirical research focuses on required model input.

Recent Developments and Open Problems

Multispecies models: Most theory on IDEs considers single-species models, but realistic ecosystem descriptions need to include several species and their interactions. Such models are relatively simple to formulate but their analysis poses great challenges. Simple two-species competition models have monotonicity properties, so that comparison theorems allow analytical results. Yet, even those models show some surprisingly anomalous spreading speeds [14]. Three-species competition or predator-prey relationships generally do not allow comparison theorems, and little is known about spreading speeds and travelling waves in these models. Novel phenomena arise, such as the formation of spatial stable patterns [19] or cyclic and chaotic behaviour in the wake of an invasion [9]. Some of these emergent patterns are well understood in reaction-diffusion equations but they are only starting to be developed IDEs and many phenomena remain unexplored.

Movement behaviour: When multiple life stages are included in the model, there are often sessile stages, during which individuals do not move. The resulting next generation operator in the IDE model fails to be compact, and classical existence theorems for travelling waves fail. There is much recent interest in travelling wave theory for non-compact and non-monotone operators [3]. A completely new challenge from a modelling and analysis perspective is to include movement behaviour that depends on the presence of other species. In reaction-diffusion systems, such questions lead to cross-diffusion models that are notoriously difficult to analyze. For IDEs only a single numerical study exists that poses more questions than it answers. A potential new way to study questions of cross-diffusion in IDEs is to connect them to impulsive reaction-diffusion equations by modelling seasonal movement explicitly.

Spatial Heterogeneity: Travelling wave solutions often serve as descriptions of species invasion and range expansion processes. To study spreading speeds one typically assumes that the habitat is spatially homogeneous. In reality, most landscapes are heterogeneous on many scales. Organisms often have clear habitat preferences and adjust their movement according to resource availability, landscape features or conspecific density. Prevention programs against invasive species might create additional heterogeneity through targeted removal of resources (e.g., host plants of forest invasive insects) or localized application of pesticides. How then does one model dispersal in such environments? And what are the effects of landscape variation and spatially localized intervention on the spread of an organism?

Few papers have dealt with IDEs in heterogeneous landscapes. They all assume temporally static, spatially periodic heterogeneity, employ relatively simple dispersal kernels and assume spatially continuous solutions of the IDE [4, 8, 22, 25]. Recently, novel dispersal kernels were derived from random walk models in patchy landscapes. These kernels are discontinuous, as are the resulting solutions of the corresponding IDE

[17, 18]. A novel analytical framework is needed to study discontinuous solutions, emergent travelling waves and related spreading speeds.

Temporal variation: Temporally varying landscapes are models for global change of climatic conditions. Initial research for IDEs with moving habitat patches unveils how dispersal may facilitate or hinder a species' ability to keep up with climate change [26]. Several talks at the workshop elaborated on this topic and presented new results that include stochasticity and stage structure (see Section 16 below).

Stochasticity and data: Although deterministic models were successful in predicting the speed of invading populations, they do not capture the patchy spread and variation in invasion speeds observed in real systems. Stochastic models are crucial for quantifying the variability in spread rates, yet despite their importance, there is relatively little work on stochastic IDEs. Extrinsic stochasticity (caused by environmental factors) and intrinsic variability (based on demographic processes) affect spread rates in different ways: the latter typically reduces invasion speed while the former may increase spread [6, 11, 16].

Great efforts are underway to collect more data on individual movement and population abundance, for example with respect to climate change scenarios, in conservation settings, and in model microcosms. To link these data with models in a meaningful way, models need to incorporate the stochasticity inherent in the data. Therefore, we need to extend the analysis and tools above and developed new techniques to better understand the behaviour of stochastic IDEs and to quantify the variation in spreading speed and other ecologically significant quantities.

Non-local operators: Several other mathematical modelling frameworks are closely related to IDEs, yet researchers from these fields only recently started to interact with one another.

Reaction-diffusion equations with non-local terms appear in various places in the literature. The non-local operator may describe non-local interaction or movement, depending on the ecological question. No systematic theory of these equations is currently available, however, there is much recent interest in studying the qualitative dynamics of these equations, for example, the study of accelerating waves through tracking of level sets or the generalization of the theory of λ-ω-systems from reaction-diffusion systems.

Integral projection models (IPMs) project the density of a population forward in discrete generations, while individuals are continuously structured by state (e.g., size) [5]. IPMs are formulated very similarly to IDEs but details (e.g., typical shapes of the kernels) and research questions are quite different. In a spatial setting, when individuals are structured by continuous state and location, IPMs and IDEs are merged, and the resulting model has both aspects, e.g., kernels that represent progression through states and kernels that represent movement in space.

Impulsive reaction-diffusion equations were only recently studied in an ecological context [13]. In their simplest linear form, these equations can be equivalent to linear IDEs, but their nonlinear extensions typically are not. The study of impulsive reaction-diffusion equations in ecology is only in its infancy, but it is clear that many questions and challenges are closely related to those for IDEs.

Presentation Highlights

Mathematical theory of population dynamics of invasion in a static environment: Analytical advances

Since the pioneering results of Weinberger (1982) much of the analytical advancements in IDEs focused on establishing results pertaining to travelling wave solutions of IDEs. This is still an active area of research and new analytical approaches to tackle the emergent challenges were presented in the meeting. Maximum principles are at the heart of proofs for many wave speed results and as such a one of the biggest challenges in this area is establishing travelling wave results for non-monotone systems where a maximum principle can not be applied. New analytical approaches are required and we saw the emergence of some of these in the meeting.

Bingtuan Li (Multiple invasion speeds in IDE competition models) proved the existence of multiple propagation speeds in his study of competition models. Consequently, the order of invading species in competitive systems can change with the lead invader being replaced by its competitor later in the invasion process.

Xiaoqiang Zhao (Bistable travelling waves for monotone discrete time recursive systems) established the general theory for their existence waves in bistable systems and then took a dynamical systems approach to establish global stability results of these waves. He illustrated these results with a two species competition model and pointed out that an open problem in these systems is to determine the sign of the wave speed. Crucially, the sign of the wave speed determines which species is the winner of the competitive interaction. It is therefore one of the most important quantities for our understanding of species invasions.

Lenhart (Optimal control of integrodifference equations) developed the basic theory for applying spatio-temporal control to integrodifference equations. The underlying ideas were based on the Pontryagin Maximum Principle (distinct from the classical maximum principle for IDEs), best known in the context of controlling systems of ordinary differential equations. Although such optimal control theory was previously extended to spatio-temporal models such as partial differential equations, the development and application of optimal control theory to integrodifference equations is new. Applications were made to optimal harvesting and to the management of gypsy moth invasive insects.

Weinberger (Spread in a two-allele genetic system) presented developments in the new mathematical theory for spatial spread of a genotype with a two-allele genetic system. Classical theory, going all the way back to R.A. Fisher, has focused on one-allele systems. Weinberger showed very recent work on extending the theory and developing it further for the two-allele case. Although Weinberger was unable to travel to the meeting due to health reasons, it was a great honour to have him participate via Skype as one of the founders of the field of mathematical analysis of integrodifference equations.

Mathematical theory of population dynamics of invasion in a static environment: New phenomenon and methods

The modeling of biological invasions was one of the earliest uses of integrodifference equations. Indeed, this was the application that first caught the attention of ecologists and that put integrodifference equations on the modeling map.

At this point, the basic theory of biological invasions in a static environment is well established: We know how to predict invasion speeds for a variety of dispersal kernels, in both one and two dimensions, for simple unstructured populations with compensatory growth and for age- and stage-structured populations. In addition, although open questions remain, we have made substantial progress in understanding invasions with depensatory growth (Allee effects) and/or with population interactions, such as competition. In addition, integrodifference equations have now been applied to a large number invasive species ranging from weeds to trees and from birds to butterflies.

Given this background, one might think there would be little new mathematical theory about invasions in a static environment presented at this meeting. Nothing could be further from the truth. Six speakers presented stimulating talks that either offered new tools for old problems or that offered new insights regarding previously unstudied or understudied phenomena. In this section, we briefly summarize each of these talks, in the order that they were presented.

Mark Kot (Models for the spread of white pine blister rust) started the meeting by describing models for the spread of white pine blister rust. This is a fungal pathogen that is threatening a large number of valuable conifers. The pathogen has a complicated life cycle: It has two obligate hosts and numerous (spore) dispersal stages. All of the dispersal stages have different length scales. Kot showed how moment generating functions, used as integral transforms, and the method of steepest descent could be used to streamline the analysis of the complicated set of governing plant-host and pathogen equations. The resulting methods for predicting invasion speeds can be easily extended to other important systems such as host-parsitoid systems.

Many simple integrodifference equations generate invasions with constant asymptotic invasion speeds. And yet, many field studies and laboratory experiments show tremendous variability in invasion speeds. It is natural to assume that this variability is due to either demographic or environmental stochasticity. Michael Neubert (Invasion variability in "simple" integrodifference equation models) began his talk by asking whether this is necessarily the case. In particular, he asked whether it is possible to construct simple, deterministic IDEs with variable invasion speeds. He answered this question by considering three ways in which persistent variability in invasion speed can arise in scalar, deterministic, spatially homogeneous, and temporally constant models. In his most notable example, Neubert showed how overcompensation in back of an invasion can interact with the pushed waves associated with a strong Allee effect to generate variability in wave speed.

Tom Miller (Ecological dynamics of colliding populations at habitat ecotones) started his talk by telling us about ecotones. Ecotones are transition zones between adjacent ecological communities. These zones occur, for example, between forests and tundra and between shrublands and grasslands. In simple invasions, exotic species invade open space, but in ecotones, we often see the collision of invasion waves for foundational species from different communities. Miller described a system of integrodifference equations with intraspecific competition, interspecific competition, and dispersal, and he showed how this system provides insight into the dynamics of ecotones. He then connected his model back to empirical data from an ecotone involving creosote bush and black grama grass from the Chihuahuan desert of New Mexico.

Nathan Marculis (Neutral genetic patterns for expanding populations) asked "How do growth and dispersal affect genetic diversity in expanding populations?" He then showed that one can study the inside dynamics of an integrodifference equation and track the fate of neutral genetics fractions as the population expands. For pulled waves arising from compensatory growth and mesokurtic kernels, the neutral fraction at the front of the wave dominates the solution for all time. This leads to a strong founder effect. In contrast, pushed waves arising from strong Allee effects lead to slower spread but higher genetic diversity. Finally, fat-tailed dispersal kernels give rise to complicated patterns of genetic diversity that are still poorly understood.

In working with age- and stage-structured IDEs, we often calculate annual spreading speeds for invasive organisms. Mark Lewis (Generational spreading speeds for integrodifference equations) argued that we may, in fact, find it easier and more useful to calculate generational spreading speeds. In particular, Lewis showed how the next-generation operator of demography can be generalized to spatiodemographic models by separating the spatial fecundity and survival operators. This procedure can dramatically reduce the degree of the dispersion relation that one must then solve for the spreading speed. Lewis illustrated these ideas using spatiodemographic data for the weed teasel.

Finally, Frithjof Lutscher (Spreading phenomena in integrodifference equations with overcompensatory growth function) reported on his detailed studies and analyses of the complicated invasion dynamics that occur with overcompensatory growth. He showed that one can observe several traveling wave profiles with differing speeds. For example, one can observe a metastable solution for invasion into open habitat that is followed by stable traveling two-cycle. Lutscher focused his attention on the second-iterate operator for the integrodifference equation. He showed that this operator has a pair of stacked fronts. He generalized the concept of spreading speed to this operator, and he related his observations to the phenomenon of dynamic stabilization.

Although the basic theory of invasions in a static environment is indeed well understood, the above talks illustrate that there is still much work to be done. The presence of differing growth dynamics (compensation, overcompensation, and depensation), age and stage structure, genetic diversity, population interactions, and short and long-distance dispersal all leave us with many challenges.

Mathematical theory of population dynamics of invasion in a dynamic environment

A growing body of literature, empirical and theoretical, studies spreading phenomena as necessary adaptations for populations to keep track of their preferred climate zones under global change scenarios. Current estimates predict a global mean shift in the location of a population's suitable habitat by 0.42 km/yr. The mathematical modelling of invasions in dynamic habitats is a rapidly emerging area of IDE research, with significant results being reported at the meeting.

Ying (Joy) Zhou (Integrodifference equation models for populations in dynamic habitats) pioneered some of the initial work in this area and presented her most recent advances covering two scenarios: 1) habitat location shifting with climate change and 2) a seasonally expanding and contracting habitat as associated with, for example, dry and wet seasons. In both cases the problems can be formulated as an IDE of the following type

$$n_{t+1} = \int_{-\infty}^{\infty} \underbrace{k(x,y)}_{\text{dispersal kernel}} \quad \underbrace{Q_t(y)}_{\text{habitat suitability}} \quad \underbrace{f(n_t(y))}_{\text{population dynamics}} \quad dy \tag{16.0.2}$$

This equation can be transformed into a time-autonomous system in both cases, allowing for population persistence conditions to be determined. Moreover, the presenter demonstrated that these persistence results, which apply to compact operators on a bounded domain (fixed habitat range) can now be extended to unbounded domains allowing for more realistic habitat suitability models to be considered. Importantly, these results also extend to 'fat' tailed dispersal kernels provided $Q_t(y)$ decays fast enough. Analytical results for IDEs in which dispersal kernels are not exponentially bounded are still quite sparse, so progress in this area is an important advancement.

Juliette Bouhours (Climate change and integrodifference equations in variable environments) extended the framework given by equation (16.0.2) to the stochastic case in which habitat suitability depends on a random variable, as does population growth rate. The random variable describes the uncertainty associated to the speed of climate change. The theorem she presented stated the dichotomy that a population in such a stochastically varying environment would go extinct with probability either 0 or 1. The condition determining extinction depended on the geometric average of the local population growth rate and dispersal kernel.

In contrast to Zhou and Bouhours, who focus on long-time asymptotic behaviour of equation (16.0.2) Austin Phillips (Will transient dynamics help or hurt species during climate change?) instead focussed on the short-term transient dynamics. In doing so, Austin presented four metrics for quantifying transient dynamics. Importantly, he demonstrated that long transients can be associated with critical slowing down in invasion speeds, which has important implications for species ability to keep pace with a moving environment.

Complementing the approaches of Zhou, Bouhours and Phillips, James Bullock (Population spread and the velocity of climate change) constructed virtual species from a statistical model of life-history and dispersal data. This new approach tackles the challenge of sparse population data. The synthesised life-history and dispersal parameters for the virtual species enabled the presenter to estimate invasion speeds from IDE models enabling him to broadly identify classes organism that would struggle to keep pace with climate change.

Taken together, these results suggest IDE research in there area is able to offer many important ecological insights regarding the vulnerability of species to climate change. Moreover, the talks demonstrated that there are number of key mathematical challenges that emerge from looking at this particular ecological problem.

Integrodifference equations: heterogeneity, data and numerics

The focus of sections 16-16 has been on IDEs which are solved on spatially homogeneous domains. Real spatial domains are typically a long way from this idealised homogeneous view. Moving away from this abstraction one has to decide how much detail to then include. Alan Hastings (Ecosystem engineering and IDEs) and Elizabeth Crone (Combining models and data to set guidelines for butterfly conservation) both advocated for the value of including the 'key features' of the landscape and biology and discussed periodic landscapes of alternating habitat types to address questions of species persistence in such landscapes. Both gave examples where the IDE theory was used to successfully inform conservation policy. In the case of Hastings this was in the design of marine reserves and in the case of Crone this was to quantify the minimum size of protected habitat areas for an endangered butterfly species. These talks illustrate that there are many questions beyond those related to spread that can be usefully addressed using IDEs.

With the increasing availability of fine resolution satellite data detailing the location of habitat types and GPS data of movement at large spatial scales we would also like to be able to understand population dynamics at a landscape level and move away from the idealised heterogeneous landscape. Two of the speakers specifically addressed the challenges associated to studying IDEs on such large spatial scales (of the order of 100s of kilometres). James Powell (Invasion speed in highly variable landscapes: multiple scales, homogenization and the migration of trees) presented a seed dispersal problem whereby seeds were dispersed by animals that made movement choices on the scale of the local habitat changes (10s metres), but moved distances of 10s kilometres. At the landscape scale the heterogeneous habitat resulted in an IDE describing seed and tree distribution which contained an anisotropic dispersal kernel. Analytical results for IDEs with anisotropic kernels are limited and many proofs rely heavily on the kernel symmetry. Powell addressed this challenge by using the technique of homogenisation to "average" out the effects of the local scale and produce an IDE description of the landscape scale dynamics. The resulting homogenised IDE contained an isotropic dispersal kernel allowing classical travelling wave solution results to be employed in the study of the tree invasion speeds.

The homogenisation approach offers a powerful method of including crucial biological detail while maintaining analytical tractability. However, one would still like to numerically simulate the dynamics of movement over landscapes and this challenge was picked up by Steven White in his talk on Predicting species spread in heterogeneous landscapes with IDEs. The usual approach to numerically solve an IDE on a 2D heterogeneous landscape is to define a fixed spatial grid and to use either fast Fourier transforms or quadrature to carry out the integration in equation (16.0.1). The problem with this approach is that it places heavy demands on RAM and CPU time. and to overcome this the presenter illustrated a new adaptive algorithm. The adaptive algorithm used a coarse spatial grid at the far front or far rear of an travelling wave solution where the solution will shows little spatial variation and so can be be accurately solved with only a few grid points, but at the wave front where the solution changes rapidly a fine spatial grid is used. As time is updated the location of the fine spatial grid is also updated. Similar ideas are used to solve PDEs, but this is the first time an adaptive approach has been developed for IDEs. The technique leads to an order of magnitude improvement in computational efficiency.

White also presented the results of a recent IDE literature search which demonstrated an exponential increase in the number of IDE publications over the last 30 years. Still a relative small number of these papers specially addressed applications. White posited that the poor uptake of IDEs by ecologists may be due to the difficulties in numerically solving IDEs, and unlike working with ODEs or PDES the availability of off the shelf code is very limited. Harsch (Increased applicability and engagement through interactive web application) presented an 'Rstudio' and 'Shiny' interactive web application to allow and encourage users to solve IDESs and gain a greater appreciation of their potential use. The idea of developing web resources for people interested in working IDEs was met with a lot of interest at the meeting and became the topic of one the breakout discussion sessions (see section 16).

Relationships to other model types

A number of different modelling approaches are closely related to IDEs, and their analysis poses similar problems. One of the goals of this workshop was to bring the various communities closer together and make them aware of ideas and results available in related fields.

Nonlocal reaction-diffusion equations are similar to IDEs in that (long-distance) dispersal is modelled by an integral operator instead of (or in addition to) the diffusion operator. Jérôme Coville (Propagation phenomena in nonlocal reaction-diffusion equations: An overview of the recent developments) gave a much appreciated overview presentation of the state of the art of analyzing spreading phenomena with these equations. His presentation began with an exercise in model selection that demonstrated that these nonlocal equations may provide a better fit to observed data than diffusion equations. Then he presented past and recent theory on spreading speeds in homogeneous and periodic environments.

Integral projection models formally look like the IDE in (16.0.1) but instead of spatial location (x) consider the 'location' of an individual in state space, i.e., x can denote size of an individual. In contrast to IDEs, the state space is typically compact and the 'kernels' that arise have qualitatively different properties. Stephen Ellner (IDEs as models for individuals: who gets into the 1%, and why?) was instrumental in developing the theory and applications of integral projection models and recently published the first book on the subject. In his talk, he used several long-term data sets and demonstrated how these models can yield much more information that is currently obtained from them. His results show that 'success' in many plant and bird populations is more a matter of luck than of individual traits.

Bill Fagan (Perceptual ranges, information gathering, and foraging success in dynamic landscapes) presented yet another application of integral equations in ecology. He modelled movement of individuals as a mix of random diffusion and directed movement, where the direction was determined by the amount of resources that an individual perceived in a certain detection radius. He compared the effectiveness of different strategies in different landscapes of resource distribution.

Scientific Progress Made

Scientific progress was made via break-out sessions. Here conferees discussed new results, scientific challenges and future directions in an informal environment. We kept a record of these discussions with the view that they can help inform future research. Summaries are given below.

Dispersal kernels and inverse problems

Dispersal kernels are central to integrodifference equations. The use of kernels to describe dispersal is the key feature that makes IDEs attractive to ecologists. The varied shapes of dispersal kernels lead, in turn, to many interesting phenomena, such as accelerating invasions.

The use of dispersal kernels does, however, entail serious empirical and theoretical challenges. For many systems, for example, invasion speeds are determined by the tail behavior of the dispersal kernel, and the tail of the distribution is precisely where data is often missing. We held a break-out session to discuss the problems, progress, and prospects associated with dispersal kernels. The discussion was wide-ranging. We briefly summarize some of the highlights of the discussion below.

Everyone quickly agreed that the old approach of fitting a pre-defined curve (e.g., a Gaussian), or even a pre-defined family of curves to dispersal data is insufficient, because of the importance of tail data. This led to a discussion of the usefulness of empirical methods, such as empirical moment generating functions and empirical saddle-point methods, that avoid pre-selecting the dispersal kernel. Several participants then pointed out, however, these empirical methods do not really solve the problem of tail data.

This led to a free-wheeling discussion regarding available methods for extrapolating kernels to regions where we don't have much data. We don't, after all, want to simply make up the tail. One suggestion that was emphasized was the usefulness of purposely censoring one's dispersal data in order to plot the estimated invasion speed as a function of the sampling radius. Investigators can then make sure that their estimates are leveling off with sampling radius. If they are, there is then some hope that the estimates can be trusted. Several people asked whether bootstrapping, Bayesian methods, and information theoretic methods could help. Others asked if there was some way to weight available tail data more heavily.

This was followed by quick discussion regarding the need for tail data. Everyone agreed that the question matters. Tails appear to be important for invasion data, but even here, time scales matter. Are we talking about transient dynamics or asymptotic behavior? In contrast, tails do not appear to matter for critical patch size problems.

At this point, the discussion shifted to questions regarding the realism of dispersal kernels. Most of the dispersal kernels that we use are simple and idealized. Can we construct more realistic dispersal kernels without adding too much complexity? There was great enthusiasm for further work on mechanistic models, and several people spoke well of the Wald distribution for wind dispersal. Jim Powell then gave us a preview of his upcoming talk on homogenization methods for highly variable landscapes.

Several people asked about kernels for resource-based dispersal and there was a fair amount of discussion regarding both spatial heterogeneity and two-dimensional kernels and anisotropy. Animal ecologists expressed some pessimism about the usefulness of fitted dispersal kernels, while plant ecologists expressed a great deal of optimism. This striking contrast helped highlight the important distinction between goal-directed movement and passive dispersal.

The discussion then turned to the inverse problem. What can we say about the shape of dispersal kernels given the shape, for example, of the invasion wave? The slope of the invasion wave does, for example, depend on the invasion speed, which is controlled, in turn, by the growth rate and the dispersal kernel. A few people were hopeful regarding the inverse problem but, in ongoing discussions after the session, several people were more negative. Indeed, one person suggested we can no more determine the shape of dispersal kernel from the invasions wave then we can determine a Leslie matrix from a stable age distribution.

Finally, we ended with a short discussion on other possible methods for incorporating dispersal data into integrodifference equations. This discussion included such topics as extreme value statistics, branching random walks, and gravity models.

Spatial heterogeneity

Spatial heterogeneity is ubiquitous in nature and often enhanced by human activities. How exactly the many forms of heterogeneity influence the processes of population growth and individual dispersal is still a source of many open questions. These ecological questions have spurred several analytical and numerical developments and much has been learned from reaction-diffusion models and from IDEs (see Section 2). The participants of the break-out session on spatial heterogeneity discussed where these developments could and should lead in the future.

Heterogeneity in local dynamics

The simplest possible scenario to study the effects of spatial heterogeneity is that of a static heterogeneous environment that affects only the local growth dynamics but not the dispersal of individuals. There is recent work on periodic landscapes (persistence of the population, existence of spreading speed), with special emphasis on piecewise constant periodic landscapes of two types of habitat (binary) where many explicit calculations are possible. The analytical tools of homogenization seem particularly helpful to gain ecological insights.

A first forward step towards a more generally applicable theory would be to extend the calculations to non-binary landscapes, for example to three habitat types such as "preferred habitat", "secondary habitat" and "unsuitable habitat". Based on previous experience, this step seems doable in a relatively short time period.

A second question is to extend the framework from periodic to random landscapes. This extension would be very important ecologically to explore questions of how landscape autocorrelation affects population distribution and spread. Non-periodic landscapes pose a huge analytical challenge since, for example, a spreading speed cannot exist. There is recent progress on this question for reaction-diffusion equations by using level-set methods. Developing corresponding theory for IDEs would be a great step forwards. For more applied questions, averaging over a random landscape could provide a reasonably simple theory and give relatively good answers to ecologically relevant questions. Some work on sparse landscapes has already been done.

A third question that emerged in this context asks how structured populations or multi-species assemblies respond to environmental heterogeneity. The analytical theory for structured populations often follows the theory for a single species closely (since the equations are order-preserving). For two competing species, the order-preserving structure remains so that some theory should be reasonably easy to obtain (one more applied study exists already). But theory for consumer-resource interactions is an even greater challenge than in homogeneous habitats, where there are still many open questions.

Heterogeneity in dispersal

The much more challenging aspect of heterogeneity in IDEs is to adequately model how environmental variation affects individual dispersal as represented by the dispersal kernel. One approach that was explored recently is closely related to impulsive reaction-diffusion equations (see above). The dispersal phase is modelled by a linear diffusion equation with spatially varying coefficients. From this diffusion equation, a dispersal kernel is derived. The kernels that arise this way are typically not convolution kernels. While some analysis has been carried out, many analytical questions are still open. An alternative approach could see the individual probing the environment in a number of places before deciding where to settle. This approach was implemented once, but has since not been used or studied in detail.

Once it is clear how to describe the outcome of dispersal in a heterogeneous landscape, the task is to study the effects of heterogeneity on population patterns. Can dynamic instabilities be caused by habitat heterogeneity? Can travelling waves or pulses or more complicated patterns be induced? How does heterogeneity affect spreading speeds? Homogenization approaches can be very helpful in the study of these questions.

Understanding the causes of patterns

Once we have a good understanding of which patterns can be created or influenced by heterogeneity and how, we can try and tackle an old yet every new and crucial question in ecology: How to distinguish between endogenous and environmentally driven variability? It is a mystery. So what should we actually do?

Typically, we cannot measure processes directly. Focusing on a few cases where the distinction between endogenous and environmental drivers matters and figuring out – via modelling – what we would need to know to distinguish, could be a first step. Finding a unique, measurable signature of a generating process would be the holy grail. In the meantime, one could look at the work on signature patterns in time series and see whether some of that can be translated into spatial signatures. But even if we cannot distinguish causes behind patters, there may be value in pointing out that endogenous patterns are possible. More often than not, it is assumed that environmental variability is the cause of static patchiness.

The evolution of dispersal

Finally, a question that has garnered great interest in reaction-diffusion equations is that of the evolution of dispersal. Work on the corresponding question for IDEs has only barely been begun. There are some insights on the evolution of dispersal kernel shape using function-valued traits and the evolution of dispersal for persistence on bounded habitats. A yet unanswered question is whether there is an equivalent of the ideal free distribution for IDEs. Any progress towards this question would be a great achievement. The first step may be to study corresponding impulsive reaction-diffusion equations as an intermediate step.

Temporal variation and stochasticity

External temporal variation and stochasticity are crucial elements to any reasonable ecological model. This is because ecological environments are notoriously variable in time. Therefore it is natural to ask how classical results regarding spreading speeds and critical domain size might change in temporally variable environments. Typical forms for the variation are either temporally periodic, or random.

However, these elements make the analysis of integrodifference equations a real challenge because the dynamics are nonautonomous. The simplest form of the model allows for the dispersal kernel K_t and growth rate r_t to be random variables, indexed by time t, taken from given distributions. Using these, the classical questions of spreading speeds and critical domain size can then be revisited for such models in the context of impacts of temporal variation and stochasticity.

Seminal work on the critical domain size problem came from a series of papers by Hardin, Web and Takac (eg. [7]). This work has been recently rediscovered and extended into new contexts, particularly in the context of species survival under climate change. Here the stochasticity means that it does not make sense to analyze eigenvalues of a linearized operator. Rather, calculations give rise to something similar to a Lyapunov exponent, from which it is possible derive threshold conditions for persistence. Recent there have been attempts to extend this theory to infinite domains, but this work is ongoing.

Early work on the spatial spread problem for scalar models, concave down growth functions, and uncorrelated spatial variation was developed by Neubert, Kot and Lewis [20]. Here there is no longer a fixed spreading speed. Rather there is a distribution of spreading speeds, and under certain conditions it is possible to show that the distribution follows a normal distribution arising from a Central Limit Theorem result. However, these results assumed that the spreading speed nonlinear system could be characterized by the operator linearized about zero. Although this appears to be the case, there is no rigorous proof. Recent work has extended these results to stage-structured models and complex correlations in temporal fluctuations [23]. However, rigorous proof for conditions under which the spreading speed for the nonlinear stochastic operator can be determined by it's linearization remains an open problem.

An alternate form of variability comes from variability in the behaviour of dispersing individuals. For example, each individual could disperse via a Gaussian dispersal kernel, with with a different variance. Thus the population dispersal kernel would no longer be Gaussian, but would require convolution of the Gaussian with the distribution for the variance. This kind of individual variability can then change the shape of the overall dispersal kernel from Gaussian to fat-tailed, leading to a dramatic effect on spreading speeds [21].

New model formulations

One of the main applications of the integrodifference framework has been to describe dispersing populations, the kernel in equation (16.0.1) then describes the probability of dispersal. For such dispersal models we are frequently interested in travelling wave solutions however integral projection models (IPMs) in which the kernel describes ageing or stage transitions results in an IDE problem where instead other questions emerge, such as finding the stable age distribution (equilibrium distribution) of the IDE. We held a break-out session to discuss other biological problems that might give rise from new types of IDEs and the corresponding new questions associated to the resulting equations.

Kernel-type functions can be used to represent a range of biological phenomena such as non-local interactions, whereby the kernel may be some description of distance-dependent decay in competition or communication among animals. The spatial influence of individuals can result in a kernel that changes sign as a function of proximity - this is a new type of dispersal kernel that has not been studied in the context of IDEs, although these kernels have been adopted in non-local PDE models of group aggregation, migration, schooling and flocking. Exactly how many of the results from non-local PDE theory can be transferred across to the IDE framework remains an open question.

Other examples novel kernels that could generate new classes of IDE model include non-local transmission of parasites (i.e., representing an unobserved process), perception / information gathering / information exchange, kernels describing interactions with individuals of remote age / stage classes and kernels describing non-local but non-global interactions. Indeed in Alan Hastings' talk he presented a model of ecosystem engineers in which the kernel depended on the amount of occupied habitat which was determined by the density of the ecosystem engineers this also gave rise to an interesting new class of kernels and IDE problems.

Another area of discussion was formulating models of intermediate complexity to bridge from individual based models (IBMs) to IDEs and IPMs. IBM are close cousins of IDEs and allow detailed biological stochastic processes to be described, but lack some of the analytical tractability of an IDE. Bridging between the two frameworks of IBMs and IDEs could allow us to exploit the strengths of both frameworks. A mean field model of the IBM with a simple representation of deviations due to finite population size is one way in which this could be achieved. Another approach is to derive a stochastic difference equation with a Gaussian approximation to binomial process in IBM. Studies of stochastic IDEs are still in there infancy although there has been a recent push in the last few years to make progress in this area (see section 16).

Finally examining the underlying assumptions of IDEs one can ask how breaking these assumptions can also offer new classes of models not previously studied. For example, mixing continuous-time dispersal and population dynamics with discrete-time events gives rise to impulsive reaction-diffusion models, whose analysis is just in it's infancy [13]. Also, by including vital rates or dispersal that depends on local conditions or time and including heterogeneity among individuals that affects rates or kernels lead to IDE models that are biologically important but are not addressed by the current theory and open up exciting new avenues of research.

Outcome of the Meeting

The most important outcomes from this meeting are most likely those that are currently intangible: the inspiration that occurred in talks and conversations over tea and coffee, between researchers who would otherwise never have met. Analysts recognized that many ecologists have used (variants of) integrodifference equations without calling them that. Ecologists recognized that theory is being developed to deal with scenarios that seemed to be accessible only to simulation before. The fruits of mutual inspiration will be visible only in a few years when new collaborations yield published results.

We are currently working on two review papers that shall summarize the results and open questions from this workshop and include the larger community in the quest to solve them. We aim one of the papers at a mathematical/analytical audience and the other at an ecological audience.

Equally important is the effort to collect current literature and simulation tools online through a mixture of website and wiki. Several possibilities of implementing such a resource were discussed at the meeting. Several researchers indicated that they are willing and able to fund some of these efforts.

All in all, we believe that this workshop will soon be viewed as the galvanizing event in the theory and application of integrodifference equations in spatial ecology. Its effects will be felt in novel collaborations, exciting research and increased visibility and accessibility of the field.

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Chapter 17

New Trends in Graph Coloring (16w5120)

October 16 - 21, 2016

Organizer(s): Zdeněk Dvořák (Charles University), Bojan Mohar (Simon Fraser University), Luke Postle (University of Waterloo), Robin Thomas (Georgia Institute of Technology)

Overview of the Field

Graph coloring is one of the oldest studied topics in graph theory. Its roots date back to 1852 with the first statement of the celebrated Four Color Conjecture: can the countries of any map on a globe be colored with at most four colors so that no two countries that share a common boundary have the same color? It took over 100 years to prove this conjecture, and the attempts to do so gave rise to many other important concepts in graph theory and motivated the study of graph colorings in greater generality. Further motivation for the concept comes from wide-ranging applications of many variants of graph coloring in algorithm design, scheduling and resource allocation.

For these reasons, the study of graph colorings is a very active research subject with many interesting open questions. While many of these questions are notoriously difficult, a plethora of new and innovative methods have been developed in graph coloring over the last decade.

Let us mention some of the most promising directions that were explored during the workshop.

Density of critical graphs

A graph is $(k+1)$-critical if its chromatic number is $k+1$, but every proper subgraph is k-colorable. Every graph that is not k-colorable contains a $(k+1)$-critical subgraph, which would make their description important in many graph coloring problems. Of course, a nice exact description is unlikely to exist in general, however, the situation is more promising when restricted to sparse graphs such as graphs embedded in a fixed surface.

Indeed, Thomassen [39] proved that for any $k \geq 5$, there are only finitely many $(k+1)$-critical graphs embedded in any fixed surface. This in particular implies that k-colorability of such graphs can be decided in polynomial time. This result was later strengthened by giving explicit bounds on the sizes of $(k+1)$-critical embedded graphs, even in the list coloring setting [35], which has further consequences such as much improved bounds on their edgewidth, number of distinct k-colorings, or precoloring extension properties. A similar theory for 3-colorability of embedded triangle-free graphs was developed by Dvořák, Král and Thomas, leading to results such as resolving a conjecture of Havel regarding 3-colorability of planar graphs with triangles far apart [13].

The results mentioned so far fundamentally rely on the properties of embedded graphs. Rather surprisingly, Kostochka and Yancey [32] recently gave a much more general result showing that for some of these problems, only the density of the graph matters. Their lower bound on density of critical graphs almost completely resolved a longstanding question of Ore. The bound is based on an interesting new idea (called potential method) that has since been a basis of a number of results both for ordinary proper graph coloring and its variants; for example, Dvořák and Postle [15] were able to apply it in the circular coloring setting. Furthermore, using this method it is possible to obtain best known bounds on the sizes of 4-critical graphs of girth five, as well as critical graphs with bounded clique number.

New probabilistic tools

Probabilistic methods found many applications in graph coloring theory, both in constructions of graphs with large chromatic number and as tools to find proper colorings. Let us in particular mention the celebrated construction of Erdős [18] demonstrating graphs with arbitrarily large girth and chromatic number, as well as Johansson's bound [26] on the chromatic number of triangle-free graphs of bounded maximum degree.

The probabilistic tools that make it possible to show a lower bound on the probability of the conjunction of many almost independent events are of great interest in study of chromatic properties of bounded degree graphs. The basic such tool is Lovász Local Lemma. Based on a new constructive proof of Lovász Local Lemma, Gonçalves, M. Montassier, and A. Pinlou [23] formulated a new technique to show existence of colorings, called entropy compression method. The idea of this method is to analyze a random process that might eventually result in a proper coloring based on a record of its run, and to show that the number of such records of runs that fail is eventually dominated by the number of random choices in the run of a given length. Many bounds previously proved using Lovász Local Lemma can be significantly improved using this technique. Bernshteyn [4] provides an analysis of this method that makes it possible to reframe it in an entirely combinatorial fashion (without having to design and analyze the algorithm), stated as Local Cut Lemma.

χ-boundedness

Chromatic number of a graph is lower bounded by its clique number, but other than that there is no relation in general; indeed, there exist triangle-free graphs with arbitrarily large chromatic number [18]. On the other hand, the class of perfect graphs (whose chromatic number is equal to their clique number, hereditarily) has received much attention, especially given the recent proof of Strong Perfect Graph Theorem characterizing these graphs.

It is natural to ask about graph classes in that the chromatic number is bounded by a function of their clique number, called χ-bounded graph classes. In his paper introducing the concept, Gyarfás [25] raised a number of questions about χ-bounded graph classes that till recently defied all attempts to solve them. This was changed in a series of break-through results of Seymour and Scott [36, 37].

Although we now have a large body of work regarding χ-bounded graph classes and their properties, there is a little of what would amount to a coherent theory of the concept of χ-boundedness, and even in the cases where χ-boundedness of a class has been established, the quantitative relationship between the chromatic number and clique number is poorly understood (e.g., we have no examples of classes where a superpolynomial lower bound would be proved, but no lack of classes where only singly or doubly exponential upper bounds are known). The results of Seymour and Scott renewed the interest in the concept and bring some hope of rectifying the situation.

Old problems approached via new variants of colorings

Occasionally, a problem that appears hard can be easier to solve by first generalizing and then using extra tools available in the generalized setting. This might be the case for some of list coloring problems; here, a list coloring is a proper coloring where each vertex is restricted in its choice of colors to a list assigned to it. Many techniques for ordinary proper coloring do not translate to this setting, especially if they involve identification of vertices (which may not be possible for vertices with different lists).

To overcome this difficulty for a particular list coloring problem, Dvořák and Postle [16] introduced the notion of correspondence coloring where assigning a color to a vertex may prevent usage of different colors at the neighbors; this makes some transformations that do not work in the list coloring setting possible, of course at the expense of potentially making the considered problem harder. Other applications were found since, and the correspondence coloring seems to be interesting as a concept of its own as well.

Workshop Programme and Presentation Highlights

We took the opportunity given by presence of some of the most renowned researchers in the area and asked them to give in-depth tutorials on some of the methods and hot topics we mentioned in the introduction.

- Paul Seymour, Alex Scott, and Ingo Schiermeyer contributed a two-part tutorial on the recently very active topic of χ-boundedness.

  Paul Seymour provided an in-depth survey on the known results in the area as well as the basic techniques used to obtain them, culminating in an overview of the ideas of their recent results with Scott on graphs without odd holes and without long holes.

  Alex Scott continued with a detailed explanation of more difficult results for graphs excluding a tree of radius two, and for graphs excluding a subdivision of a given tree.

  Ingo Schiermeyer briefly discussed the few classes for that somewhat sharp bounds on the function bounding the chromatic number with respect to the clique number are known.

- Luke Postle and Matthew Yancey gave a detailed tutorial on the application of the potential method for coloring sparse graphs, which they co-developed in the last few years.

  Matthew Yancey described step-by-step the process of finding a proof by this method and demonstrated it in a detail on the problem of finding a decomposition of a sparse graph into an independent set and a forest.

  Luke Postle surveyed some of the more advanced results obtained using this technique, especially regarding the density of critical graphs without cliques of a given size, as well as applications to other kinds of coloring such as the circular coloring.

Many of presentations at the workshop also focused on recently developed methods and notions.

- Alexander Kostochka gave a survey talk on the properties and applications of correspondence coloring. As we mentioned in the introduction, correspondence coloring was originally developed to deal with list coloring problems, and many of the properties mentioned in the talk either mirror those known for list coloring. Nevertheless, he also highlighted many differences, such as a quantitatively different behavior with respect to the average degree—average degree d forces the list chromatic number to be at least $\Omega(\log d)$, while the correspondence chromatic number is much larger, at the order of $\Omega(d/\log d)$.

- David Wood gave talk showcasing several methods used to solve problems related to non-repetitive colorings. The arguments included the use of probabilistic tools such as Lovász Local Lemma and its recently developed strengthening—the entropy compression method. He also gave examples of use of tree decompositions not restricted by width, but by other properties of the bags.

- Anton Bernshteyn gave a short tutorial on the usage of Local Cut Lemma, a strengthening of the entropy compression method, providing a step-by-step demonstration on the problem of density of critical hypergraphs.

- Paul Wollan gave a talk outlining several approaches to coloring graphs avoiding a fixed graph as an immersion, including the usage of Kempe chains to serve as edges of the immersed graph.

- Louis Esperet's talk showed some methods for coloring geometric graphs and the connections to bounding the integrality gap for the maximum packing of cycles in planar digraphs.

- Daniel Cranston's talk gave an introduction into methods used for edge-coloring problems, including a detailed argument showing the properties of Kierstead paths, and their far-reaching generalization into Tashkinov trees (both of these are basic tools that enable one to extend an edge-coloring of a subgraph to an edge coloring of the whole graph). He also described an application of these tools to coloring linegraphs of multigraphs.

Several other presentations focused on particular problems and results of interest:

- Zdeněk Dvořák's talk outlined the equivalence of Thomassen's conjecture on exponentially many 3-colorings of planar triangle-free graph to new conjecture on satisfiability of a constant fraction of prescribed restrictions for such 3-colorings. The latter problem seems of independent interest in other settings.
- Marthe Bonamy presented a lower bound for the algorithmic complexity of multicolorings (relative to Exponential Time Hypothesis), showing an interesting way how a SAT instance can be succinctly transformed into a set coloring instance.
- Chun-Hung Liu gave talk on partitions of planar graphs without certain cycle lengths. He showed a complete characterizations of excluded cycle lengths necessary to force a planar graph to have a partition into an independent set and a graph of bounded maximum degree.
- Daniel Král' presented several results on colorings of planar graphs—a progress on colorings of 3-connected graphs so that vertices incident with a common face receive different colors, and a construction of a surprising counterexample to a long-standing conjecture of Steinberg.
- Robert Šámal presented a result on 3-flows, showing that a 3-edge-connected graph has a 3-flow that is only zero on at most one sixth of the edges.

Open Problems

We solicited among the participants open problems that they consider to be of interest. We believe these collected problems give an insight into the current trends and developing topics in the area, and we are considering the publication of the collection.

The problems range from new and potentially easy to solve ones to long-standing open questions and their variations. They cover a variety of subjects, from traditional topics such as the chromatic properties of planar graphs, to several new and potentially influential notions of coloring. The topic of χ-bounded graph classes is touched on by several of the problems, reflecting recent interest and progress in the area.

Let us mention some of the problems here.

Louis Esperet: Polynomial bounds on chromatic number in hereditary χ-bounded classes

Problem: Is it true that for every hereditary χ-bounded class $\mathcal{G}$, there is a constant C such that $\chi(G) \leq \omega(G)^C$ for all graphs $G \in \mathcal{G}$?

Let us recall that a class of graphs $\mathcal{G}$ is χ-bounded if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. Thus, χ-bounded classes are "almost perfect".

This problem outlines the difficulty in proving exact functions that bound the chromatic number in such classes—while we now have fairly developed tools for proving χ-boundedness qualitatively, the provided bounds look wasteful and are likely to be far from optimal. On the other hand, finding graphs with large chromatic number subject to structural restrictions is notoriously hard.

A concrete case where the polynomial lower bound/exponential upper bound gap waits to be closed for a long time concerns circle graphs (intersection graphs of chords of a circle). The circle graphs are known to be colorable using $21 \cdot 2^{\omega}$ colors (ern [10], improving previous results [30, 31]), and Kostochka [30] constructed circle graphs requiring $\Omega(\omega \log \omega)$ colors.

Let us remark that there exist hereditary classes whose χ-bounding function is at least a polynomial of an arbitrarily large degree, namely the classes of graphs without induced stars $K_{1,k}$. Let G be a graph with $R(k, \omega + 1) - 1$ vertices that contains neither an independent set of order k nor a clique of order greater than ω. Then G does not contain an induced star $K_{1,k}$ and its chromatic number is at least $|V(G)|/(k-1) = \Omega(\omega^{k/2})$ by the lower bound on the off-diagonal Ramsey numbers of [5]. On the other hand, if a graph G with clique number ω does not contain $K_{1,k}$, then its maximum degree is less than (and the chromatic number is at most) $R(k, \omega)$, and thus these classes cannot give a counterexample to the considered problem.

A positive answer to the problem would have the following interesting consequence. Let H be a graph such that the class $\mathcal{G}$ of graphs avoiding H as an induced subgraph is χ-bounded. In that case every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq \omega(G)^C$, and thus $\alpha(G) \geq |V(G)|/\chi(G) \geq |V(G)|/\omega(G)^C$, and $\max(\alpha(G), \omega(G)) \geq |V(G)|^{1/(C+1)}$. In other words, every such graph H would have the Erdős-Hajnal property. Let us remark that there are many graphs whose exclusion is known to imply χ-boundedness (paths, trees of radius 2, ...), but for which the Erdős-Hajnal conjecture is open. In the light of this, it seems more likely that the answer to the question is negative.

Nicolas Trotignon: Polynomial χ-bounds

Esperet asks whether every hereditary χ-bounded class has a χ-bounding function that is a polynomial. This question is certainly extremely difficult. Here are two potentially easier questions.

First, let us point out one interesting special case of Esperet's problem.

Problem: For any integer $k \geq 1$, is the class of graphs of rank-width at most k χ-bounded with a polynomial dependence of the chromatic number on the clique number?

Dvořák and Král' [12] proved that for any integer k, graphs of rank-width at most k form a χ-bounded class.

For the second question, we need the following definition. For any hereditary class of graphs $\mathcal{G}$, let $f_{\mathcal{G}}$ be the optimal χ-bounding function for $\mathcal{G}$ defined by

$$f_{\mathcal{G}}(x) = \max\{\chi(G) \text{ for all graph } G \in \mathcal{G} \text{ such that } \omega(G) = x\}.$$

A general question is the following.

Problem: For a given non-decreasing function f, is there a hereditary class $\mathcal{G}$ such that $f_{\mathcal{G}} = f$?

For instance, is there a class such that $f_{\mathcal{G}}(2) = 2$, and $f_{\mathcal{G}}(x)$ is huge for $x > 2$? The answer is no. Indeed, since $f_{\mathcal{G}}(2) = 2$, any graph in $\mathcal{G}$ has no odd hole. By a theorem by Scott and Seymour [36], the class of odd-hole-free graphs is χ-bounded. It follows that $f_{\mathcal{G}}$ cannot be too "huge".

More specific question is then the following:

Problem: Given an integer k, is there a hereditary class of graphs $\mathcal{G}$ such that $f_{\mathcal{G}}(2) = 3$ and $f_{\mathcal{G}}(3) = k$?

Marthe Bonamy: Edge deletions in graphs without induced cycles of length divisible by 3

Problem: Let G be a non-empty graph without induced cycles of length divisible by 3. Does G necessarily contain an edge e such that $G - e$ also contains no induced cycles of length divisible by 3?

Graphs without induced cycles of length divisible by 3 are known to have bounded chromatic number as proved by Bonamy et al. [6]. If the assertion of the problem holds, this would imply that such graphs are actually 3-colorable by an adaptation of an argument of Wrochna (who used it to prove that graphs without any cycles of length divisible by 3 are 3-colorable).

A far-reaching strengthening of [6] was recently proved by Scott and Seymour [37], whose result implies that for every $a > b \geq 0$, a triangle-free graph of large chromatic number contains an induced cycle of length $b \pmod{a}$.

Hence, it might seem that the choice of the class (avoiding induced cycles of length divisible by 3) is quite arbitrary; curiously though, the class seems to have some interesting properties. Aharoni and Haxell [1] point out one reason why this could be the case, based on the topological properties of the complex of independent sets of such graphs. For a graph G, let $I(G)$ denote the abstract simplicial complex formed by the independent sets of G. Kalai and Meshulam conjectured that each graph with a large chromatic number has an induced subgraph H such that the sum of Betti numbers of $I(H)$ is large. They also conjectured that if G has no induced cycles of length divisible by 3, then the sum of Betti numbers of $I(G)$ is at most 1. In the former conjecture, they also proposed the variant with the sum of Betti numbers replaced by Euler characteristic. This motivates the following conjecture proposed a few years ago.

Problem: Show that if G is a graph without induced cycles of length divisible by 3, then the Euler characteristic of $I(G)$ is in absolute value at most 1. In other words, the numbers of odd and even independent sets of G differ by at most 1.

Dan Král': Weak Steinberg's conjecture

Steinberg [38] conjectured that all planar graphs without cycles of lengths four or five are 3-colorable; however, this conjecture was recently disproved [11]. On the other hand, Borodin et al. [9] proved that it suffices to forbid cycles of lengths 4 to 7. This leaves open the following question.

Problem: Is every planar graph containing no cycles of lengths four, five or six 3-colorable?

The study of chromatic number of planar graphs without cycles of given lengths was started by Grtzsch [24] who proved that excluding triangles suffices to guarantee 3-colorability. Naturally, a question arises regarding the planar graphs with triangles restricted in some way (in full generality, 3-colorability of planar graphs is NP-complete to decide [22]). Planar graphs with at most three triangles are 3-colorable [2], and non-3-colorable planar graphs with exactly 4 triangles were fully described recently [7]. An approximate description of minimal non-3-colorable planar graphs with a bounded number of triangles can be gleaned from the theory of Dvok, Krl' and Thomas, which also implies that planar graphs with triangles far apart are 3-colorable [13].

The last mentioned result answers in positive an old question of Havel [27]. However, the provided bound on the required distance is rather large (on the order of 10^{100}), while the best know lower bound (given by Aksionov and Mel'nikov [3]) is 4. Thus, the following problem still stands.

Problem: Determine the minimum integer d such that every planar graph with distance at least d among its triangles is 3-colorable.

The best partial result towards resolving this theorem is by Borodin et al. [8], who proved that distance 4 is sufficient in planar graphs without 5-cycles that share at least one edge with a triangle.

Zdeněk Dvořák: Fractional chromatic number of planar graphs of girth at least 5

Problem: Does there exist a constant $c < 3$ such that every planar graph of girth at least 5 has fractional chromatic number less than c?

An n-vertex graph with fractional chromatic number c must contain an independent set of size at least c. Supporting the assertion of the problem, it is known [14] that there exists $c < 3$ such that all n-vertex planar graphs of girth at least 5 have independence number at least n/c.

The girth restriction in the problem cannot be relaxed, since Pirnazar and Ullman [34] found triangle-free planar graphs with fractional chromatic number arbitrarily close to 3 (on the other hand, Dvok et al. [17] proved that the fractional chromatic number of every planar triangle-free graph is strictly smaller than 3). Furthermore, we cannot replace fractional chromatic number by circular chromatic number in the statement of the problem, since for example odd wheels whose spokes are subdivided once have circular chromatic number exactly 3.

More generally, it is natural to ask what is the maximum fractional chromatic number of planar graphs of given girth. Without girth restriction, there exist planar graphs of fractional chromatic number exactly 4; describing planar graphs with smaller fractional chromatic number is an open question, related to another well-known open problem of recognizing n-vertex planar graphs that have independent set larger than $n/4$. A significant source of complications in both problems comes from the fact that the only known way how to prove the bound is using the Four Color Theorem.

Penny Haxell: Strong chromatic number of graphs with maximum degree 2

Problem: Let G be a graph of maximum degree at most 2 (i.e., a disjoint union of paths, cycles, and isolated vertices) whose number of vertices is divisible by 4. Let G' be a graph obtained from G by partitioning its vertices into groups of size 4 and adding all edges within each such group. Is G' necessarily 4-colorable?

The answer to the analogous question with 4 replaced by other integer k is true for $k \geq 5$ by the result of Haxell [28], and false for $k \leq 3$ by a construction of Fleischner and Stiebitz [21].

For an integer k, a k-clique enlargement of a graph G is a graph obtained from G by adding at most $k-1$ isolated vertices to make its number of vertices divisible by k, partitioning the vertices into groups of size k, and adding all edges within each such group. We say that a graph G is strongly k-colorable if every k-clique enlargement of G is k-colorable. The strong chromatic number of G is the minimum k such that G is strongly k-colorable; a non-trivial fact that G is also strongly k'-colorable for every $k' \geq k$ was proved by Fellows [19].

One motivation for this notion comes from the well-known "cycle plus triangles" conjecture (proved by Fleischner and Stiebitz [20]), which can be stated as the claim that cycles of length divisible by 3 are strongly 3-colorable.

Haxell [28] proved a Brooks-like result for strong coloring: a graph of maximum degree Δ has strong chromatic number at most $3\Delta - 1$, improved later[29] to $\frac{11}{4}\Delta$. On the other hand, Fleischner and Stiebitz [21] constructed for each Δ an example of a Δ-regular graph with strong chromatic number at least 2Δ, and it has been conjectured that the corresponding upper bound holds as well. The presented problem thus concerns the first unresolved case of this conjecture, for $\Delta = 2$.

David R. Wood: Majority colorings of digraphs

A majority coloring of a digraph is a function that assigns each vertex v a color, such that at most half the out-neighbours of v receive the same color as v. In other words, more than half the out-neighbours of v receive a color different from v (hence the name 'majority'). Whether every digraph has a majority coloring with a bounded number of colors was posed as an open problem on mathoverflow [40]. In response, Ilya Bogdanov proved that a bounded number of colors suffice for tournaments [40]. Kreutzer et al. [33] solved the problem, showing that every digraph has a majority 4-coloring. They also found infinitely many digraphs with no majority 2-coloring (e.g. cyclically oriented K_3). The following problem naturally arises:

Problem: Does every digraph have a majority 3-coloring?

Or, indeed, the following natural generalization.

Problem For an integer $c \geq 2$, does every digraph have a vertex $(2c-1)$-coloring such that for each vertex v, at most $\frac{1}{c}\deg^{+}(v)$ out-neighbours of v receive the same color as v?

This result would be best possible for all $c \geq 2$, as shown by the cyclic orientation of K_{2c-1}.

Scientific Progress Made

Both the open problems and the presentations lead to interesting discussions among the participants, inspiring new ideas and questions. Some new results were obtained already during the workshop and several cooperations on the other questions were started.

Paul Wollan asked about a variant of the problem presented in Subsection 17: Given a graph with no induced cycles whose length is a multiple of a fixed integer m, does there necessarily exist an edge whose removal preserves this property? Paul Seymour found a counterexample for $m = 5$, which he together with Sergey Norin and Marthe Bonamy generalized to all odd $m > 3$. Dan Cranston, Felix Joos and Marthe Bonamy found a counter-example for any m divisible by 4. A construction of Robert Šámal and Paul Seymour gives a counterexample for $m = 6$. It seems likely that there exist counterexamples for $m = 4k + 2$ with $k \geq 2$ as well, while the cases of $m = 2$ and $m = 3$ might have positive answers.

Problems regarding majority coloring (see Subsection 17) also attracted some attention. Gregory Gauthier gave an approach towards resolving the first problem of majority 3-coloring, and Fiachra Knox found an elegant argument that in the second problem, a $2c$-coloring with the required properties exists.

Motivated by a talk by Zdeněk Dvořák, Luke Postle proposed several questions regarding the possibility to satisfy a constant fraction of requests in a coloring; e.g., for which $k \geq 5$ does there exist $\varepsilon > 0$ such that given an n-vertex planar graph, an assignment L of lists of size k, and a function r assigning to each vertex v a color $r(v) \in L(v)$, is it possible to find an L-coloring of the graph that matches r on at least εn vertices? Joined also by

Sergey Norin, they showed that the answer is affirmative for $k \geq 6$ and found interesting results for several other related problems. They are preparing a short note introducing this concept, establishing its basic properties and presenting their results.

Alexandr Kostochka and Anton Bernshteyn continued their discussion of correspondence chromatic number, proving that there is $f(n) \leq 5n^2/4$ such that for every n-vertex graph G, the graph $G^\star$ obtained from G by adding $f(n)$ dominating vertices has DP-chromatic number equal to $\chi(G)$. This answers the corresponding question of Xuding Zhu. On the other hand, they found an example that needs approximately $n^2/4$ extra vertices to have this property. It follows that the Noel-Reed-Wu Theorem for list coloring extends to correspondence coloring only in a very weak form. Also, they found an example of a planar bipartite graph with correspondence chromatic number 4.

With Pavol Hell, Penny Haxell and Luke Postle, they also talked about a generalization of the concept to homomorphisms, and identified a nice question about its complexity. They also discussed this question later in Bordeaux at the Bordeaux Graph Workshop and will continue thinking about.

Paul Wollan and David Wood started a collaboration regarding the problems related to the topic of David's talk (nonrepetitive chromatic number), and are now close to completing a paper entitled "Nonrepetitive colourings of graphs excluding a fixed immersion".

Lena Yuditsky, Maria Aksenovich, Ingo Schiermeyer, and Marthe Bonamy started a cooperation on improving the χ-bounding function for the family of $2K_2$ free graphs, which was one of the open problem that Ingo Schiermeyer suggested for the workshop. Initially, they obtained a modest additive improvement by 2 to the currently known bounds.

Chun-Hung Liu started a joint work with Louis Esperet for proving or disproving that every triangle-free planar graph of bounded maximum degree can be two colored in a way that every monochromatic component has bounded size, reflecting both his presentation at the workshop and one of the problems raised by Louis Esperet. He also started a cooperation with Bojan Mohar and Hehui Wu regarding the problem of finding an orientation of a d-degenerated graph with small maximum out-degree such that no pair of adjacent vertices have the same out-degree (inspired by a problem suggested by Bojan Mohar for the workshop).

Participants

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Esperet, Louis (CNRS)
Gauthier, Gregory (Princeton University)
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Le, Ngoc Khang (ENS de Lyon)
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Chapter 18

Theoretical and computational aspects of nonlinear surface waves (16w5112)

October 30 - November 4, 2016

Organizer(s): Mark Groves (Saarland University, Germany), Philippe Guyenne (University of Delaware, USA), Emilian Părău (University of East Anglia, UK), Erik Wahlén (Lund University, Sweden)

Overview

Waves on the surface of a fluid — or the interface between different fluids — are omnipresent phenomena. In water, ripples driven by surface tension affect remote sensing of underwater obstacles; waves on the surface and the interface between internal layers of water of differing densities affect ocean shipping, coastal morphology, and near-shore navigation; and tsunamis and hurricane-generated waves can cause devastation on a global scale. Hydroelastic waves are generated by man-made floating structures (especially airports) and by Antarctic exploration where heavy equipment travels over roads on floating ice. Hydromagnetic waves arise on the surface of ferrofluids in use in high-technology applications.

From a mathematical viewpoint, the surface-wave equations pose surprisingly deep and subtle challenges for rigorous analysis and numerical simulation. The governing equations are widely accepted and there has been substantial research into their validity and limitations. However, a rigorous theory of their solutions is extremely complex due not only to the fact that the water-wave problem is a classical free-boundary problem (where the problem domain, specifically the water surface, is one of the unknowns), but also because the boundary conditions (and, in some cases, the equations) are strongly nonlinear. The level of difficulty is such that the theory has merely begun to answer the fundamental questions which must be addressed before our understanding can be termed 'adequate'. The workshop was convened in response to recent advances in analysis and scientific computing, and the consequent renewed interest in this classical problem from several different mathematical directions.

The workshop was organised around five main themes related to key mathematical and numerical issues about nonlinear waves arising in a variety of free-surface problems: initial-value problems, coherent structures, stability of coherent structures, model equations and waves in domains with complex geometry. The problems under consideration (water waves, hydroelastic waves, internal waves, etc.) have similar formulations and thus can be addressed by closely related methods. The participants reflected the multidisciplinary nature of these problems: over forty researchers – including mathematical analysts, applied mathematicians, numerical analysts as well as fluid dynamicists, oceanographers and engineers – from twelve different countries took part.

Substantial progress was made in each of the themes (see below for a detailed description of the current state of the art together with the advances reported at the conference). Our main objective – catalysing progress in this classical problem by promoting direct interaction of researchers from different communities, and cross-fertilisation of ideas among them – was achieved, with many discussions taking place during and outside the scheduled talks. A roadmap for making future progress on some of the difficult open problems in nonlinear surface waves was prepared; it includes proposals for intensive research semesters at leading mathematical research institutes, setting up a new interdisciplinary scientific journal on water waves and possibly arranging a follow-up workshop at Casa Matemática Oaxaca in 2019 or 2020.

Themes and presentations

The workshop consisted of thirty-five 30-minute presentations, each of which was associated with one of the central themes.

Initial-value problems

There is now a wealth of local well-posedness results for various versions of the water-wave problem, each relying upon a different formulation (see e.g. [6] and references therein). Recently, global well-posedness results for small waves on water of infinite depth or finite depth with a flat bottom have become available in both two and three dimensions for various combinations of gravity and capillarity; see [2] for a recent overview. These results all exclude the existence of solitary waves with small energy; to go beyond this obstacle (and thus include gravity waves on water of finite depth, for example) is a challenging open problem. In the opposite direction, theories of singularity formation due to wave breaking and overturning have been developed in the last few years. At the same time, a number of accurate and efficient numerical methods have been developed to solve the full Euler equations for irrotational water waves on finite or infinite depth, based on boundary integral equations, conformal mappings and Taylor series expansions. Here the main task consists in evaluating the Dirichlet–Neumann operator associated with the fluid domain. Special effort has also been devoted to designing numerical schemes that preserve important properties of the mathematical system such as energy conservation.

- Walter Craig (McMaster University, Canada) gave the talk 'Hamiltonians and normal forms for water waves', in which he presented a generalization of Zakharov's Hamiltonian formulation for the water-wave problem to overturning wave profiles. He also discussed the question of Birkhoff normal forms for the water-wave equations in the setting of spatially periodic solutions, including the function-space mapping properties of these transformations.

- Mihaela Ifrim (UC Berkeley, USA) gave the talk 'Constant vorticity water waves', in which she considered the incompressible, infinite depth water-wave equations in two space dimensions, with gravity and constant vorticity but without surface tension. She described how to prove local well-posedness for large data, as well as cubic lifespan bounds for small data solutions, by expressing the problem in position-velocity potential holomorphic coordinates.

- Paul Milewski (University of Bath, UK) gave the talk 'Wave collapse for ripples: dynamics of CG waves', in which he discussed capillary-gravity wave packets that are described asymptotically by the focussing 2D NLS equation. This equation has solutions that blow up in finite time. In his talk, Milewski numerically explored the dynamics of blowup initial data under the full Euler equations.

- Daniel Tataru (UC Berkeley, USA) discussed 'Long time solutions for finite bottom gravity waves'. He gave an overview of recent work with Benjamin Harrop-Griffiths and Mihaela Ifrim concerning enhanced lifespan bounds for small data gravity waves in the presence of a flat bottom. This work is based on two key ideas: (i) the use of holomorphic coordinates; (ii) the modified energy method, a quasilinear alternative to normal forms.

- Sijue Wu (University of Michigan, USA) gave the talk 'On the motion of a self-gravitating incompressible fluid with free boundary', reporting joint work with Lydia Bieri, Shuang Miao and Sohrab Shahshahani. She considered the motion of the interface separating a vacuum from an inviscid, incompressible, and irrotational fluid subject to self-gravitational force and neglecting surface tension in two space dimensions. The main result was a long-time existence result for small perturbations of an equilibrium state. The proof is based on a nonlinear transformation of the unknown function and a coordinate change which eliminates quadratic nonlinear terms.

Coherent structures

Recent advances in this area include the development – from a base of almost zero knowledge – of existence theories for travelling waves with vorticity, standing waves (waves which are periodic in space and time) and three-dimensional surface waves in the context of the water-wave problem (see e.g. [1, 3, 4, 5, 9]). Various types of coherent structures in two and three dimensions have been computed in the last few years using boundary integral equations and spectral methods (see e.g. [10, 8, 7]).

- Mark Groves (Universität des Saarlandes, Germany) presented the talk 'Fully localised solitary gravity-capillary water waves', which is joint work with Boris Buffoni and Erik Wahlén. He considered the classical gravity-capillary water-wave problem in its usual formulation as a three-dimensional free-boundary problem for the Euler equations for a perfect fluid. The existence of fully localised solitary waves has been predicted on the basis of simpler model equations, namely the Kadomtsev-Petviashvili (KP) equation in the case of strong surface tension and the Davey-Stewartson (DS) system in the case of weak surface tension. The talk confirmed the existence of such waves as solutions to the full water-wave problem and gave rigorous justification for the use of the model equations.

- Philippe Guyenne (University of Delaware, USA) gave the talk 'Nonlinear waves in ice sheets' concerning the mathematical modelling and numerical simulation of waves in ice sheets as occurring, e.g., in polar regions. A three-dimensional Hamiltonian formulation for ice sheets deforming on top of an ideal fluid of arbitrary depth was presented and nonlinear wave solutions were examined. In certain asymptotic regimes, analytical solutions were derived and compared with fully nonlinear solutions obtained numerically by a pseudospectral method.

- David Henry (University College Cork, Ireland) gave the talk 'Nonlinear water waves and wave-current interactions', in which he examined the nonlinear water waves and wave-current interactions, which may be prescribed by Gerstner-like exact and explicit solutions to the geophysical $\beta-$plane equations in the equatorial region. In particular, he presented recent work which highlights the role played by the typically-neglected centripetal force terms in describing such physical processes.

- Olga Trichtchenko (University College London, UK) presented the talk 'Computing three-dimensional flexural gravity water waves', in which she discussed an efficient and accurate method for computing solutions to Euler's equations for water waves underneath an ice sheet in three dimensions (joint work with Emilian Parau, Jean-Marc Vanden-Broeck and Paul Milewski). The equations were solved via a numerically implemented boundary integral-equation method and some high performance computing techniques were utilized. Trichtchenko gave details of the current methods and compared solutions for different models of the ice sheet.

- Jean-Marc Vanden-Broeck (University College London, UK) presented 'A numerical investigation of non-symmetric nonlinear water waves'. He considered nonlinear waves travelling at a constant velocity at the surface of a fluid of finite depth, assuming the fluid to be incompressible and inviscid and the flow to be irrotational. Gravity and surface tension were taken into account and both two and three dimensional waves were studied. Classical solutions usually assume that the waves are symmetric. Vanden-Broeck showed that there are in addition an infinite number of branches of non-symmetric waves which include periodic waves, solitary waves and generalised solitary waves.

- Samuel Walsh (University of Missouri, USA) gave the talk 'Existence and qualitative theory of stratified solitary water waves', in which he reported some recent results concerning two-dimensional gravity solitary water waves with heterogeneous density obtained in collaboration with Robin Ming Chen and Miles Wheeler. The fluid domain is assumed be bounded below by an impenetrable flat ocean bed, while the interface between the water and vacuum above is a free boundary. Their main main existence result states that, for any smooth choice of upstream velocity and streamline density function, there exists a path connected set of such solutions that includes large-amplitude surface waves. Indeed, this solution set can be continued up to (but does not include) an 'extreme wave' that possess a stagnation point. Walsh also discussed a number of results characterizing the qualitative features of solitary stratified waves.

- Miles Wheeler (Courant Institute, USA) discussed 'Global bifurcation of rotating vortex patches', (joint work with Zineb Hassainia and Nader Masmoudi), in which continuous curves of rotating vortex patch solutions to the two-dimensional Euler equations are constructed. These curves are large in that, as the parameter tends to infinity, the minimum value on the boundary of the relative angular fluid velocity becomes arbitrarily small.

Stability of coherent structures

One crucial question facing all developments, both analytical and numerical, in the theory of surface waves is that of stability: Once one has found a solution of interest (e.g., a travelling or standing wave) will it be observable in the laboratory setting? Will it be found in the open ocean? There are now a range of rigorous mathematical linear instability results for two-dimensional periodic wave trains and solitary waves on deep water (Benjamin-Feir, superharmonic instabilities) and two-dimensional waves under three-dimensional perturbations. Conditional stability results for solitary waves are also available ('conditional' since they hold only over the unknown, possibly small, existence times of solutions).

- Mariana Hărăguş (Université de Franche-Comté, France) gave the talk 'Counting unstable eigenvalues in Hamiltonian spectral problems via commuting operators'. She presented a general counting result for the unstable eigenvalues of linear operators of the form JL, in which J and L are respectively skew- and self-adjoint operators. Under the assumption that there exists a self-adjoint operator K such that JL and JK commute, the result states that the number of unstable eigenvalues of JL is bounded by the number of nonpositive eigenvalues of K. As an application, Hărăguş discussed the transverse stability of one-dimensional periodic traveling waves in the classical KP-II (Kadomtsev–Petviashvili) equation.

- Christian Klein (Institut de Mathématiques de Bourgogne, France) discussed 'Multidomain spectral methods for Schrödinger equations'. A multidomain spectral method with compactified exterior domains combined with stable second and fourth order time integrators was presented for Schrödinger equations. The numerical approach allows high precision numerical studies of solutions on the whole real line or in higher dimensions. The method was compared with transparent boundary conditions and perfectly matched layers. The code can deal with asymptotically non vanishing solutions such as the Peregrine breather (currently being discussed as a model for rogue waves). It was shown that the Peregrine breather can be numerically propagated with essentially machine precision, and the localized perturbations of this solution were discussed.

- Alexander Korotkevich (University of New Mexico, USA and Landau Institute, Russia) gave the talk 'Circular instability of a standing surface wave: numerical simulation and wave-tank experiments', in which he showed that a standing wave is unstable if four-wave nonlinear processes are considered. This is joint work with Sergei Lukaschuk. Korotkevich compared numerical simulations of unstable weakly nonlinear standing waves on the surface of a deep fluid in the framework of the primordial dynamical equations and in a laboratory wave-tank experiment. Direct measurements of the spatial Fourier spectrum confirm the existence of the instability in real life conditions for gravity-capillary surface waves.

- Emilian Părău (University of East Anglia, UK) presented the talk 'Stability of capillary waves on fluid sheets'. The talk was based on joint work with Mark Blyth in which the linear stability of finite-amplitude capillary waves on inviscid sheets of fluid is investigated; superharmonic and subharmonic perturbations are considered and a conformal mapping technique is used. The instability results are also checked by time integration of the fully nonlinear unsteady equations.

- Erik Wahlén (Lund University) gave the talk 'Variational existence and stability theory for hydroelastic solitary waves' (joint work with Mark Groves and Benedikt Hewer). He presented an existence and stability theory for solitary waves at the interface between a thin ice sheet and an ideal fluid, which is based on minimising the total energy subject to the constraint of fixed total horizontal momentum. The ice sheet is modelled using the Cosserat theory of hyperelastic shells. Since the energy functional is quadratic in the highest derivatives, stronger results are obtained than in the case of capillary-gravity waves.

Model equations

An important direction of inquiry in the general field of nonlinear waves is the development and application of simplified model equations. In asymptotic regimes where small parameters can be defined, many such models can be derived at various orders of approximation. Their interest lies in the fact that they are usually more tractable analytically and numerically than the full equations. As a consequence, they have been a popular tool in the engineering community where they have been used with varying degrees of success. The well-posedness as well as rigorous justification of model equations are fundamental mathematical questions that have recently led to a surge of activity (see e.g. [6]). Such analysis is crucial in determining their precise range of validity. Their numerical simulation also requires sophisticated numerical methods that are suited to their mathematical structure. In addition, some models may exhibit blow-up which calls for careful analysis and computation.

- David Ambrose (Drexel University, USA) gave the talk 'Sufficiently strong dispersion removes ill-posedness of truncated series models of water waves' (joint work with Shunlian Liu). Previous joint work with Jerry Bona, David Nicholls and Michael Siegel, demonstrated that truncated series models of gravity water waves exhibit ill-posedness. Ambrose explained that the addition of sufficiently strong dispersion makes such a system well-posed. Physically, this strong dispersion can be relevant, for instance, for hydroelastic waves. The proof uses techniques of paradifferential calculus.

- Jerry Bona (University of Illinois at Chicago, USA) discussed 'Higher-order Hamiltonian models for water waves'. He introduced higher-order models for unidirectional propagation of long-crested water waves and sketched their analysis.

- Gabriele Bruell (NTNU, Norway) presented the talk 'On symmetry and decay of traveling wave solutions to the Whitham equation'. The Whitham equation is a nonlocal, nonlinear dispersive wave equation introduced by G. B. Whitham as an alternative to the Korteweg-de Vries equation, describing the wave motion at the surface on shallow water. Bruell showed that any supercritical solitary waves decays exponentially, is symmetric and has exactly one crest. She also presented a result stating that, conversely, any classical, symmetric solution constitutes a traveling wave. The latter result holds true for a large class of partial differential equations sharing a certain structure.

- John Carter (Seattle University, USA) discussed 'Frequency downshifting in a viscous fluid'. Frequency downshift, i.e. a shift in the spectral peak to a lower frequency, in a train of nearly monochromatic gravity waves was first reported by Lake et al. (1977). Even though it is generally agreed upon that frequency downshifting (FD) is related to the Benjamin-Feir instability and many physical phenomena (including wave breaking and wind) have been proposed as mechanisms for FD, its precise cause remains an open question. Dias et al. (2008) added a viscous correction to the Euler equations and derived the dissipative NLS equation (DNLS). In his talk, Carter introduced a higher-order generalization of the DNLS equation, which he called the viscous Dysthe equation. He outlined the derivation of this new equation and presented many of its properties. He established that it predicts FD in both the spectral mean and spectral peak senses. Finally, he demonstrated that predictions obtained from the viscous Dysthe equation accurately model data from experiments in which frequency downshift occurred.

- Christopher Curtis (San Diego State University, USA) discussed 'Surface and interfacial waves over currents and point vortices'. The computation of surface and interfacial waves is a central problem in fluid mechanics. While much has been done, the effect of vorticity on surface and internal wave propagation is still poorly understood. Curtis first looked at shallow-water propagation in density stratified fluids with piecewise linear shear profiles. He showed that by allowing for jumps in the shear across the interface, strong nonlinear responses can be generated, resulting in phenomena such as dispersive shock waves. Thus depth varying currents could play a larger role in interface dynamics than is currently understood. Second, he studied the problem of collections of irrotational point vortices underneath a free fluid surface. He presented a derivation of a model and numerical scheme which allows for arbitrary numbers of vortices in a shallow-water limit and argued that this might give some hint as to how underwater eddies can generate free surface waves.

- Mats Ehrnström (NTNU, Norway) gave the talk 'On waves of greatest height in fully dispersive equations', in which he discussed the existence of large-amplitude periodic traveling waves to the nonlocal Whitham equation. In particular, he discussed the existence of a highest, $C^{1/2}$-cusped, traveling wave solution, which is obtained as a limiting case at the end of the main bifurcation branch of P-periodic traveling wave solutions. He proved that this regularity is optimal. Given that the Euler equations admits a highest wave that is not cusped, but Lipschitz continuous, it is an interesting question whether a bidirectional Whitham equation, which carries the full two-way dispersion relation from the Euler equations, could encompass a Lipschitz wave as well. In his talk, Ehrnström showed however that the highest wave for the bidirectional Whitham equation is not Lipschitz. He characterized its behaviour near the wave crest. Finally, he outlined the first steps towards a more general theory. The talk comprised joint work with Erik Wahlén, Mathew Johnson and Kyle Claassen.

- Anna Geyer (University of Vienna, Austria) presented the talk 'On periodic traveling waves of the Camassa-Holm equation', in which she considered the wave length λ of smooth periodic travelling wave solutions of the Camassa-Holm equation. The set of these solutions can be parametrized using the wave height a. Her main result establishes monotonicity properties of the map $a \mapsto \lambda(a)$ i.e., the wave length as a function of the wave height. She explained how to obtain the explicit bifurcation values, in terms of the parameters associated with the equation, which distinguish between the two possible qualitative behaviours of $\lambda(a)$, namely monotonicity and unimodality. The key point is to relate $\lambda(a)$ to the period function of a planar differential system with a quadratic-like first integral, and to apply a criterion which bounds the number of critical periods for this type of system.

- Henrik Kalisch (University of Bergen) gave the talk 'On existence and uniqueness of singular solutions for systems of conservation laws', in which he discussed existence and admissibility of delta-shock solutions for hyperbolic systems of conservation laws. One of the systems discussed is fully nonlinear, and does not admit a classical Lax-admissible solution to certain Riemann problems. By introducing complex-valued corrections in the framework of the weak asymptotic method, he showed that a compressive delta-shock wave solution resolves such Riemann problems. By letting the approximation parameter tend to zero, the corrections become real valued and the resulting distributions fit into a generalized concept of singular solutions. In this framework, it can be shown that every 2×2 system of conservation laws admits delta-shock solutions. As an example, he showed that the combination of discontinuous free-surface solutions and bottom step transitions naturally leads to singular solutions featuring Dirac delta distributions in the context of shallow-water flows.

- Vera Hur (University of Illinois at Urbana-Champaign, USA) discussed 'Wave breaking and modulational instability in full-dispersion shallow water models'. In the 1960s, Benjamin and Feir, and Whitham, discovered that a Stokes wave would be unstable to long wavelength perturbations, provided that (the carrier wave number) $\times$ (the undisturbed water depth) > 1.363.... In the 1990s, Bridges and Mielke studied the corresponding spectral instability in a rigorous manner. But it leaves some important issues open, such as the spectrum away from the origin. The governing equations of the water wave problem are complicated, and one may resort to simple approximate models to gain insights. Hur began by discussing Whitham's shallow water equation and the wave breaking conjecture, and then moved on to the modulational instability index for small-amplitude periodic traveling waves, the effects of surface tension and constant vorticity. She also discussed higher order corrections as well as extensions to bidirectional propagation and two-dimensional surfaces. The talk was based on joint work with Jared Bronski, Mathew Johnson, Ashish Pandey and Leeds Tao.

- Dag Nilsson (Lund University, Sweden) discussed 'Solitary waves of a class of Green-Naghdi type systems' (work in in progress with Erik Wahlén and Vincent Duchêne). He considered a class of nonlocal Green-Naghdi type systems and presented an existence result for solitary wave solutions. The solutions are constructed as constrained minimizers of a certain scalar functional. A key component of the proof is the use of the concentration compactness principle.

- Daniel Ratliff (University of Surrey, UK) discussed 'The emergence of higher order dispersion from periodic waves'. Following the method of Bridges (2013), he demonstrated how one may derive PDEs with fifth order dispersion from periodic waves (and, in general, relative equilibria). Many of the coefficients of the emerging nonlinear approximations are directly related to the system's conservation laws, and those of the dispersive terms are tied to a Jordan chain analysis. Applications of the theory were also discussed.

- Steve Shkoller (UC Davis, USA) discussed 'Nonuniqueness of weak solutions to the SQG equation'. This is joint work with Tristan Buckmaster and Vlad Vicol, in which it is proved that weak solutions of the inviscid surface quasi-geostrophic (SQG) equation are not unique, thereby answering an open problem posed by De Lellis and Szekelihidi. Moreover, he showed that weak solutions of the dissipative SQG equation are not unique, even if the fractional dissipation is stronger than the square root of the Laplacian.

Waves in domains with complex geometry

In view of real-world applications, it is crucial to consider wave problems in domains with complex geometry or complex boundary conditions, e.g. surface waves propagating over topography near the shore or generated by a wavemaker in a laboratory basin. The modelling of surface-wave propagation under wind forcing and bottom friction also poses many theoretical challenges. Other examples in fluid mechanics include hydroelastic waves in ice of variable thickness and hydromagnetic waves under the influence of a multidirectional magnetic field. These effects add considerably to the technical difficulties of the problem and also require special treatment in the numerical schemes.

- Harry Bingham (Technical University of Denmark) discussed 'Stable, high-order finite difference methods for nonlinear wave-structure interaction in a moving reference frame' which are used for solving potential flow approximations of nonlinear surface waves interacting with marine structures. Of special interest is the loading and wave-induced response of sailing ships. The work builds on the numerical solution strategy described in Bingham and Zhang (2007); in contrast to a simple upwinding strategy, the approach is found to give accurate and stable solutions for all combinations of wave celerity and ship forward speed. An immersed boundary method based on weighted least-squares difference approximations was developed to introduce the ship geometry into the numerical solution. Challenges with respect to tracking the body-free surface intersection line, and treating wave-breaking in a rational way were raised for discussion.

- Onno Bokhove (University of Leeds, UK) gave the talk 'Variational coupling of nonlinear water wave and ship dynamics: continuum and finite element modelling'. He reported on the mathematical and numerical modelling of (non)linear ship motion in (non)linear water waves, and derived a coupled model for the wave-ship dynamics following a variational methodology, in order to ensure zero numerical damping which is important for wave propagation. The final system of evolution equations comprises the classical water-wave equations for incompressible and irrotational waves, and a set of equations describing the dynamics of the ship. The novelty in this model is in the presence of a physical restriction on the water height under the ship, which is enforced through an inequality constraint via a Lagrange multiplier. The model is solved numerically using a variational (dis)continuous Galerkin finite element method with special, new and robust time integration methods. Bokhove showed numerical results for the dynamics of the coupled system in a hierarchy of increasing complexity: linear water-wave and linear ship dynamics, and potentially also fully coupled (non)linear water-wave and nonlinear ship dynamics.

- John Grue (University of Oslo, Norway) presented 'Ship generated tsunamis: linearity vs. nonlinearity'. The motivation for the talk was recent observations of very long waves running ahead of ships in the Oslofjord in Norway. The waves are triggered when the new, very large and relatively fast cruise ferries run across significant depth changes in the shallow fjord. An asymptotic linear analysis expressing the upstream wave field in terms of a pressure impulse at the depth change is complemented by fully dispersive calculations of the upstream waves. At very shallow positions the local ship speed becomes critical where the effect of nonlinearity is analyzed. A wave length of 700 m, wave height at the shore of 1 m, average depth of the fjord of 35 m, depth change similar to the average depth, ship length of 200 m, moving at subcritical speed, are typical characteristics.

- David Lannes (Université de Bordeaux, France) gave a talk entitled 'On the dynamics of floating structures'. He derived some equations describing the interaction of a floating solid structure and the surface of a perfect fluid. This is a double free boundary problem since in addition to the water-wave problem (determining the free boundary of the fluid region), one has to find the evolution of the contact line between the solid and the surface of the water. The so-called floating body problem has been studied so far as a three-dimensional problem. Lannes' first goal was to reduce it to a two-dimensional problem that takes the form of a coupled compressible-incompressible system. It was also shown that the hydrodynamic forces acting on the solid can be partly cast into the form of an added mass-inertia matrix, which turns out to be affected by the dispersive terms in the equations.

- Katie Oliveras (Seattle University, USA) discussed 'Relationships between pressure, bathymetry, and wave-height' (joint work with Vishal Vasan and Daniel Ferguson). A new method was proposed to relate the pressure at the bottom of a fluid, the shape of the bathymetry, and the surface elevation of a wave for steady flow or traveling waves. Given a measurement of any one of these physical quantities (pressure, bathymetry, or surface elevation), a numerical representation of the other two quantities is obtained via a nonlocal nonlinear equation derived from the Euler formulation of the water-wave problem without approximation. From this new equation, a variety of different asymptotic formulas were derived. The nonlocal equation and the asymptotic formulas were compared with both numerical data and physical experiments.

- Rosa Vargas-Magana (Universidad Nacional Autónoma de México) talked about 'A Whitham-Boussinesq long-wave model for variable topography' (joint work with Panayotis Panayotaros). They studied the propagation of water waves in a channel of variable depth using the long-wave asymptotic regime, using the Hamiltonian formulation of the problem in which the non-local Dirichlet-Neumann (DN) operator appears explicitly. They performed an ad-hoc modification of these terms using a pseudo differential operator (PDO) associated with the bottom topography. A Whitham-Boussinesq model for bidirectional wave propagation in shallow water that involves a PDO and which consider explicitly the expression for the depth profile was proposed in the talk. The model generalizes the Boussinesq system, as it includes the exact dispersion relation in the case of constant depth. An accurate and efficient numerical method was developed to compute this PDO and some results for the normal modes and eigenfrequencies of the linearized problem for families of different topographies were presented.

Participants

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Chapter 19

Permutation Groups (16w5087)

November 13 - 18, 2016

Organizer(s): Cheryl E. Praeger (University of Western Australia), Katrin Tent (University of Münster), Donna Testerman (Ecole Polytechnique Fédérale de Lausanne), George Willis (University of Newcastle)

Overview of the Field

Permutation groups are a mathematical approach to analysing structures by studying the rearrangements of the elements of the structure that preserve it. Finite permutation groups are primarily understood through combinatorial methods, while concepts from logic and topology come to the fore when studying infinite permutation groups. These two branches of permutation group theory are not completely independent however because techniques from algebra and geometry may be applied to both, and ideas transfer from one branch to the other. Our workshop brought together researchers on both finite and infinite permutation groups; the participants shared techniques and expertise and explored new avenues of study.

The permutation groups act on discrete structures such as graphs or incidence geometries comprising points and lines, and many of the same intuitions apply whether the structure is finite or infinite. Matrix groups are also an important source of both finite and infinite permutation groups, and the constructions used to produce the geometries on which they act in each case are similar.

Counting arguments are important when studying finite permutation groups but cannot be used to the same effect with infinite permutation groups. Progress comes instead by using techniques from model theory and descriptive set theory. Another approach to infinite permutation groups is through topology and approximation. In this approach, the infinite permutation group is locally the limit of finite permutation groups and results from the finite theory may be brought to bear.

The progress in "modern" times on what one would traditionally call "permutation groups" is characterised by the interplay of these rich and varied areas and approaches. The workshop brought together people working in (1) finite permutation groups, acting on graphs, geometric objects, *i.e.* more classical areas of the theory, (2) a model theoretic approach to infinite permutation groups and related questions, (3) topological groups, in particular, totally disconnected locally compact (t.d.l.c.) groups and, (4) matrix groups, particularly simple groups of Lie type and their primitive permutation actions.

There is no single perspective from which an overview of the workshop can be taken because it brought together researchers from four fields, each with its own methods and goals. The next section discusses the current state of research on these topics at a general and fairly high level. The remainder of the report discusses recent progress on the topics which were emphasized during the workshop and indicates connections made between the different fields.

Recent Developments and Open Problems

The four fields of research on permutation groups have reached different levels of maturity, as indicated by the fact that while those studying finite permutation groups and algebraic groups speak of 'classical' results and even of 'the classical groups', those studying infinite permutation groups and model theory or totally disconnected locally compact groups usually do not. A significant part of the benefit of bringing researchers from these fields together derives from this disparity. Those investigating totally disconnected local compact groups and infinite permutation groups seek to emulate the approaches successful for finite permutation and algebraic groups. Moreover, classical results about finite and linear groups underpin further work not only in those fields but also form the basis of current work in all four fields and, in the other direction, questions about totally disconnected locally compact or infinite permutation groups may lead back to unsolved problems in the more established areas. Researchers in these fields therefore can learn much from each other.

Much of the ongoing success of the study of finite permutation groups and matrix groups is based on decompositions of these groups into simple factors and on classifications of the simple groups in these classes. A decomposition theory for t.d.l.c. groups is only now emerging, following the appearance of [6], and techniques that would aid a classification of simple groups, including [7, 21], are still being developed. It is known [22] that compact generation is a necessary additional hypothesis if there is to be hope of a classification. Several classes of compactly generated simple groups are known, among them Lie groups over local fields and groups of automorphisms of trees [20], an up-to-date list of the known sources of simple groups is given in [7]. It is possible that other sources of large numbers of simple groups remain to be found. Gaps remain and it is not known, for example, whether simple groups exist with certain of the local structures identified in [7] or whether there can be compactly generated simple groups having flat-rank 0 (where the flat-rank is an invariant for t.d.l.c. groups associated with the scale [21] and equal to the familiar rank in the case of simple Lie groups). To resolve these difficulties either new examples of simple t.d.l.c. groups or new theorems are needed.

Developing a decomposition theory and classifying simple groups is not the end of the story of permutation groups however. Recent results (many still unpublished) suggest that the finite simple groups still hold secrets about simply stated properties of finite permutation groups and the structures on which they act. It has been shown, for example, that in most primitive permutation groups there is an unexpectedly close relationship between the orders of elements in the group and the size of the set on which it acts (at least four times the order) and the size of its orbits (there is an orbit of size equal to the order). These relationships are all the more remarkable for the fact that there are a few exceptions, which can all be described.

Primitive permutation actions of a group can be studied by classifying maximal subgroups of the group. Although much progress has been made with the maximal subgroups of almost simple finite groups, this program is still ongoing. Work on maximal subgroups of infinite groups must of necessity be confined to particular groups or classes of groups because no decomposition theory or classification of simple groups is possible for general infinite groups. One such natural class under consideration are the closed subgroups of $\mathrm{Sym}(\mathbb{N})$. These are exactly the automorphism groups of countable homogeneous structures and the maximal closed subgroups correspond to structures having no proper reducts in the model theoretic sense. There has been a lot of interest in this question even from the point of view of theoretical computer science.

Transitive permutation actions of groups may be studied by analysing the subgroup structure of the group in general. One approach to doing that is to seek information about various generating sets for the groups or for its subgroups. This approach can be applied to both finite and infinite groups. For this question too, the investigation is more advanced in the case of finite groups. There has been considerable progress in recent years, particularly for simple groups, where knowledge of good generating sets is useful for representing groups by permutations or matrices in computer algebra packages. In the case of infinite groups it is often useful to consider topological groups and topological generating sets.

Presentation Highlights

Many participants reported their immense appreciation at being able to interact with colleagues, to explore new questions to make progress on existing major research projects. In particular, they mentioned seeing unexpected connections between the various research directions that were represented, and finding discussions with the speakers after their lectures very inspiring and thought-provoking. Here are a few highlights and examples. More details may be found in the section on scientific progress.

Csaba Schneider's exposition of his theory of Cartesian decompositions left invariant by finite (and some families of infinite) permutation groups led to vigorous discussions, initiated by Alejandro Garrido, on whether there was a useful inverse limit concept for Cartesian decompositions and the wreath products which act on them.

Simon Smith's structure theory for subdegree-finite permutation groups, and his new fundamental box product construction/decomposition related to trees is a significant addition to the theory of t.d.l.c. groups. In his talk he showed how, as one outcome, his construction produces the first known uncountable family of compactly generated, non-discrete simple t.d.l.c. groups.

Pierre-Emmanuel Caprace gave a clear survey of recent work on automorphism groups of trees from the point of view of 2-transitive group actions on compact spaces. These groups are one of the key topics in the theory of t.d.l.c. groups and he described an emerging systematic analysis of them.

Zoë Chatzidakis presented a remarkable link between the theory of difference fields and t.d.l.c. groups in which the scale function on these groups helped to motivate the formulation of a new invariant for the fields. Her talk sparked a lot of interest and discussion among participants.

Colin Reid introduced the theory of decompositions of t.d.l.c. groups which he and Phillip Wesolek are developing. This theory can be expected to become a standard part of the analysis of t.d.l.c. groups and his talk helped to promote awareness of important new technique.

Alejandra Garrido's talk described the maximal subgroups in a class of branch groups and showed the their number is countable. This description for groups of intermediate growth breaks new ground because groups for which it had been done previously were either solvable or contained a non-abelian free subgroup.

André Nies spoke about a new direction of research inspired by the work of Dugald Macpherson and others on bi-interpretability of structures and their reducts. He addressed the question of the logical difficulty of determining conjugacy or topological isomorphism of two groups (represented as permutation groups) within a particular class. This is the same question as addressed by Caprace for automorphism groups of trees.

Dugald MacPherson described work with Cheryl Praeger and Simon Smith on subgroups of the full symmetric group on a countable set equipped with the permutation topology, which is totally disconnected. These included locally compact groups and groups which are maximal with respect to various properties and as a result his talk provoked discussions with a number of the participants.

Tim Burness gave an account of his current joint work with Martin Liebeck and Aner Shalev on generating sets for maximal and second maximal subgroups of finite almost simple groups which is yet to appear in print. His talk thus provided participants with tools and up-to-date knowledge for their own research. The talk by Inna Capdeboscq addressed similar questions for (infinite) Kac-Moody groups.

David Craven's survey of the current state of knowledge of the classification of subgroups of exceptional algebraic groups also brought participants up-to-date with that endeavour.

Gabriel Verret spoke his work on finite vertex-primitive graphs of valency 5 and 6 and there is a productive interaction between that work and David's resulting from the difficult problem of determining the conjugacy classes of extremely small maximal subgroups of the finite almost simple exceptional groups of Lie type. This problem about the nonabelian simple groups is forming the back-bone of David's recent research program (involving simple groups theory and representation theory), and some of David's results have already fed into the classification which Gabriel spoke about. A bonus from having such a diverse audience was the questions which led to research bridging the areas. In this case questions from Simon Smith about infinite vertex-primitive graphs having vertices with similar neighbourhoods prompted Gabriel to take a new look at the problem. He discovered that methods which had seemed only to work in the finite case could be modified to apply to infinite groups.

Scientific Progress and Interactions

The advances discussed and interactions which occurred at the workshop will now be reported in more detail from the points of view of the four subfields represented.

Finite permutation groups, graphs, geometries, and simple groups

Questions concerning groups, or the structures on which they act, often reduce to fundamental questions about primitive permutation groups. A pertinent example is the theme of the talk given by Michael Aschbacher inspired by the Palfy-Pudlak question: is each finite lattice an 'intermediate subgroups lattice', that is, is it isomorphic to the lattice of subgroups of a some group G which contain some subgroup H? At the conference Aschbacher considered the question for the lattice of overgroups of a given subgroup in an exceptional Lie type simple group.

In previous work of Aschbacher [1, 2] on this problem, some in collaboration with Shareshian, the context considered was sub-lattices of the overgroup lattice of a finite primitive permutation group. A major question was to decide when a given finite primitive group G is 'product indecomposable', in other words, to determine whether or not G preserves a Cartesian decomposition of the underlying point set. Knowledge of the lattice of overgroups of a given permutation group can also be important in combinatorics or geometry, for example, if the given group is a group of automorphisms of some structure and we wish to determine the full automorphism group. Csaba Schneider's talk addressed the fundamental question of finding all Cartesian decompositions preserved by a given permutation group G, in the case where G has a transitive minimal normal subgroup [21]. This theory is most powerful in the case where G is finite, as it exploits the finite simple group classification. However it was shown also to apply to a large family of infinite permutation groups; this extension was inspired by recent work of Smith [19] and by Neumann, Praeger and Smith [15].

Recent results about finite permutation groups (many still unpublished) suggest that the finite simple groups still hold secrets about simply stated properties of finite permutation groups and the structures on which they act: for example, in most primitive permutation groups each element has a regular cycle (of maximum possible length, equal to the element order), and in most primitive groups each element has order less than a quarter of the number of points; and in both cases we can describe the exceptions.

New studies of symmetric structures demand strong new theoretical tools from permutation groups. This was illustrated in Michael Giudici's talk which reported on application of the structure theory of quasiprimitive permutation groups to study s-arc transitive graphs, and showed how the subgroup structure of certain finite simple groups yielded new constructions of s-arc transitive digraphs with almost simple automorphism groups. This answered a 30 year old question of Ito and Praeger. In a similar vein, Gabriel Verret's talk explained how detailed knowledge of small maximal subgroups of simple groups of Lie type was required to complete a classification of primitive valency 5 graphs - the classification applied recent work of David Mason on maximal A_5 subgroups of $E_8(q)$. A different example was provided by Joanna Fawcett's talk on partial linear spaces. These are point–line incidence structures in which each point-pair is incident with at most one line. The most symmetrical partial linear spaces admit a group transitive on both collinear point-pairs and non-collinear point-pairs. Thus, provided the space is non-degenerate, in the sense that non-collinear point-pairs exist, such symmetrical spaces admit an automorphism group acting as a rank 3 permutation group on points. The work of Alice Devillers (2005, 2008) reduced classification of primitive rank 3 linear spaces to the case where the group is of affine type. Joanna Fawcett reported on an almost completed classification of the remaining case, the affine primitive rank 3 partial linear spaces.

From the opposite point of view, breakthroughs in combinatorics and geometric group theory lend new methods for studying group actions, and throw up new group theoretic challenges. For instance, Cai Heng Li's studies of edge-transitive embeddings of vertex-primitive graphs in compact Riemann surfaces pose new factorisation problems for finite almost simple groups. Also the exciting results of Helfgott on expansion in groups, extended by Pyber and Szabó, and independently by Breuillard, Green and Tao, have shown not only that Cayley graphs of bounded rank finite simple Lie type groups are expanders, but have also yielded proof of the Weiss Conjecture of 1978 for locally primitive graphs involving only bounded rank composition factors, [17]. This highlights the classical Lie type groups of unbounded rank as those for which greater understanding is needed.

The Weiss and Praeger conjectures for locally primitive and locally quasiprimitive graphs predict bounds, in terms of the valency, on the number of graph automorphisms fixing a vertex. More recent studies showed that the only local actions which could possibly result in such a bound are semiprimitive actions: a permutation group is semiprimitive if each normal subgroup is either transitive or semiregular. Potočnik, Spiga and Verret [16] conjecture that, for a locally semiprimitive graph of valency d, the vertex stabiliser order should be bounded in terms of d. Studies by Bereczky and Maróti [3] on semiprimitive groups give some information about them, but more information is needed about the general structure of semiprimitive groups: this is on-going work of Giudici and Morgan [9], and Luke Morgan's lecture gave a helpful report on recent progress.

Thus the talks, even in this central area of permutation groups, addressed both progress with the fundamental theory of permutation groups, and many of their applications. There were obvious bridges spanning the different areas with new questions posed, and we saw collaborations newly formed or strengthened.

Algebraic groups, subgroup structure and group actions

An effective approach to solving problems in finite group theory is to reduce to the case of quasisimple groups. The large majority of these are finite groups of Lie type, which in turn are obtained as fixed point subgroups of certain endomorphisms of reductive algebraic groups. Hence one is led to include this family of infinite groups in the study of finite groups.

One of the themes of current research on finite and infinite reductive groups aims at understanding the subgroup lattice of these groups, knowledge of which is required for the study of permutation actions of the groups. The by-now classical reduction theorems of Aschbacher and Liebeck-Seitz have motivated the study of so-called irreducible triples (X, Y, V), where X is a subgroup of Y and V is an absolutely irreducible module for Y on which X acts irreducibly. Recent progress on the classification of such triples has been achieved in the work of Husen, Hiss and Magaard (an AMS Memoir on imprimitive actions for finite groups), Burness, Ghandour, Marion and Testerman, (two AMS Memoirs on irreducible triples for classical algebraic groups), and further work of Tiep, Kleshchev, Magaard, Röhrle, Testerman, and others for certain configurations of finite irreducible triples. Magaard and Testerman have an ongoing project to extend this work and the workshop provided an occasion for discussing this work. In a related project, Liebeck, Seitz and testerman are studying multiplicity free actions of simple algebraic groups and the meeting gave Seitz and Testerman the opportinuty to make progress and to discuss their work with other participants.

In a different direction, there has been significant progress in recent work of Stewart and Thomas, Thomas and Litterick, and Craven on understanding the subgroup lattice of the exceptional type algebraic groups. Stewart and Thomas consider irreducible subgroups, that is, those not lying in any proper parabolic subgroup, and non completely reducible subgroups, that is, those which lie in a proper parabolic subgroup but do not lie in any Levi factor of the parabolic. The techniques here involve some delicate calculations in the first and second cohomology groups of the embedded subgroup, techniques which have been developed and exploited in Stewart's work on the subgroups of F_4 and the non completely reducible subgroups of the exceptional groups. Thomas and Litterick have a joint project whose aim is to classify the non-completely reducible subgroups of the exceptional groups. Litterick reported on their progress in the case of good characteristic.

During the workshop Aschbacher reported on work on the Palfy-Pudlak question for exceptional groups, concerning the lattice of overgroups of a given subgroup in an exceptional type Lie group. Craven reported on work on Lie primitive subgroups of exceptional type groups. Further work on the subgroup structure of the exceptional type algebraic groups is being carried out by Craven and independently Burness and Testerman, in particular, the existence and uniqueness of certain A_1 type subgroups. The workshop provided an opportunity for several of these people to discuss this work.

Further work on the subgroup lattice of classical type groups is related to questions about the modular representation theory of these groups. Malle reported on recent joint work with Robinson giving a conjectural upper bound for the number of irreducible characters in a p-block of a group is evidence for some of the progress in this area. This very active area of research has seen considerable progress in the past 5-10 years, and has been the topic of varous conferences, workshops and research semesters. While there is a deep connection to the subgroup structure of Lie type groups, this was not the central theme of the meeting.

There is continuing work being done on the monodromy groups of compact, connected Riemann surfaces of genus at most 2, beyond the Frohardt-Magaard proof of the Guralnick-Thompson conjecture. In particular, Frohardt, Guralnick, and Magaard have recently shown that any such configuration arising from the action of a classical group on Ω, the set of 1-spaces of its natural modules, must satisfy $|\Omega| \leq 10^4$. In two recent papers Magaard and Waldecker analyze the structure of transitive permutation groups that have trivial four point stabilizers, but some nontrivial three point stabilizer and consider permutation groups that act nonregularly, such that every nontrivial element has at most two fixed points. The applications they have in mind are to the monodromy groups problem.

A further geometric construction which provides a rich family of interesting examples is that of letting a finite group act on a product of Riemann surfaces and quotienting out by this action to obtain a so-called Beauville surface. This then translates to studying goups with a Beauville structure, which is also related to the study of triangle groups. The recent work of Larsen, Marion and Lubotzky using deformation theory essentially proves Marion's conjecture concerning triangle generation of low-rank groups of Lie type. Their work led to the notion of "saturation"; roughly, a given hyperbolic triangle group is "saturated" with finite quotients of type Φ (here Φ is an irreducible root system) if it has finite quotients isomorphic to $\Phi(p^\ell)$ for infinitely many p and ℓ. While they have established a very strong result, there remain some interesting open questions. This is related to the question of generation in finite groups; here Burness, Liebeck and Shalev have recently completed some interesting work on the number of generators of maximal, second maximal, and third maximal subgroups in finite simple or quasisimple groups. This also has some interesting connections to number theoretic questions and may open new research directions. This also highlights one of the "cross-overs" between the different topics of the conference: Capdeboscq's talk on generation questions in Kac-Moody type groups took as a starting point the classical results on generation in finite quasisimple groups.

Infinite permutation groups and model theory

What might have been somewhat surprising, but certainly hoped for, was the discovery that even though the participants belonging to different groups in many cases did not previously know each other at all, it turned out during the talks that there were many interconnections and inspiring recurrences of themes in different settings.

Over the last years there have been major developments in infinite permutation group theory. One aspect here is the interaction between permutation group theory, combinatorics, model theory, and descriptive set theory, typically in the investigation of first order relational structures with rich automorphism groups. The connections between these fields are seen most clearly for permutation groups on countably infinite sets which are closed (in the topology of pointwise convergence) and oligomorphic (that is, have finitely many orbits on k-tuples for all k); these are exactly the automorphism groups of ω-categorical structures, that is, first order structures determined up to isomorphism (among countable structures) by their first order theory.

The use of group theoretic means (O'Nan-Scott, Aschbacher's description of maximal subgroups of classical groups, representation theory) to obtain structural results for model-theoretically important classes (totally categorical structures, or much more generally, approximable structures, finite covers of well-understood structures). Recent progress by Macpherson, Kaplan and Simon towards showing that natural classes of automorphism groups are maximal closed subgroups of the symmetric group give rise to hope for a complete classification in certain settings. These questions were addressed in both the talks by Macpherson and Simon. It was very inspiring to see the progress combining aspects from very different fields. Simon presented his joint work with Itay Kaplan, showing that $AGL_n(Q), n > 1$ and $PGL_n(Q), n > 2$ are maximal amongst closed proper subgroups of the infinite symmetric group.The proof relies on Adeleke and Macpherson's classification of infinite Jordan groups.

Properties which the full symmetric group S on a countable set shares with various other closed oligomorphic groups. We have in mind such properties as: complete description of the normal subgroup structure; uncountable cofinality (that is, the group is not the union of a countable chain of proper subgroups); existence of a conjugacy class which is dense in the automorphism group, or, better, comeagre (or better still, the condition of 'ample homogeneous generic automorphisms'); the Bergman property for a group (a recently investigated property of certain groups G, which states that if G is generated by a subset S, then there is a natural number n such that any element of G is expressible as a word of length at most n in $S \cup S^{-1}$); the small index property. These questions were also addressed in Macpherson's talk.

Reconstruction of a first order structure (up to isomorphism, up to having the same orbits on finite sequences, up to 'bi-interpretability') from its automorphism group, typically, presented as an abstract group. Partially successful techniques here include the description of subgroups of the automorphism group of countable index (the 'small index property'), and first order interpretation of the structure in its automorphism group. This leads to a new, yet very natural direction of research which was presented in Nies' talk. He addressed the question of how complicated it is for a class $\mathcal{C}$ of closed subgroups of S to determine whether $G, H \in \mathcal{C}$ are conjugate or topologically isomorphic. He explained how to consider such classes of groups as Borel sets and how to study Borel reducibility in this context. Using a result of Lubotzky's he showed that the isomorphism relation between finitely generated profinite groups is Borel-equivalent to id_R and that for a prime $p \geq 3$, graph isomorphism can be Borel reduced to isomorphism between profinite nilpotent class 2 groups of exponent p.

In a more unexpected vein, Chatzidakis' talk presented yet a different connection between model theory and totally disconnected locally compact groups: if (K, f) is a difference field with a distinguished automorphism f, and a is a finite tuple in some difference field extending K, and such that $f(a)$ is algebraic over $K(a)$, then we define $dd(a/K) = \lim[K(fk(a), a) : K(a)]^{1/k}$, the distant degree of a over K. This is an invariant of the difference field extension of the algebraic closure $K(a)^{alg}$ of $K(a)$ over K. They show that there is some b in the difference field generated by a over K which is equi-algebraic with a over K, and such that $dd(a/K) = [K(f(b), b) : K(b)]$. This means that for every $k > 0$, we have $f(b)$ in $K(b, f^k(b))$. Viewing $Aut(K(a)^{alg}/K)$ as a locally compact group, this result is connected to results of Willis on scales of automorphisms of locally compact totally disconnected groups.

Totally disconnected locally compact groups

The inspiration for a number of talks at the workshop was a link between totally disconnected locally compact groups and permutation groups first seen in the paper [4] by M. Burger and S. Mozes, in which they related the structure of the group of tree automorphisms to its "local action" on the neighbours of vertices. Properties of the finite permutation group formed by this local action were thus seen to control properties of the infinite topological group of tree automorphisms.

Automorphism groups of trees that are 2-transitive on the boundary of the tree were the subject of Caprace's talk. Infinitely many topologically simple t.d.l.c. groups with flat-rank 1 arise as closed subgroups of the automorphism groups of semiregular trees and he surveyed recent classification and characterization results for these groups obtained by himself and his students Radu and Stulemeijer. The valency of the tree and properties of the local actions such as primitivity are important features in these results and establish a link with finite permutation groups. He also presented his approach to proofs of simplicity of groups through the existence of what he has called micro-supported actions of a group. This approach unifies and provides a clear insight into numerous simplicity proofs obtained in previous decades and will be an important method for future proofs.

Smith showed how to construct an uncountably infinite number of topologically simple compactly generated t.d.l.c. groups, all having flat-rank 1, using a generalization of the ideas of Burger-Mozes to invent what he calls a "box product" of permutation groups. He also described his classification of infinite primitive permutation groups having finite sub-orbits. As a variation of Cayley's Theorem shows, each compactly generated t.d.l.c. group may be represented as a permutation group with finite sub-orbits and since each such permutation group embeds densely in a t.d.l.c. group, and the topologically simple groups are an important part of his classification.

In his talk, Reid outlined a theory of decompositions of t.d.l.c. groups developed in collaboration with Wesolek. An important conclusion of the theory is that each compactly generated t.d.l.c. group may be broken down into a finite number of chief factors. The key idea, which again is based on the paper of Burger and Mozes [4] and its extension to the general situation by Caprace and Monod [6], is to represent the group as acting on a graph and to consider its local action. The point is then that, when the group is decomposed, the valency of this local action strictly decreases and the number of pieces in the decomposition must therefore be finite. Reid and Wesolek also show how to compare the factors occurring in any two decompositions of the same group.

Ramagge's talk concerned a direct link between t.d.l.c. groups and permutation groups established by Möller when he gave a new proof of basic results about the scale function and established the spectral radius formula for the scale which was referred to in the talk by Chatzidakis. She explained this link and then went on to describe current joint work with Praeger and Willis which is extending an equivalence found by Möller for a single group element to flat subsemigroups of elements. This extension involves yet another cross-over, this time to the field of operator algebras, because the relevant structure needed to establish the equivalence turns out to be that of a P-graph, a generalization of the notion of graph originally made in that field. The collaboration continued at the workshop and this work is now near completion.

Other talks on t.d.l.c. groups described recent advances on several fronts. Castellano spoke about a cohomology theory [8]. Wesolek and Garrido both gave talks about branch groups: Garrido characterizing primitive permutation actions of these groups and Wesolek showing the rigidity theorem that just-infinite groups in this class have no non-trivial commensurated subgroups. Capdeboscq talked about the size of generating sets for Kac-Moody groups over finite fields. Kac-Moody groups are closely related to algebraic groups and this work, which is related to corresponding investigations for finite simple groups and groups of Lie type, is another aspect of the connections between the different types of permutation groups.

Links between finite permutations, algebraic groups and t.d.l.c. groups were anticipated. Less anticipated were the links with model theory and infinite permutation groups which emerged. Chatzidakis described Galois groups which are t.d.l.c., thus providing a new class of these groups which remains to be explored, and how the scale function inspired results about field extensions. MacPherson also talked about how potentially new examples of t.d.l.c. groups that are infinite permutation groups might be constructed using model theoretic methods. Nies' talk on the complexity of classes of groups is relevant to attempts to classify simple t.d.l.c. groups up to isomorphism and other discussions regarding the complexity of computation in profinite groups should be relevant to attempts to evaluate the effectiveness of certain algorithms and arguments in t.d.l.c. groups with a view towards automating them.

Outcome of the Meeting

The workshop brought together both senior and mid- to early-career scientists, from North and South America, Europe, the Middle East and Oceania. Notably, of the 24 talks, 10 were given by post-doctoral fellows or early-career academics. Participants were drawn from four distinct, albeit closely related, fields and there was a chance that there would be little interaction between them. The risk that the organisers took in bringing together researchers from these distinct fields paid off in terms of transfer of knowledge between the fields and the potential for the formation of new collaborations.

Because of the nature of the conference, the majority of the audience for each talk had only a general interest in the field and only a broad knowledge of it. The talks presented significant recent advances in their respective fields, informing the experts who were present but also giving the non-experts an insight into the field. Verret, Fawcett, Schneider, Morgan and Giudici spoke about primitivity and classifications of finite permutation groups and geometries; Aschbacher, Craven, Litterick, Burness and Malle about classifications and structure of finite and, in particular, matrix groups; MacPherson, Smith, Garrido, Simon and Nies about infinite permutation groups and model theory; and Caprace, Wesolek, Castellano, Ramagge and Reid about totally disconnected, locally compact groups. There was often lively discussion following a talk, with participants from different specialties asking questions from their own perspectives. For example, questions asked by Smith about infinite graphs prompted Verret to apply methods used for finite graphs with some success, and MacPherson's talk on closed subgroups of the topological group $\mathrm{Sym}(\mathbb{N})$ provoked a lot of questions from the t.d.l.c. group community. In addition to these talks in particular fields, talks by Chatzidakis, Capdeboscq, Tiep and Segev bridged the interests of many participants, in some cases transferring ideas from one field to another and in other cases simply delivering an excellent talk for the general audience. The interest aroused by these talks is ongoing and will likely lead to future work. Chatzidakis' talk has prompted a series of working seminars on Galois groups to be held in Newcastle, Australia, for example.

Ample time was provided in the schedule of the workshop for participants to collaborate. This opportunity was taken up to established collaborations to continue existing projects, as with Testerman and Magaard for example, or to plan future work, as with Caprace, Reid and Willis for example. There were many discussion following talks, as mentioned in the previous paragraph, and new collaborations were initiated or are envisaged. Examples here are a planned collaboration between Ramagge and Garrido, and proposed exchange visits between Nies and Willis.

It is probably true to say that every participant, including the organisers, met several mathematicians, and their work, for the first time and learned many new ideas which will influence their future research.

Participants

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Chapter 20

Analytic versus Combinatorial in Free Probability (16w5025)

December 4 - 9, 2016

Organizer(s): James Mingo (Queen's University), Alexandru Nica (UWaterloo), Roland Speicher (Saarland University), Dan-Virgil Voiculescu (UC Berkeley)

Overview of the Field

Free probability theory is an area of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out of attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost thirty years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability, the theory of large deviations, and algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics and quantum information theory, as well as applications in statistics and wireless communications.

There exist two different approaches to free probability theory at a very basic level; one is analytic and the other one is combinatorial. These approaches complement each other, and in many situations it is the interaction between both of them which drives the subject forward.

A main theme of the workshop was the discussion of a number of important extensions of free probability that were studied during the recent years, and continue to provide an intense active topic of research at present. In the basic theory of free probability both the combinatorial and the analytic structure, as well as the interaction between them, are quite well understood. However, in these recent extensions of the setting of free probability, the development of these features is not yet clear; progress on these subjects will surely rely on an interaction between analytic and combinatorial considerations.

In the following we present (not exhaustively) several such developments that were covered by the workshop. Most of the sections below are based on write-ups made by workshop participants who work in those directions, and we thank Octavio Arizmendi, Serban Belinschi, Camille Male, Paul Skoufranis, and Moritz Weber for their contributions to this report.

Recent Developments and Open Problems

Bi-freeness and pairs of faces

Although the simplest way to define free independence is via the "alternating centred moments vanish" condition, the connection between free probability and operator algebras is most easily seen by defining free independence as the ability to represent algebras on reduced free product spaces using the left regular representation. As there is also a right regular representation on reduced free product spaces, in [21] Voiculescu introduced a generalization of free independence, known as bi-free independence, which is defined as the ability to represent pairs of algebras on reduced free product spaces where one algebra from each pair acts via the left regular representation and the other acts via the right regular representation. This ability to simultaneously study the left and right regular representations allows for a wider variety of behaviours to be observed and modelled thereby permitting bi-free probability to investigate problems untouched by free probability.

In Voiculescu's inception of bi-free probability [21, 23] it was demonstrated that bi-free independence is well-defined, bi-free probability has a notion of cumulants, and there exists a bi-free partial R-transform. Due to the difficulty of working with reduced free product spaces, Mastnak and Nica in [17] began the process of introducing combinatorial techniques into bi-free probability. The combinatorial approach to bi-free probability reached fruition in [7, 8] where Charlesworth, Nelson, and Skoufranis demonstrated that lattices consisting of permutations of non-crossing partitions produced the correct cumulants to characterize bi-free independence. Since free independence is characterized using non-crossing partitions, many of the combinatorial results from free probability directly had bi-free analogues.

The influx of combinatorial techniques has allowed for a rapid expansion of the theory of bi-free probability, with a flurry of activity going on throughout 2015 and 2016. Here are some recent contributions made in this direction.

- Skoufranis demonstrated that all five natural notions of independence (classical, free, Boolean, monotone, and anti-monotone) can be studied through bi-free independence.
- Freslon and Weber developed bi-free de Finetti theorems to study bi-free independence over tail algebras.
- Gu, Huang, and Mingo developed the theory of bi-free infinitely divisible distributions for bi-partite systems, which was then generalized to arbitrary systems by Gao.
- Skoufranis developed the bi-free analogue of random matrix models by demonstrating that several pairs of matrices tend to bi-free independent distributions and by characterizing the notion of bi-freely independent pairs amalgamated over diagonal matrices.

An overview of combinatorial techniques in bi-free probability was presented at the workshop by Paul Skoufranis, with ample illustrations drawn from his recent paper [18]. These combinatorial techniques will very likely continue to be important in further developing the theory of bi-free probability.

There have also been many advances via analytical techniques.

- Voiculescu developed the bi-free partial S-transform, with Skoufranis later discovering a combinatorial proof.
- Voiculescu also developed the notion of bi-free extremes in the plane, thereby computing the distributions of the maximum and minimum of bi-partite bi-free pairs.
- Dykema and Na obtained the principal function of non-normal bi-free central limit distributions.
- Huang and Wang developed analytical aspects of the bi-free partial R-transform.

At the workshop, the opening talk was given by Dan Voiculescu on his work [22] on bi-free extremes. Another talk on the analytic side of bi-free probability was given by Jiun-Chau Wang, who reported on recent joint work with Hasebe and Huang on analytical aspects of additive and multiplicative bi-free convolution.

Further developments of analytical techniques in bi-free probability are ongoing, and are likely to lead to many interesting avenues of inquiry.

One-sided free probability was known to have intimate connections to other brands of non-commutative probability, and two such connections were extended to the bi-free framework in recent work by Gu and Skoufranis.

One of these connections is with the theory of "conditional free independence" (upgraded by Gu and Skoufranis to "conditional bi-free independence"), which is a notion of independence with respect to pairs of states on an algebra of random variables; the framework of conditional bi-free independence widens even more the range of behaviours that can be displayed by two-faced systems of noncommutative random variables, and offers many new examples that are now possible to study. The other connection, which was the topic of the talk given by Yinzheng Gu at the workshop, concerns the relation of free probability to Boolean probability – now upgraded to bi-Boolean probability.

Finally, we mention the interesting point that (unlike the development of one-sided free probability, where the whole theory started from the "alternating centred moments vanish" condition), a description of bi-free indpendence in terms of moments was only found very recently by Ian Charlesworth, who reported on this in his talk at the workshop. It is still unknown to what extent the moment condition of Charlesworth extends to the operator valued setting. Nevertheless, this condition promises to offer a new approach, in future developments, for proving that pairs of algebras are bi-freely independent.

Bi-free probability is a rapidly expanding area of free probability. As there are many interesting questions and potential applications to this theory, bi-free independence will be an active area of research in free probability for the foreseeable future.

Finite Free Probability

The theory of finite free (polynomial) convolutions is very recent. It started with the paper by Marcus, Spielman, and Srivastava [16], where they established a connection between different polynomial convolutions and addition and multiplication of random matrices, which in the limit is related to free probability. The new feature of this convolution is that instead of looking at distributions of eigenvalues of random matrices, one looks at the (expected) characteristic polynomial of a random matrix.

To be precise, for a matrix M, let $\chi_M(x) = \det(xI - M)$ be the characteristic polynomial of the matrix M. Then, for $d \times d$ Hermitian matrices A and B with characteristic polynomials p and q, respectively, one defines the finite free additive convolution of p and q to be

$$p(x) \boxplus_d q(x) = \mathbf{E}_Q[\chi_{A+QBQ^T}(x)],$$

where the expectation is taken over orthogonal matrices Q sampled according to the Haar measure.

Similarly, when A and B are positive semidefinite, the finite free multiplicative convolution of p and q is defined to be

$$p(x) \boxtimes_d q(x) = \mathbf{E}_Q[\chi_{AQBQ^T}(x)],$$

where, again, the expectation is taken over random orthogonal matrices Q sampled according to the Haar measure.

Both these convolutions turn out to not depend on the specific choice of A and B, but only on p and q.

The connection with free probability is that, because of the concentration of measure phenomenon for random matrices, as $d \to \infty$, these polynomial convolutions approximate free additive convolution and free multiplicative convolution. This connection is quite remarkable since, as proved in [16], these convolutions have appeared before and there are very explicit formulas to calculate the coefficients of the finite free convolutions of two polynomials.

The original motivation in [16] was to obtain new inequalities between polynomials by using free probabilistic tools. In [16] they proved the following inequalities between the R-transform and the S-transform of the resulting polynomials:

$$R_{p \boxplus_d q}(w) \leq R_p(w) + R_q(w), \qquad S_{p \boxtimes_d q}(w) \leq S_p(w) S_q(w).$$

These inequalities are translated into inequalities of polynomials and were used, in [15], to prove the existence of bipartite Ramanujan graphs on an arbitrary number of vertices, improving on results from [14].

There is not yet much literature on finite free convolution. Indeed, apart from the original work of Marcus, Spielman and Srivastava [16], there are presently only two more papers [1, 13] on the subject.

The purpose of [13] is to make precise the above mentioned connections to free probability. In this paper, the analytic machinery of finite free additive convolution and finite free multiplicative convolution were introduced, in the sense that finite versions of Voiculescu's R transform and S-transform which "linearize" finite convolutions were found. To be precise, there are transforms R^d and S^d such that

$$R^d_{p\boxplus_d q}(w) = R^d_p(w) + R^d_q(w), \qquad S_{p\boxtimes_d q}(w) = S^d_p(w)S^d_q(w).$$

In particular, using the transform R^d, the law of large numbers, central limit theorems and law of rare events have been established for finite free additive convolution. It was an important highlight of the workshop to have Adam Marcus as a speaker presenting these exciting new developments; his presentation prompted many private discussions around further possibilities of continuing this line of research.

On the other hand in [1] the authors gave a combinatorial treatment of finite free additive convolution by introducing cumulants for finite free additive convolution and by deriving moment-cumulant formulas. These cumulants approximate free cumulants, when the degree of the polynomials tends to infinity. This allowed them in [1]

- to give criteria for infinite divisibility;
- to show that there exist $T > 1$ such that for all $t > T$, the polynomial $p^{\boxplus_d t}$ is well defined and real rooted;
- to prove the existence of a counterexample for Cramer's Theorem.

Such examples of properties, that were known before for free additive convolution and that are now shown to already appear in the level of finite free additive convolution, may lead to a better insight on free convolution. These ideas were presented in the talk given at the workshop by Octavio Arizmendi.

There are many open problems and interesting directions related to this topic, which will surely be the subject of future research. Some of them are:

- Combinatorial aspects of finite free multiplicative convolution.
- Free entropy and free Fisher information in the finite setting.
- Extensions of finite free probability, like freeness with amalgamation or second order freeness.
- Multivariate finite convolution or a notion of finite independence.

More specific problems that should serve as leading guides for understanding differences and similarities between finite free convolution and free convolution are the following.

- Give qualitative and quantitative aspects of the central limit theorem in finite free additive convolution, such as superconvergence or a Berry-Esseen Theorem.
- Describe properties of free convolution with "gaussian law", γ_d, appearing in the central limit theorem. In particular it is expected that γ_d has no multiple roots.
- Find the minimum $T > 1$ such that for all $t > T$, the polynomial $p^{\boxplus_d t}$ is well defined and real rooted.
- Describe the multiplicity of the roots of $p \boxplus_d q$ in terms of the multiplicity of the roots of p and of q.
- Improve the relation between the R-transform and finite free additive convolution as follows: for $k \in \mathbb{N}$ we expect to have
$$R^d_{p\boxplus_d q}(w) \leq R^{kd}_{p^k}(w) + R^{kd}_{q^k}(w),$$
and similarly for the S-transform and the finite free multiplicative convolution.

Spectral theory for large random matrices.

One of the most important results in free probability is the asymptotic freeness of large unitarily invariant random matrices, obtained by Voiculescu in his 1991 paper [20]. The result roughly states that if $\{A_1(N), \ldots, A_j(N)\}$

are properly normalized $N \times N$ independent Gaussian random matrices or independent Haar distributed unitary random matrices and $(D_1(N), \ldots, D_k(N))$ is a k-tuple of deterministic $N \times N$ diagonal matrices which converges in distribution to a k-tuple $(d_1, \ldots, d_k)$, then there are noncommutative random variables $a_1, \ldots, a_j$ such that $\{A_1(N), \ldots, A_j(N), (D_1(N), \ldots, D_k(N))\}$ converges in distribution as N tends to infinity to $\{a_1, \ldots, a_j, (d_1, \ldots, d_k)\}$, and the sets $\{a_1\}, \ldots, \{a_j\}, \{d_1, \ldots, d_k\}$ are free. This result has been strengthened numerous times by, among others, Voiculescu, Speicher, Mingo-Speicher, Haagerup-Thorbjørnsen, Male, Collins-Male. One obvious candidate for improvement in the above result is the quality of convergence. The most spectacular first result in this direction is due to Haagerup and Thorbjørnsen [9]: they show that if $\{A_1(N), \ldots, A_j(N)\}$ are properly normalized independent Gaussian random matrices and $P = P^* \in \mathbb{C}\langle X_1, \ldots, X_j\rangle$ is a selfadjoint polynomial in j non-commuting selfadjoint indeterminates, then $\|P(A_1(N), \ldots, A_j(N))\|$ converges almost surely to $\|P(a_1, \ldots, a_j)\|$, where $a_1, \ldots, a_j$ are free identically distributed semicircular random variables. This kind of convergence is referred to as <u>strong convergence</u>, and the strong convergence to a free family of random variables as <u>strong asymptotic freeness</u>. Male improved this result to include strongly converging deterministic matrices: if the k-tuple $(D_1(N), \ldots, D_k(N))$ of deterministic selfadjoint $N \times N$ matrices (not necessarily diagonal) converges <u>strongly</u> to $(d_1, \ldots, d_k)$ as $N \to \infty$, then so does the family $\{A_1(N), \ldots, A_j(N), (D_1(N), \ldots, D_k(N))\}$ to the family $\{a_1, \ldots, a_j, (d_1, \ldots, d_k)\}$. A further extension to include independent Haar unitary random matrices has been obtained in joint work by Collins and Male.

The recent paper [3], presented by Mireille Capitaine at the workshop, extends the above result of Male to Wigner matrices, i.e. selfadjoint random matrices X whose entries satisfy the conditions that X_{ii}, $1 \leq i \leq N$, $\sqrt{2}\Re X_{il}$, $\sqrt{2}\Im X_{il}$, $1 \leq i < l \leq N$, are all centered, independent, identically distributed, and of variance $1/N$. Gaussian random matrices are a particular case of Wigner matrices. The main result of [3] can be stated as follows.

Theorem: Assume that $\{A_1(N), \ldots, A_j(N)\}$ are independent $N \times N$ Wigner matrices whose entries have finite fourth moments. Let $D_1(N), \ldots, D_k(N)$ be deterministic $N \times N$ matrices and assume that $\{D_1(N), \ldots, D_k(N), D_1(N)^*, \ldots, D_k(N)^*\}$ converges strongly to a $2k$-tuple of bounded random variables. Then $\{A_1(N)\}, \ldots, \{A_j(N)\}$, $\{D_1(N), \ldots, D_k(N), D_1(N)^*, \ldots, D_k(N)^*\}$ are strongly asymptotically free.

The requirement of finite fourth moments can be replaced by a slightly weaker, but more technical condition. The proof involves a fairly broad array of methods and tools coming from both classical and free probability, matrix theory and analytic noncommutative functions theory. The main steps involve a truncation of the entries of the Wigner matrices (idea due to Bai-Yin and Bai-Silverstein), a linearization trick that reduces the study of noncommutative polynomials of arbitrary degree and complex coefficients to the study of linear polynomials with coefficients that are $n \times n$ selfadjoint matrices for a fixed n depending on the polynomial (this idea was first introduced in free probability by the above-mentioned work of Haagerup and Thorbjørnsen [9], but was already well-known in other fields) and an application of some analytic properties of Voiculescu's operator-valued subordination functions.

As byproducts of the proof of the main result, a spectral inclusion property is also obtained. It can be roughly outlined as follows: assume that $\{a_1, \ldots, a_j\}$ are free semicircular random variables and the k-tuple $(d_1(N), \ldots, d_k(N))$ is free from $\{a_1, \ldots, a_j\}$ for all $N \in \mathbb{N}$. Assume also that $(d_1(N), \ldots, d_k(N))$ has the same distribution as the constant selfadjoint matrices $D_1(N), \ldots, D_k(N) \in M_N(\mathbb{C})$. If $P = P^* \in \mathbb{C}\langle X_1, \ldots, X_{j+k}\rangle$ is such that for all N sufficiently large the spectrum of $P(d_1(N), \ldots, d_k(N), a_1, \ldots, a_j)$ does not intersect the interval $[b, c]$, then for any $\delta > 0$, the spectrum of $P(D_1(N), \ldots, D_k(N), A_1(N), \ldots, A_j(N))$ does not intersect $[b + \delta, c - \delta]$ almost surely as $N \to \infty$. This yields also a characterization of the outliers generated by spikes of the constant matrices.

In this context, it is worth to point out that it is a quite surprising, and also very exciting, realization in the last couple of years (see, eg, [4]) that free probability is able to address also asymptotic properties of special single eigenvalues; this is surely a direction which will be followed up in the future.

Traffics and their independence

Random matrices are important in free probability since canonical models of random matrices are free in the large dimension limit. The most basic of these results is that a collection of independent Wigner matrices converges to free semicircular variables, see [20]. Nevertheless, the notion of non-commutative distribution is sometimes too restrictive to treat certain models of large matrices. For any $N \geq 1$ let $X_{\ell,N}, \ell = 1, \dots, L$, be independent symmetric random matrices with i.i.d. sub-diagonal entries distributed according to the Bernoulli distribution with parameter $\frac{c_\ell}{N}$, $c_\ell > 0$. Then $(X_{\ell,N})_{\ell=1,\dots,L}$ converges toward non-free random variables [12]. If moreover Y_N is a sequence of deterministic matrices, then the possible limiting distributions of $X_{\ell,N}$ and Y_N depend on more than the limiting non-commutative distribution of Y_N. A similar problem appears when considering matrices $Z_{\ell,N} = V_{\ell,N} A_{\ell,N} V_{\ell,N}^*$, $\ell = 1, \dots L$, where the matrices $V_{\ell,N}$ are independent random permutation matrices uniformly chosen. When $A_{\ell,N}$ are diagonal, so is $Z_{\ell,N}$ and so $(Z_{\ell,N})_{\ell=1,\dots,L}$ cannot be asymptotically free.

It was in order to remedy this point that Male [11] introduced another non-commutative framework: the traffic spaces, which come with an associated notion of distribution and of independence. Traffics are non-commutative variables, with additional structure, given by a generalization of polynomials called graph polynomials. In particular, given a collection A_ℓ, $\ell = 1, \dots, L$, of N by N random matrices, the traffic distribution of $\mathbf{A}_N = (A_\ell)_{\ell=1,\dots,L}$ encodes the data, for any finite connected graph $T = (V, E)$ and any map γ from E to $\{1, \dots, L\}$, of the observable

$$\tau_{\mathbf{A}_N}(T, \gamma) = \mathbb{E}\Big[\frac{1}{N} \sum_{\phi: V \to \{1,\dots,N\}} \prod_{e=(v,w)\in E} A_{\gamma(e)}\big(\phi(w), \phi(v)\big)\Big].$$

For instance let T be a simple cycle, with edges $e_p = (v_p, v_{p+1})$, $p = 1, \dots, n$ where $v_{n+1} = v_1$ and $v_1, \dots, v_n$ are pairwise disjoint. Then $\tau_{\mathbf{A}_N}(T, \gamma)$ is the normalized trace $\mathbb{E}\, \frac{1}{N}\mathrm{Tr}$ of the monomial $A_{\gamma(1)} \cdots A_{\gamma(n)}$. For another example, if T denotes a graph consisting of two simple cycles that are connected at one vertex, then $\tau_{\mathbf{A}_N}(T, \gamma)$ is the normalized trace of the the entry-wise product of monomials. The convergence in traffic distribution of $\mathbf{A}_N$ is the pointwise convergence of $\tau_{\mathbf{A}_N}$.

Traffic distributions come together with a notion of independence, which is more complicated to introduce since it involves non-algebraic (combinatorial) formulas. However, this notion of independence allows one to describe uniquely the joint traffic distribution of several families of variables, in terms of the separate distributions of those families. Traffic-independent variables can be freely independent, but in general we obtain a different relation.

Traffic independence applies to a large class of random matrices:

- Independent matrices with i.i.d. entries that converge in non-commutative distribution (for instance when the entries are truncated heavy tailed variables) are asymptotically traffic-independent [12]. Their limiting distribution is characterized by a kind of Schwinger-Dyson system of equations.

- Traffic is the good notion at the intersection of free probability and random graph theory. For sparse random graphs (when the degree of a generic vertex is bounded), the traffic convergence of the adjacency matrix of a graph is equivalent to the so-called local weak convergence of Benjamini-Schramm [11]. Independent adjacency matrices are asymptotically traffic-independent, but not asymptotically freely independent in general.

- A class of large graphs with large degree was studied in joint work of Male and Peché. In particular adjacency matrices of independent regular graphs uniformly chosen with growing degree are asymptotically freely independent.

- In 2015, Gabriel introduced the concepts of $\mathcal{P}$-algebras and of $\mathcal{P}$-independence, which slightly generalized traffic independence, and he defined a notion of cumulant for traffic distributions. His work implies a result of convergence for the additive and multiplicative matricial Lévy processes invariant in law by conjugation by permutation matrices, and then for random walk on the symmetric group.

- Cébron, Dahlqvist and Male studied in [5] canonical constructions associated to traffic spaces. In particular they proved the positivity of the free product of distributions. They also proved that every non-commutative probability space can be endowed with a structure of traffic space. Benson Au and Camille Male are presently studying the non-commutative structure of this canonical construction; both of them presented their current work in talks at the workshop.

Free quantum groups

The study of symmetries has always been a central topic in the history of mathematics. Since about two hundred years ago, symmetries are mostly modelled by actions of groups. However, modern mathematics requires an extension of the symmetry concept to highly non-commutative situations. This was the birth of quantum groups in the 1980's due to the pioneering work of Drinfeld and Jimbo in the purely algebraic setting, and Woronowicz in the topological one. The latter one is more relevant for this workshop. His approach to quantum groups is based on the concept of "non-commutative function algebras" by Gelfand-Naimark, using C^*-algebras as underlying algebras. Main features of compact quantum groups are:

- Every compact group is a compact quantum group, but the converse is not true; hence compact quantum groups are honest generalizations of compact groups.
- Every compact quantum group possesses a Haar state. This is an analogue of the well-known result that every compact group admits a Haar measure. Hence, Haar integration is possible both on compact groups as well as on compact quantum groups.
- We have a quantum version of Schur-Weyl or Tannaka-Krein duality. Given a compact matrix quantum group, its space of finite-dimensional unitary representations is a tensor category of a certain kind; conversely, to any such tensor category we may find a universal compact matrix quantum group whose representation theory is exactly given by this tensor category.

In the past few decades, the investigation of compact quantum groups has developed into a rapidly growing field of mathematics with many links to other domains. The link to free probability is mainly given by compact matrix quantum groups of combinatorial type. Their construction relies on the Tannaka-Krein duality. The main idea is to identify a certain set of combinatorial objects equipped with operations resembling those of a tensor category; then to associate a tensor category to it; and then to obtain a compact quantum group by the Tannaka-Krein Theorem. The examples of combinatorial type discussed in the workshop were the following.

1. The so called Banica-Speicher quantum groups (also called easy quantum groups), were introduced in 2009 in [2]. Given a partition p of the set $\{1,\ldots,k+l\}$ into disjoint subsets (called blocks), we may associate a linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ to it by sending a basis vector $e_{i_1} \otimes \cdots \otimes e_{i_k}$ to the sum over all $e_{i_{k+1}} \otimes \cdots \otimes e_{i_{k+l}}$ such that the indices $(i_1,\ldots,i_{k+l})$ match with the partition p in a suitable way (i.e. $i_s = i_t$, if s and t are in the same block of p). If a set of partitions is closed under certain natural operations, the linear span of the associated maps T_p forms a tensor category in Woronowicz's sense, and we obtain a compact matrix quantum group, a so called Banica-Speicher quantum group. The definition of Banica and Speicher (requiring self-adjoint entries of the matrices) has been extended by Tarrago and Weber to unitary easy quantum groups (non-selfadjoint case).

   Easy quantum groups are linked to free probability theory by de Finetti theorems: Köstler and Speicher proved in [10] a characterization of free independence by distributional invariance under the quantum symmetric group S_n^+. This is parallel to the classical de Finetti theorem, where independence is characterized by distributional invariance under the symmetric group S_n. This result has been extended also to other quantum groups, see also the case of Boolean independence.

2. Spatial partition quantum groups were introduced by Cébron and Weber in 2016 in [6]. The idea is similar to the one of Banica-Speicher quantum groups, but the underlying objects are three-dimensional partitions rather than two-dimensional ones. This allows for finding new examples of quantum subgroups of Wang's free orthogonal quantum group; there are new kinds of products of quantum groups coming from new products of categories of partitions; and there is a quantum group interpretation of certain categories of partitions which do neither contain the pair partition nor the identity partition.

3. Partially commutative quantum groups have been introduced by Speicher and Weber in 2016 in [19]. They fit with the mixtures of classical and free independence by Młotkowski and Speicher-Wysoczański. The main idea of partial commutation is the following. Given a symmetric matrix $\varepsilon = (\varepsilon_{ij})_{i,j=1}^n$ with $\varepsilon_{ij} \in \{0,1\}$

and $\varepsilon_{ii} = 0$, two coordinates x_i and x_j shall commute if and only if $\varepsilon_{ij} = 1$. Imposing such partial commutation relations has a long history in various contexts, for example, on the level of groups partial commutation relations have been studied extensively under names such as "right angled Artin groups", "free partially commutative groups", or "graph groups"; on the level of monoids "Cartier-Foata monoids", "trace monoids" are common notions; there is also the general notion of a "graph product of groups" introduced by Green in the 1990's; recently, a corresponding version of a graph product for von Neumann algebras was introduced and investigated by Caspers and others.

A quantum probabilistic version of the idea of imposing partial commutation relations is given by the notion of Λ-freeness. This concept was defined by Młotkowski in 2004 and revived and refined by Wysoczański and Speicher in 2016. This mixture of classical and free independences goes as follows. Let $\varepsilon = (\varepsilon_{ij})$ be a symmetric matrix as above. If variables $x_1, \ldots, x_n$ are ε-independent, then:

- x_i and x_j are free in the case $\varepsilon_{ij} = 0$
- x_i and x_j are independent in the case $\varepsilon_{ij} = 1$ (in particular, $x_i x_j = x_j x_i$ in this situation).

If all entries of ε are zero, we obtain Voiculescu's free independence; if all non-diagonal entries of ε are one, we obtain classical independence. The notion of ε-independence arises naturally in the context of graph products of groups by considering subgroups with respect to the canonical trace state.

The partially commutative quantum groups defined by Speicher and Weber in [19] relate to Λ-freeness in a de Finetti sense, hence characterizing the distributional symmetries related to Λ-freeness. However, the study of these quantum groups reveals many other aspects. For instance:

- they act maximally on noncommutative spheres with partial commutation relations for the coordinates;
- they provide new quantum versions of the orthogonal group;
- they contain Bichon's quantum automorphism groups of graphs as quantum versions of the symmetric group.

A survey of the ideas mentioned above was presented in the talk given at the workshop by Moritz Weber. Some further recent developments concerning Λ-freeness were presented by Frédéric Patras, who reported a recent work on this topic done jointly with Ebrahimi-Fard and Speicher.

New developments around older concepts

Second order freeness

Second order freeness was introduced by Mingo and Speicher around 2003, and has had some impact on describing global fluctuations of random matrices. Second order freeness should be seen as a refinement of the question about global eigenvalue distributions of random matrices. (The later can be described by the usual, or first order, free probability theory.) Whereas we have by now a quite advanced and satisfactory combinatorial theory of second order freeness (by Collins, Mingo, Sniady, Speicher), we have no good grasp on positivity problems in this context. In particular, we do not know a satisfying answer to the relevant "moment problem" here, i.e., to the question which fluctuations can really arise in a random matrix context. Building an analytic theory of the Cauchy transforms for this theory will be an important step. As a long term goal one also hopes to find an operator-valued version of second order freeness, which would be of relevance for dealing with fluctuations of more general classes of random matrices. A first promising step in this direction was reported at the workshop by Mario Diaz.

Free stochastic analysis

Free stochastic analysis, the foundation of which was laid quite early in the development of free probability, has become a very active area in recent years. On one hand, it can be seen as the large N limit of stochastic analysis on $N \times N$ matrices – and many new results on making this rigorous for various quantities have been derived by Cébron, Kemp, and Ulrich, following the basic work of Biane from the 90s. On the other hand, working directly with the free version provides powerful tools via stochastic differential equations to investigate the structure of operator algebras which are given as the large N limit of random matrix models. We mention here just recent break-throughs by Guionnet and Shlyakhtenko and by Dabrowski, which resulted, amongst others, in the solution of the 20 year old problem on the isomorphism of the q-deformed free group factors. In another direction, work of Kemp, Nourdin, Peccati, and Speicher transferred theorems from classical stochastic analysis to the free setting, thus giving criteria for the convergence of a sequence of variables, under some constraints (for example living in a fixed chaos with respect to free Brownian motion) to specific limits, like the semicircular distribution. These results rely on a mixture of diagrammatic and analytic considerations. However, quantitative versions of these convergence results are not yet well understood and we are, for example, in need of a better understanding of adequate notions of distances between non-commutative distributions. Refinements of the technical tools that are being used (like free Malliavin calculus) will also be instrumental for making more precise statements about the regularity of the involved non-commutative distributions. First progress on such questions, namely the absence of atoms for various distributions, was achieved recently by Shlyakhtenko and Skoufranis and by Mai, Speicher, and Weber. Another interesting progress on such questions, on improving the free logarithmic Sobolev inequality, was reported at the workshop by Brent Nelson.

Outcome of the Meeting

The topics highlighted above (which cover not all, but a substantial part of the workshop programme) indicate very clearly that free probability theory, having reached a quite mature status in some of its original directions, is rejuvenating itself with the emergence of new problems, ideas, and concepts.

Free probability is a very active area, with many unsolved problems ahead, as well as various recent new exciting developments. This meeting brought together various mathematical backgrounds – in particular, analytic and combinatorial – with an emphasis on the connections. The meeting was very timely and useful, and will surely have a strong impact on the further developments of free probability and related subjects.

The organisers were in particular pleased to observe the large number of young people interested in the subject; many of them have already made substantial contributions (as witnessed by the large number of talks given by junior researchers) and will surely continue to advance the subject.

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Two-day Workshop Reports

Chapter 21

Big Data Tsunami at the Interface of Statistics, Environmental Sciences and Beyond (16w2669)

March 11 - 13, 2016

Organizer(s): Yulia R. Gel (University of Texas at Dallas), Vyacheslav Lyubchich (University of Maryland Center for Environmental Science), L. Leticia Ramirez Ramirez (Centro de Investigacion en Matematicas)

Overview of the Field and Recent Developments

The rampant growth of digital technologies and information storage have revolutionized the volume, velocity and variety of collected information, leading to the so-called "Big Data" paradigm. In turn, this alters the way in which scientists sense and analyze the available information, and ignites the interest in Big Data phenomenon virtually everywhere, from climate research to omics studies to business analytics.

One of the greatest challenges of massive data (and the one that typically constitutes the primary focus of all Big Data workshops) is an increasing demand for developing innovative computational methodology and more powerful computational tools. What is typically less discussed at many computational workshops on Big Data is the whelm of ground-breaking new interdisciplinary links that emerge from these massive information volumes. The primary goal and the key difference of this workshop from other Big Data events is that it aimed to bridge together disciplines and methodologies that typically never meet and interact at other conferences and scientific gatherings but which are in fact intrinsically close. In particular, the workshop highlighted three tightly woven themes: climate, infectious epidemiology and social media; weather, climate and complex socio-ecological networks, and climate change vulnerability, risk mitigation and adaptation.

The speakers presented a variety of modern statistical and machine learning methods to tackle big data in a spatio-temporal context, including such approaches as:

- causal discovery;
- Bayesian networks;
- dynamic networks and graphs;
- statistical compression;
- statistical downscaling and data assimilation;

- distributional calibration;
- penalized likelihood.

Presentation Highlights

Saturday, March 12: The presentations in the morning session focused on the topic of *methods for big data in climate science*. The conference was opened by talks of **Imme Ebert-Uphoff** and **Dorit Hammerling**. The first speaker introduced a new framework for constructing climate networks [9] based on causal relationship, competing with the existing methods based on correlations, mutual information, and phase synchronization. One of the main advantages of the new approach is the ability to discover patterns of climate interactions [2]. The second speaker presented and illustrated a new method for high-performance computing in climate sciences, the multi-resolution approximation (MRA).

Presentations by **Joe Guinness** and **Daniel Griffith** continued the session. Guinness targeted the emerging problem of increasing memory requirements for the data storage, by suggesting a framework to compress the data in a form of statistical model with all the parameters obtained from original data. The decompression would result in a surrogate data set with the same statistical qualities, or a model-generated sample. Griffith closed the session with a talk on spatial correlations and uncertainties associated with remotely sensed data [3], which can be seen as an extension of spatial statistics to the remote sensing work of [1].

The first afternoon session incorporated three presentations on forecasting of infectious diseases, highlighting different transmission pathways of diseases and analysis outcomes. **Teresa Yamana** presented extensions of their work with J. Shaman on probabilistic prediction of seasonal influenza outbreaks that is rooted in ensemble methodology for weather and climate forecasting [10]. **Lilia Leticia Ramirez Ramirez** discussed new results on probabilistic forecasting of influenza, based on initializing epidemic models on networks of contacts [7] with online social media. **Chris McMahan** developed Bayesian hierarchical space-time methods for long-term prediction of zootonic and vector-borne diseases using vaccination data in the conterminous United States [6].

The conference continued with presentations by **Ola Haug** and **Andrew Finley**. Haug discussed the application of climate model outputs in forecasting of insurance claims [8], and stressed the demand for reliable procedures to calibrate the climate model outputs, for each spatial location. In a case study of forest biomass prediction across Alaska, Finley showcased a highly scalable Nearest Neighbor Gaussian Process (NNGP) to provide model-based spatial inference within a hierarchical modeling framework.

The last session of the day featured **Elizabeth Martinez-Gomez** and **Robert Lund** who focused on the big data problems in particular applications. Martinez-Gomez discussed the data volumes produced by modern telescopes and statistical challenges [5] associated with timely analysis of the large amounts of data, and utilization of the information for comprehensive investigations of the Universe. Lund described a novel approach of identifying regime shifts, coupled with a genetic algorithm to more efficiently analyze long climate time series [4].
Sunday, March 13: The final morning was filled with informal discussions.

Outcome of the Meeting

The main goal of the workshop was to bring researchers tackling the big data problems at the interface of statistics and the broad field of environmental science. The meeting presented a unique opportunity to exchange the ideas and methodological advancements across the countries (Canada, Mexico, Norway, and USA) and the areas of applications (climate science, astronomy, disease surveillance, remote sensing, and insurance).

Participants

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Chapter 22

Robustness Theory and Methodology: Recent Advances and Future Directions (16w2693)

September 2 - 4, 2016

Organizer(s): Giseon Heo (University of Alberta)

Overview of the Field

While Box [1] used the word robust for the first time to mean insensitivity to the violation of assumptions, Tukey [2] and Huber [3], in the 1960s, made substantial contributions to establish robustness as a major sub-discipline of statistics. John Tukey at Princeton University invited Peter Bickel, Peter Huber and Frank Hampel in 1970-71 to make cooperative efforts for the further progression and development of robust statistics. This period was later called the Princeton robustness year and became known for its extensive Monte Carlo study of robust location estimates under symmetric long-tailed distributions [4]. The team focused on the theory of robust estimation based on specified properties and estimators under certain conditions. Since then, many review articles (e.g., Stigler [5], 2010) and books (e.g., Rieder [6], 2012) have been published and the definition of robustness largely refined. One more recent definition of robustness might be the "stability theory of statistical procedures [that] systematically investigates the effects of deviations from modeling assumptions on known procedures and, if necessary, develops new, better procedures" [7].

Recent Developments and Open Problems

Robustness has spread widely to other fields of statistics in the robust design of experiments, correction of misspecified regression functions, violation of error structures, and furthermore against wild observations or misspecified underlying distributions. Recently, robustness has expanded even further to the robust design of experiments for quantile regression and machine learning for robust active learning [8].

Presentation Highlights and Scientific Progress

The first speaker, Xiaojian Xu, presented on the historical development of robust statistics as a sub-discipline of statistics. Some landmark articles and studies for such developments were emphasized. In particular, the advances of robustification at the stage of experimental design were discussed and categorized. Xu also summarized Wiens' contributions to robust statistics, distribution theory and robustness, robust optimal designs [9], [10], robust sampling designs [11], [12], and robust active learning [8].

Xu's talk was followed by Julie Zhou, who discussed mimimax regression designs and challenges. Considered was a true regression model,

$$y_i = g(\mathbf{x}_i;\theta) + f(\mathbf{x}_i) + \varepsilon_i,\;\; i = 1, \cdots, n,$$

where $f \in \mathcal{F}$ is the unknown disturbance function (a neighbourhood of functions), $E(\varepsilon_i) = 0$, and $\mathrm{Cov}((\varepsilon_1, \cdots, \varepsilon_{\mathrm{n}})^\top) = \Sigma$. Let $\mathrm{MSE}(\xi, f, \Sigma)$ be the mean squared matrix of an estimator $\hat{\theta}$ of θ and $\phi(\xi, f, \Sigma)$ a scalar function of $\mathrm{MSE}(\xi, f, \Sigma)$, where ξ is the design distribution of $\mathbf{x}_1, \cdots, \mathbf{x}_n$. Zhou formulated three minimax design problems,

$$\min_{\xi} \max_{f \in \mathcal{F}} \phi(\xi, f, \sigma^2 I),$$

$$\min_{\xi} \max_{\Sigma \in D_\Sigma} \phi(\xi, 0, \Sigma),$$

$$\min_{\xi} \max_{f \in \mathcal{F}} \max_{\Sigma \in D_\Sigma} \phi(\xi, f, \Sigma),$$

where D_Σ is a neighbourhood of covariance matrices. The challenges presented by these problems lie in the fact that their objective functions may not be convex nor smooth and it is thus difficult to construct minimax designs.

Rui Hu, the "youngest" of Wiens' PhD students, presented her research on "maxmin" designs (which are unlike the traditional minimix designs). She considered discriminating two rival models, f_0 and f_1, in hypothesis testing, namely,

$$H_0 : f_0(y|\mathbf{x}, \mu_0) \;\; \text{versus} \;\; H_1 : f_1(y|\mathbf{x}, \mu_1),$$

where y is the response variable and $\mathbf{x}$ is the covariate vector. Hunter and Reiner [13], in 1965, constructed optimal designs when both f_0 and f_1 are fixed normal densities, and many researchers have since extended these classical optimal designs. In particular, López-Fidalgo et al [14], 2007, proposed a classical optimal design when the two rival models are non-normal. All of these models, however, assume that one of f_0 and f_1 is the true model. Hu and Wiens [15], in 2016, proposed a robust optimal design by assuming that the correct model lies within only an approximately known class (Hellinger neighborhood) of $f_0 \in \mathcal{F}_0$ or $f_1 \in \mathcal{F}_1$. Robust design is to find an optimal design measure ξ that maximizes the minimum power of the test over $\mathcal{F}_1$. Hu presented an analytical solution and the maximization portion was solved by proposing a sequential design. She also showed that this sequential design is indeed optimal. Hu and Wiens [15] further considered constructing a robust optimal design for discriminating between two models. A current open problem is an extension of their work to more than two rival models.

Zhou's student, Lucy Gao, proposed an extension of the distributionally robust logistic regression [16] to multinomial logistic regression [17]. Gao defined a distributionally robust optimization problem as minimizing a convex loss function over logistic regression coefficients on a family of probability measures within the Wasserstein metric ε. She also highlighted a method of estimating the misclassification rate of robust multinomial logistic regression by solving a tractable convex optimization problem and considering the best and worst risk scenarios.

Xiaojian Xu wrote in her abstract that "since statistics is the science of a process of data collecting, data analyzing, drawing conclusions, and (re)evaluating the process, the consideration of robustification can encompass any stage of a statistical process." Indeed, Matthew Pietrosanu, an undergraduate student of Heo, alluded to the effectiveness of robustness in estimating high dimensional surfaces from noisy point cloud data.

Outcome of the Meeting

Four academic generations participated the workshop, namely, John Collins (Wiens' PhD supervisor), Doug Wiens, Wiens' students, and their students. This fourth generation is not only continuing to solve fundamental research problems in robustness but is also branching its applications to shape analysis and statistical machine learning. Zhichun Zhai and Doug Wiens plan to lay groundwork in the applications of robust theory in machine learning. During the workshop, Xiaojian Xu, Linglong Kong, and Doug Wiens also discussed some new research problems on the robust designs of experiment for composite quantile regression.

In many statistical problems including robustness theory, it is common to fix certain parameter(s) before solving a given optimization problem. In recent years, it is becoming more popular to study the trend of solutions as the parameter(s) vary. In robustness theory, these problems are often formulated in a neighborhood of certain functions and/or spaces of probability measures. The size of the neighborhood ε must be set before solving a given loss function. However, the size ε can be built into the optimization problem itself. Rather than treating ε as fixed, one can view the pattern of optimization solutions as ε varies. During the workshop, a number of the researchers have observed this phenomenon. Julie Zhou and Giseon Heo plan to meet in February 2017 to discuss novel ways to view robust regression problems.

Participants

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Focused Research Group Reports

Chapter 23

The inverse eigenvalue problem of a graph (16frg677)

June 5 - 12, 2016

Organizer(s): Shaun Fallat (University of Regina), Leslie Hogben (Iowa State University), Bryan Shader (University of Wyoming, participated remotely)

Overview of the Field

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering, and refer to determining all possible lists of eigenvalues (spectra) for matrices fitting some description. The inverse eigenvalue problem of a graph refers to determining the possible spectra of real symmetric matrices whose pattern of nonzero off-diagonal entries is described by the edges of a given graph (precise definitions of this and other terms are given in the next paragraph). This problem and related variants have been of interest for many years and were originally approached through the study of ordered multiplicity lists. It was thought by many researchers in the field that at least for a tree T, determining the ordered multiplicity lists of T would suffice to determine the spectra of matrices described by T. When it was shown in [2] that this was not the case, the focus of much of the research in the area shifted to the narrower question of maximum multiplicity, or equivalently maximum nullity or minimum rank of matrices described by the graph. While the maximum multiplicity has been determined for many families of graphs, including all trees, in general it remains an open question and active area of research (see [4, 5] for extensive bibliographies). More recently, there has been progress on the related question of determining the minimum number of distinct eigenvalues of matrices described by a given graph [1, 3]. Maximum nullity, minimum number of distinct eigenvalues, and ordered multiplicity lists all provide information that can in some cases be used to solve the inverse eigenvalue problem for a specific graph, but the question of the structures or properties that are necessary to allow this to be done more generally is fundamental.

For a (simple, undirected) graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set E, the set of symmetric matrices described by G, $\mathcal{S}(G)$, is the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E$. The maximum multiplicity of G is

$$M(G) = \max\{\operatorname{mult}_A(\lambda) : A \in \mathcal{S}(G),\ \lambda \in \operatorname{spec}(A)\}$$

and the minimum rank of G is $\operatorname{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}$. It is easily seen that $M(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$ and $\operatorname{mr}(G) + M(G) = |G|$, where $|G|$ is the number of vertices of G. For a real symmetric matrix A, $q(A)$ denotes the number of distinct eigenvalues of A and for a graph G, the minimum number of distinct eigenvalues of G is $q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}$. Given a real symmetric matrix A with distinct eigenvalues $\lambda_1 < \cdots < \lambda_r$ having multiplicity m_i for $\lambda_i, i = 1, \ldots, r$, the ordered multiplicity list of A is $(m_1, \ldots, m_r)$. For a graph G, the set of ordered multiplicity lists of G is the set of all ordered multiplicity lists of matrices $A \in \mathcal{S}(G)$.

Observe that $q(G)$ is the minimum number of entries in an ordered multiplicity list of G and $M(G)$ is the maximum value of an entry in an ordered multiplicity list of G.

Recent Developments

At a research group meeting in Iowa in July 2015 attended by most of the participants of this FRG, several new tools were developed to attack the inverse eigenvalue problem of a graph [3]. These include the Strong Spectral Property (SSP) and Strong Multiplicity Property (SMP), matrix properties that generalize the Strong Arnold Property (SAP) (see [3] for precise definitions of the SSP and the SMP, and [6] for the SAP). All graphs having $q(G) \geq |G| - 1$ were characterized in [3].

Scientific Progress Made

While at BIRS we established several additional tools for the inverse eigenvalue problem of a graph: We defined and proved precise forms of minor monotonicity of SMP and SSP that preserve the multiplicities or eigenvalues of the minor in the larger graph but have some restrictions on additional multiplicities/eigenvalues. For matrices with SSP, we developed a method to produce another matrix with the same spectrum except one of the multiple eigenvalues has been split into one or more distinct eigenvalues.

We used minor monotonicity and subgraph monotonicity established in [3] to solve the inverse eigenvalue problem for graphs of order 5 (4 or less was known previously), and to determine the forbidden minors for graphs having at most one multiple eigenvalue; the latter are shown in Figure 23.1.

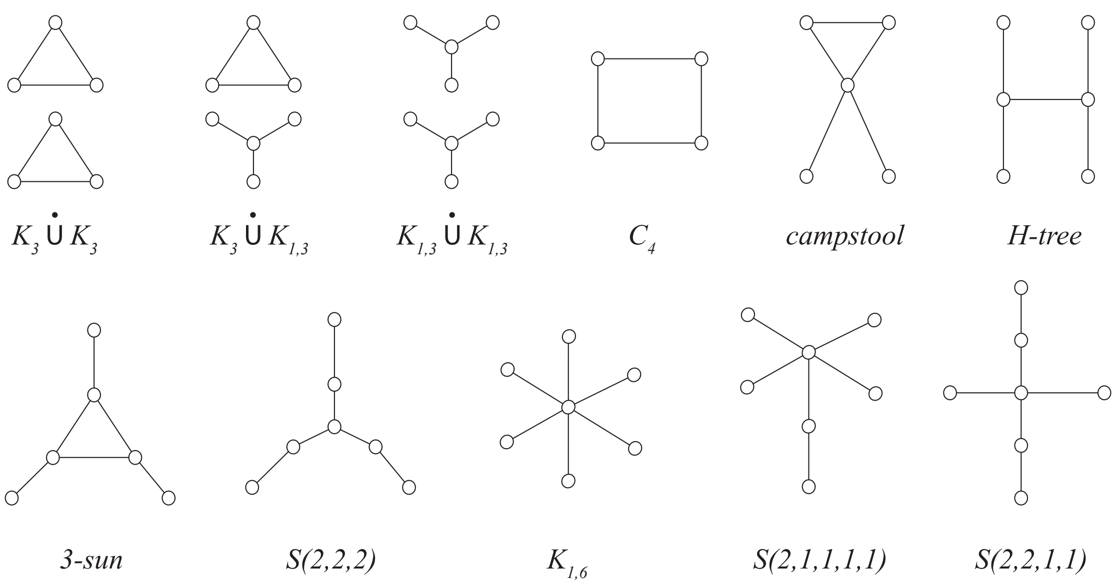

Figure 23.1: Any graph that has at least two multiple eigenvalues must have one of these 11 graphs as a minor.

Outcome of the Meeting

We are in the process of preparing a paper for submission to a journal describing the research discussed in Section 23; we expect to post a draft on arxiv by the end of 2016. We also made plans to work on extensions of SSP and SMP.

Acknowledgement We thank BIRS for providing a wonderful environment in which to conduct mathematical research.

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Chapter 24

Topological Methods in Model Theory (16frg676)

July 12 - 19, 2016

Organizer(s): Xavier Caicedo (Universidad de los Andes), Eduardo Dueñez (The University of Texas at San Antonio), Christopher Eagle (University of Victoria), José Iovino (The University of Texas at San Antonio), Franklin Tall (University of Toronto)

The event was quite productive, and the participants are grateful for BIRS for providing an excellent environment for their daily meetings during the week of June 12-19, 2016. The participants examined problems where topological methods can be used to improve or bring new insight into results in model theory. The following were among the projects considered.

Establishing a precise relationship between topological properties and model-theoretic properties.

The Baire Category Theorem and the Omitting Types Theorem for logics. One of the fundamental theorems of first-order model theory is the classical omitting types theorem. The proof of this which appears in textbooks uses the method known as "Henkin construction". However it is relatively well known that the theorem can be proved topologically, by using the Baire Category Theorem.

The topological approach to omitting types has the advantage that it can be adapted for contexts more general than first order. For instance, Morley [Mor74] used it to prove the Omitting Types Theorem for countable fragments of the infinitary logic $L_{\omega_1\omega}$, and Caicedo-Iovino [CI14] used it to provide a characterization of continuous logic. Later, Eagle [Eag14] used it to extend Morley's result to real-valued logics.

These developments have shown that the omitting types theorem holds in any abstract logic that has a space of structures which is Baire (i.e., it satisfies the Baire category theorem) and has a fairly simple syntax. It is then natural to ask for the most general situation under which conditions an abstract logic with a Baire space of structures satisfies the omitting types theorem, and whether the converse is true, i.e., whether every Baire space occurs as a space of structures of an abstract logic that satisfies the omitting types theorem.

During our week at BIRS, it became clear that merely assuming Baire is not sufficient [ET], and ongoing work by Eagle and Tall was sketched which associates logics to systems of topological spaces which aims to answer the second question.

Proposed project: To understand the logical contents of the game versions of Baire's theorem. As with the Banach-Mazur game, where Baire is equivalent to Player I not having a winning strategy while Player II having a winning strategy is equivalent to the stronger property of weak alpha-favourablity, one can similarly play games to omit types. The question of whether the omitting types theorem is equivalent to the prima facie stronger game version turns out to be strongly related to a long-unsolved problem in set theory, namely, the existence of P-filters on the set of natural numbers which are non-meagre subsets of the Cantor set. See [Mar98] and [MZ15].

Topological compactness and model-theoretic compactness. The expressive power of first-order logic is limited for some contexts, its model theory is powerful and mature. One of the characteristics of first-order model theory that make it particularly powerful is compactness. Proper extensions of first-order are typically not compact; however, several important extensions of first-order satisfy the weaker property of $[\kappa, \lambda]$-compactness, for infinite cardinals κ and λ.

The remarkable abstract compactness theorem, originally proved by Makowsky and Shelah in the 80's [MS83], states roughly that a logic $\mathscr{L}$ satisfies $[\kappa, \lambda]$-compactness if and only if there exists a (κ, λ)-regular ultrafilter D that provides an abstract notion of ultraproduct for structures of $\mathscr{L}$. The proof given by Makowsky and Shelah uses nontrivial model theory, and it supposes that the underlying logic $\mathscr{L}$ satisfies certain restrictive properties, e.g., $\mathscr{L}$ must be closed under negation.

At BIRS, we went through a simpler proof, due to Caicedo, that replaces the combinatorics in the Makowsky-Shelah analysis by elementary topological ideas related to preservation of topological $[\kappa, \lambda]$-compactness under products [Cai95]. This topological approach turns out to be not only simpler, but it also covers wider classes of logics, e.g., logics without classical negation.

Proposed project: Explore the relationship between this approach and current set-theoretic research on strong logics.

Topology and categoricity in model theory.

Morley's categoricity theorem [Mor65] is one of the groundbreaking theorems of model theory. It was the first instance of a categoricity transfer theorem. Roughly, a categoricity transfer result states that if a theory T is categorical in one cardinality (i.e. has only one model of that size), then T is categorical in other cardinalities. Research in this direction involves difficult combinatorial ideas [Bal09, She09b, She09a].

The motivation behind several of the methods originally used by Morley was topological; however, the spaces that he dealt with are 0-dimensional, so he was able to formulate them combinatorially for the benefit of non-topologically-minded model theorists.

We believe that approaching categoricity transfer results from a topological point of view, as Morley originally did, but using contemporary topological technology that was not available to him, will shed light onto this research direction in model theory. In particular, we noted an unexpected connection between the key model theoretic notion of stability and a topic extensively studied by topologists and set-theorists: superatomic Boolean algebras (see [Kop89, Section 17 of Chapter 6] and [MB89, Chapter 19]); the latter notion has a purely topological meaning when transferred to Boolean spaces of types, namely: the power of every quotient space is identical to its weight, which makes sense in non-Boolean contexts.

Proposed project: Explore the topic of superatomic Boolean algebras and their relation to categoricity.

Continuous operations between spaces of structures.

As shown by Caicedo [Cai95], the topology of elementary classes in spaces of structures associated to any Boolean model theoretic logic is uniformizable, and the uniform continuity of natural operations arising between them (Cartesian products, quotients, all sort of algebraic constructions, etc.) reflect the model theoretical properties of the logics. Thus, Lindström's axioms for logics, relativisations, interpretations, and Feferman-Vaught uniform reduction properties [FV59] are uniform continuity phenomena. The same is true of Beth's definability theorem and other definability properties.

Proposed project: Explore these connections in the realm of stronger logics.

Development of projective model theory and measurable logic.

The development of continuous logic in recent years has brought about fruitful interaction between model theory and other areas of mathematics, especially functional analysis. Continuous logic is finitary. However, the ideas of continuous model theory have been recently extended to the infinitary realm [BYI09, Eag14]. In finitary continuous logic, it is required that all the functions and relations of the structures involved be continuous, and that all the logical connectives be continuous functions as well. During the BIRS sessions, we realized that for an important property of the infinitary extension, namely, the omitting types property, the continuity of the structures is not needed (so the continuity of the connectives suffices). We intend to investigate the consequences of replacing the continuity of the logical connectives by a weaker condition without eliminating the (desirable) omitting types property. A natural candidate seems to be requiring that the connectives be projective in the sense of descriptive set theory [Kec95]. Formulas will then be closed under negation and the existential quantifier.

This project intersects with another: the development of "measurable logic". This has been a project in model theory for decades, since such a logic would be natural for applications to analysis. During our BIRS sessions, it was suggested that for the recent Dueñez-Iovino work on metastability [DI1, DI2], the continuous model theory formulation may be unnecessary, and projective logic would provide a finer foundation.

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Chapter 25

SYZ mirror symmetry (16frg673)

June 19 - 26, 2016

Organizer(s): Naichung Leung (The Chinese University of Hong Kong), Siu-Cheong Lau (Boston University)

Overview of the Field

Strominger-Yau-Zaslow (SYZ) [27] proposed that mirror symmetry can be understood in terms of duality of special Lagrangian torus fibrations. Namely, the mirror manifold can be constructed by taking fiberwise torus dual of the original manifold, and homological mirror symmetry can be understood by a real version of Fourier-Mukai transform. Fukaya [15] proposed that the mirror transform can be realized by family Lagrangian Floer theory of fibers of the SYZ fibration, which was further studied by Tu [28, 29] and Abouzaid [1, 2].

The SYZ program has led to a series of breakthroughs in geometric understanding of mirror symmetry. Based on Kontsevich-Soibelman's theory of wall-crossing [19], Gross-Siebert developed a general reconstruction program of mirror geometries using toric degenerations [18]. Auroux [4], Chan-Lau-Leung [8] and Abouzaid-Auroux-Katzarkov [3] realized the SYZ construction using symplecto-geometric methods and computed SYZ mirrors explicitly for local Calabi-Yau geometries. Moreover, Fukaya-Oh-Ohta-Ono [16] reconstructed the mirrors of toric manifolds based on the work of Cho-Oh [13] and proved close-string mirror symmetry for their geometrically constructed mirrors. Furthermore Aganagic-Vafa [5] proposed a generalized SYZ construction of mirror conifolds which uses non-compact Lagrangians constructed from knots in place of torus fibrations. Motivated from this and the works of Seidel [22, 23] and Sheridan [24, 25] on homological mirror symmetry, Cho-Hong-Lau [11, 12] developed a generalized SYZ program using Lagrangian immersions in place of torus fibrations.

Recent Developments and Open Problems

1. **Talk: Open Gromov-Witten invariants and augmentation varieties** (by Garrett Alston)
   Talk: SYZ and Fourier-Mukai transform (by Naichung Leung)
   Aganagic-Ekholm-Ng-Vafa [6] proposed a mathematical approach to prove that the generalized SYZ mirror of Aganagic-Vafa [5] is the augmentation variety associated to a knot. Briefly, the idea is the following. To a knot K in $\mathbb{S}^3$, one can associate a Legendrian submanifold in the unit sphere bundle of the cotangent bundle of $\mathbb{S}^3$. To this Legendrian is associated a differential graded algebra, which is a knot invariant. In turn, invariants associated to the dga, such as the augmentation variety, are also knot invariants. In another direction, the dga is related to non-compact Lagrangians in the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$, whose quantum-corrected moduli should give the mirror geometry. It was conjectured by [5] that the augmentation variety equals to the generalized SYZ mirror of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Alston and Leung in the group are working to prove the conjecture.

2. **Talk: SYZ and quantum deformations** (by Kwokwai Chan)
Chan-Leung-Ma [9] defined a differential graded Lie algebra in terms of Witten-Morse theory on a lattice bundle over the base of a Lagrangian torus fibration, which is isomorphic to the Kodaira-Spencer DGLA on the mirror Calabi-Yau manifold $\check{X}$ under the SYZ transform. It was speculated that the DGLA should control the deformation theory on the symplectic side, namely it should describe some kind of "quantum" deformations of the symplectic structure on X.

   Talk: Scattering diagrams and Witten-Morse theory (by Ziming Ma)
Moreover, the work [9] related scattering diagrams, which play a key role in Gross-Siebert program, with solutions to the B-model Maurer-Cartan equation using semiclassical expansions. However, the correspondence is away from the singular locus of the SYZ fibration. The ultimate goal is to extend it through singular locus to obtain a complete understanding of scattering in terms of Maurer-Cartan equation.

3. **Talk: SYZ and noncommutative deformations** (by Hansol Hong)
Kontsevich showed that there exists a canonical deformation quantization of the algebra of holomorphic functions on X with respect to a given Poisson structure α. On the other hand, one can transform α to be a 2-form on the symplectic manifold Y via SYZ transformation, which can be used to deform the symplectic structure on Y. The original torus fibers in Y may no longer be Lagrangian with respect to the deformed symplectic form. It is expected that one obtains a Lagrangian foliation after the deformation. Applying Connes theory, one obtains a noncommutative algebra from a foliation, which is expected to be mirror to the deformation quantization of X.

4. **Talk: Topological Fukaya category and localized mirror functor** (by Cheol-Hyun Cho)
Fukaya category of a symplectic manifold, which is a central object in homological mirror symmetry, is difficult to formulate. Recently, Kontsevich proposed that Fukaya category of a Stein manifold can be understood by gluing local data associated to those of a Lagrangian skeleton, which has the structure of cosheaf of dg categories. For punctured Riemann surfaces, Dyckerhoff-Kapranov [14], Sibilla-Truemann-Zaslow [26] and Pascaleff-Sibilla [21] constructed a notion called topological Fukaya categories and found applications in mirror symmetry. However, it is not yet proved that the topological Fukaya category is equivalent to the original Fukaya category.

5. **Talk: Generalized SYZ** (by Siu-Cheong Lau)
Cho-Hong-Lau [11, 12] developed a generalized SYZ program using Lagrangian immersions in place of Lagrangian torus fibrations. The program can be applied even in the absence of toric degenerations and Lagrangian fibrations, for instance rigid Calabi-Yau manifolds. Currently the program only concerns about formal deformations. Geometric deformations and family of immersions should be incorporated in the future. Moreover the program has more applications, for instance BHK transpose mirror symmetry [7, 20] and Hitchin systems [17].

Scientific Progress Made

1. Alston and Leung worked on an ongoing project to prove that the open Gromov-Witten potential function of a non-compact Lagrangian is related to the augmentation variety. They also discussed follow-up projects and applications to work on once this project is complete. The group discussed possible relations with the work [12] of Cho-Hong-Lau and non-commutative augmentations recently defined by [10]. There should exist a natural mirror functor from the Fukaya category of resolved conifold to the derived category of the moduli space of augmentations, which descends to the derived category of the augmentation variety.

2. Chan tried to get some idea on quantum deformations of symplectic geometry by studying J. Tu's recent works [28, 29, 30] on the reconstruction problem and the interpretation of homological mirror symmetry as a Koszul duality. The key idea is to make use of Fukaya's earlier work on constructing a sheaf of A_∞ algebras on B. We expect that this sheaf of A_∞ algebras is encoding the quantum information of the symplectic manifold (X, ω) because the symplectic structure ω appears as part of the curvature term in the family of

A_∞ structures. By studying the deformation theory of this sheaf of A_∞ algebras, perhaps also applying results analogue to that of Fukaya-Oh, one should be able to recover the DGLA we discussed above.

Moreover Ma found that it is better to consider the polyvector fields governing the extended deformation of holomorphic volume form. Barannikov gave a construction of B-model Frobenius structure on the extended complex moduli of a smooth Calabi-Yau manifold, from the universal solution to the extended Maurer-Cartan equation. For a one parameter family of K3 surfaces degenerating to the large complex structure limit, we use a resolution of the total space to set up a Barannikov type differential BV algebra, and explore the relation between tropical diagrams and the log-Frobenius manifold structure near the large complex structure limit.

3. Hong and Leung investigated the relation between noncommutative deformations of SYZ (Strominger-Yau-Zaslow) mirror pair, X and Y. More precisely, they considered the situation where X and Y admit Lagrangian torus fibations over the same base B, and the fibers in X and Y are dual to each other. They studied the detailed construction Lagrangian foliation structure on Y, and found that there is a way of expanding the foliation in Fourier series in a suitable sense. The observation is based on the fact that SYZ transformation is, fiberwisely, a Fourier transform. While the genuine foliation could be too complicated to handle, one may still be able to find a construction of an noncommutative algebra (in the sense of Connes) directly from the Fourier expansion data.

4. Cho, Hong and Lau discussed the relationship between topological Fukaya category and localized mirror functors in [11]. Although the topological Fukaya category is made from perfect complexes of A-type quiver representations, we have learned that localized mirror functor formalism in the Calabi-Yau setting should be very helpful to find the isomorphism between the original Fukaya category and the topological one, after one establishes the definition of appropriate Fukaya category of a Riemann surface with non-empty boundaries together with marked points.

5. They also discussed a way to understand BHK transpose mirror symmetry using the general theory of [11]. Given a CY hypersurface defined by an invertible polynomial, they constructed Lagrangian immersions in the hypersurface and discussed how transpose polynomial arises by counting certain orbi-polygons bounded by the Lagrangian immersions. It was found that deformations by twisted sectors are necessary in order to have the transpose terms. This may give a hint on understanding transpose mirror symmetry for general-type hypersurfaces as well.

Participants

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Research in Teams Reports

Chapter 26

Metastring Theory and Generalized Geometries (16rit683)

March 20 - 27, 2016

Organizer(s): Laurent Freidel (Perimeter Institute), Robert G. Leigh (University of Illinois at Urbana-Champaign), Djordje Minic (Virginia Tech)

Overview of the Field

Generalized geometry is a very active area of research both in mathematics and in mathematical physics, especially in the context of double field theory and string theory (for review, consult, [1, 2]). From a purely mathematical point of view, generalized geometry is based on the replacement of the tangent bundle TM of a manifold M by $TM \oplus T^*M$, and the Lie bracket by the Courant bracket [1]. On the physics side, the mathematical structure of generalized geometry comes about when one considers the phenomenon of T-duality in string theory, and in particular, the supergravity limit of string theory [2]. The ideas of generalized geometry and double field theory have inspired hundreds of papers in recent years, and many research meetings on these topics have been held in forms of workshops and conferences, including the recent prominent ones at CERN and at the Simons Center for Geometry and Physics.

Recent Developments and Open Problems

The field of generalized geometry has been thriving for a number of years. Similarly steady progress has been achieved in using the techniques of generalized geometry in the fields of supergravity and double field theory. However, the real underpinnings of these developments remain largely hidden. In particular, the uniqueness of certain mathematical structures in the context of generalized geometry is not at all apparent. Similarly, the physical viability of the double field theory limit in string theory is not clear. Recently, we have provided a new insight on some of the underlying conceptual issues both from the physics and mathematics perspective [3, 4, 5, 6]. In particular, we have introduced new concepts of metastring theory, modular space-time and Born geometry, which reveal the deeper structures behind T-dualty in string theory, as well as illuminate the appearance of generalized geometry in string theory and point to further generalizations of generalized geometry. Further development of this approach was the central reason for our research-in-teams meeting at BIRS. The outcome of this meeting is summarized in our new paper [7] and an upcoming publication [8]. The results developed during our research conducted at Banff were most recently presented at the meeting on generalized geometry and string theory at the Simons Center for Geometry and Physics, and the same work will be reported at various forthcoming conferences in Amsterdam, Sao Paulo, Prague, Wroclaw, Pretoria and Perimeter Institute.

Presentation Highlights

The concept of locality in space-time is one of the cornerstones of modern physics. It is one of the key properties underlying effective field theory, which in turn is widely considered a universal tool to describe fundamental physics and captures the main features of disparate physical systems at low energy scales. Nevertheless, it is becoming increasingly clear that non-locality may play a central role in solving some of the most outstanding puzzles in theoretical physics, such as the vacuum energy problem, the black hole information paradox, as well as the deep understanding of the central non-local features of quantum theory. If this is the case, then the tools of effective field theory become inadequate, and we must develop new ways of thinking. Similarly, the fundamental concepts of differential geometry constitute the basic mathematical language of general relativity, our deepest theory of space and time. However, these tools seem to be just a limiting case of the mathematical language of generalized geometry required to talk about various new phenomena encountered in the areas of supergravity and string theory. In particular, in string theory T-duality is one of the central features, and it also points to the underlying non-local foundations of that field. The concept of non-locality is brought to the forefront in our work on new string constructions involving Born geometry in the context of the so-called metastring theory we have proposed in [5]. This subject ties together disparate ideas in quantum gravity, invoking a notion of relative locality [9] as well as the crucial features of string dualities. The metastring formulation of quantum gravity introduces a new concept of quantum space-time called modular space-time which sheds new light on the foundational issues in quantum theory. Finally the new concept of Born geometry provides a quantum foundation for generalized geometry and opens a new and fruitful area of research in mathematics.

Scientific Progress Made

Our meeting in Banff was immensely productive. We worked out the details of the quantum origins of spatial geometry, which has since appeared in [7], and we developed tools for the generalization of the new mathematical structures needed to understand space-time geometry, and their application to metastring theory [8]. This work extends the usual notions of quantization to the most general commutative subgroups of the Heisenberg group, leading to a novel notion of space, which we refer to as modular space. This purely quantum geometry comes automatically equipped with additional metrical structures encountered in the context of metastring theory, and provides the quantum foundations for some open questions in the field of generalized geometry. We have also elucidated how the usual classical notion of space comes out as a singular limit of modular space in the process we call extensification. The week we spent at Banff was crucial in these developments as it gave us a chance to concentrate on these topics for an uninterrupted period of time. In addition to this work, the basic elements of another upcoming publication [8] were developed during the same extremely productive week.

Outcome of the Meeting

The outcome of the meeting at BIRS is published in a 34 page paper [7], as well as in the upcoming publication [8]. The results of our meeting at Banff were presented at the meeting on generalized geometry at the Simons center for Geometry and Physics in May of 2016, and are going to be presented at various meetings on string theory, generalized geometry and quantum gravity in Amsterdam, Sao Paulo, Prague, Wroclaw, Pretoria and Perimeter institute, during the summer of 2016.

Participants

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Chapter 27

Random Partitions and Bayesian Nonparametrics (16rit674)

April 17 - 24, 2016

Organizer(s): Stefano Favaro (University of Torino and Collegio Carlo Alberto), Shui Feng (McMaster University)

Overview of the Field

Random partitions arise naturally in many subjects including but not limited to Bayesian nonparametric, ecology, machine learning, number theory, physics, population genetics, and the theory of random matrices.

The basic framework for a Bayesian nonparametric model involves a two stage generation of data: a probability P is first chosen from the support of a prior distribution on the space of probability measures, followed by a sample of (conditionally) independent and identically distributed random variables from P. The main goal consists in determining and investigating the posterior distribution, that is the conditional distribution of P given the observable sample. The prior in a nonparametric Bayesian model corresponds to the law of the random probability measure P, and probability theory provides a large arsenal for studying distributional properties of P, especially under the assumption that P is discrete almost surely.

The natural link between random partitions and Bayesian nonparametric is through the celebrated de Finetti representation theorem. Exchangeable random partitions are the cornerstone of Bayesian nonparametric inference for a broad class of statistical problems, referred to as species sampling problems.

Recent Developments

Let $\mathbb{N}$ denote the set of natural numbers. A partition π of $\mathbb{N}$ is a collection of disjoint subsets $\{\pi_i : i \geq 1\}$ of $\mathbb{N}$ ordered by their least elements with $\cup_{i=1}^{\infty} \pi_i = \mathbb{N}$. Denote the collection of all partitions of $\mathbb{N}$ by Π. For any $n \geq 1$, a partition π^n of $\mathbb{N}^n = \{1, 2, \ldots, n\}$ is defined similarly. The set of all such partitions is denoted by Π^n. A random partition of $\mathbb{N}^n$ or $\mathbb{N}$ is a probability on Π^n or $\mathbb{N}$. Under certain consistency assumptions, one is able to construct a random partition on $\mathbb{N}$ from those on $\mathbb{N}^n$ by letting n tend to infinity. When the random partition depends only on the number of subsets and the size of each subsets of a partition π^n, it corresponds to a family of probability partition functions

$$\{p(n_1, n_2, \ldots, n_k) : 1 \leq i \leq k \leq n, 1 \leq n_i \leq n, \sum_{i=1}^{k} n_i = n\}$$

where k is the number of subsets and n_i is the size of the i-th set.

The most studied family of probability partition functions is the Ewens sampling formula ([4]) describing in the genetics context the sampling distribution of a neutral population. This is followed by the study of Kingman's partition structures and coalescent ([9],[10]). After the discovery of Pitman sampling formula ([12]) and the coalescent with multiple collisions ([13],[15]), there have been intensive studies on various generalizations of these models ([2],[14]).

In the Bayesian nonparametric settings, one starts with a discrete random probability measure $P = \sum_{i\geq 1} p_i \delta_{\xi_i}$ on an arbitrary space S, where $(p_i)_{i\geq 1}$ are nonnegative random weights such that $\sum_{i\geq 1} p_i = 1$ almost surely, and $(\xi_i)_{i\geq 1}$ are S-valued random locations, or random labels, independent of $(p_i)_{i\geq 1}$ and independent and identically distributed according to a nonatomic distribution. By virtue of de Finetti representation theorem, there exists an exchangeable sequence of random variables $(X_i)_{i\geq 1}$ such that

$$\begin{aligned} X_i \,|\, P \quad &\sim iid \quad P, \qquad i = 1, 2, \ldots, \\ P \quad &\sim \quad \mathcal{P}, \end{aligned} \tag{27.0.1}$$

for any $n \geq 1$, with $\mathcal{P}$ being the distribution of $\lim_{n\to+\infty} n^{-1} \sum_{1\leq i\leq n} \delta_{X_i}$. Due to the discreteness of P, we expect ties in $(X_1, \ldots, X_n)$. Let $K_n = k \leq n$ denote the number of different types or species in the sample, labelled by $X_1^*, \ldots, X_{K_n}^*$, with corresponding frequencies $(N_{1,n}, \ldots, N_{K_n,n}) = (n_1, \ldots, n_k)$ such that $\sum_{1\leq i\leq K_n} N_{i,n} = n$. The sample $(X_1, \ldots, X_n)$ induces a random partition Π_n of the set $\{1, \ldots, n\}$, in the sense that any index $1 \leq i \neq j \leq n$ belongs to the same partition set if and only if $X_i = X_j$. As shown by Kingman [9], for any $n \geq 1$ the distribution of the random partition is exchangeable.

Exchangeable random partitions are the cornerstone of Bayesian nonparametric inference for a broad class of statistical problems, referred to as species sampling problems, where samples are assumed to be drawn from a population of individuals belonging to an (ideally) infinite number of species $(X_i^*)_{i\geq 1}$ with unknown proportions $(p_i)_{i\geq 1}$. In such a species sampling framework, (27.0.1) takes on the natural interpretation of a Bayesian nonparametric model, where $\mathcal{P}$ is the prior distribution on the unknown species composition $(p_i)_{i\geq 1}$ of the population. Species sampling problems have originally appeared in ecology, and their importance has grown considerably in recent years, driven by challenging applications arising from bioinformatics, genetics, linguistics, design of experiments, machine learning, etc. Given an initial sample $(X_1, \ldots, X_n)$ featuring $K_n = k$ species with frequencies $(N_{1,n}, \ldots, N_{K_n,n}) = (n_1, \ldots, n_k)$, interest lies in making inference on certain statistics of the random partition induced by an additional unobserved sample of size m. Statistics of interest are, among others, the $K_m^{(n)}$ new species and the $M_{l,m}^{(n)}$ species with frequency l to be observed in the additional sample. Given that, Bayesian nonparametric inference for species sampling problems relies on the study of the conditional distribution of the random partition induced by the additional sample given $(X_1, \ldots, X_n)$, i.e., the distribution of the random partition of a sample of size m from the posterior distribution of P. More details are found in [5] and [6] and references therein.

Objectives and Outcome of the Meeting

The random partitions discussed above are all associated with random measures describing the equilibrium behaviour of certain population. But in probabilistic literature, there are a plethora of random measures arising from stochastic processes. Random partitions constructed from these measures are thus associated with the non-equilibrium population structures. Models in this aspects include but not limited to the Fleming-Viot process ([8]), infinitely-many-neutral-alleles model ([3]), Petrov diffusion ([11]), coagulation and fragmentation processes ([2]), GEM process ([7]), and general coalescents ([1]). From a statistical perspective, these more general random structures suggest potential applications in modelling samples arising from populations with more complex compositional structures.

Our first objective is concerned with the equilibrium random partitions. Under the prior assumption that P is the two parameter Poisson Dirichlet process, we want to study the large m asymptotic behaviour of the posterior distribution of $K_m^{(m)}$ and $M_{l,m}^{(n)}$, given an initial observed sample of size n. From a Bayesian nonparametric perspective, this asymptotic analysis is mainly motivated by the need of deriving approximations of the posterior distributions $K_m^{(m)}$ and $M_{l,m}^{(n)}$. Indeed, while the two parameter Poisson Dirichlet prior leads to explicit expressions for these posterior distributions, such expressions involve combinatorial coefficients and special functions whose evaluation for large m is cumbersome, thus preventing their concrete implementation. In [5] and [6] we studied the large m asymptotic behaviour, in terms of fluctuations and large deviations, of the posterior distribution of $K_m^{(m)}$ and $M_{l,m}^{(n)}$. Recently we made progresses on the related problem of deriving central limit theorems and moderate deviation principles, for the posterior distribution of $K_m^{(m)}$. Of course such a result is of direct applicability for deriving large m asymptotic credible intervals for the Bayesian nonparametric estimator of $K_m^{(n)}$. We are able to complete this project during our stay at BIRS. We also intend to discuss the problem of deriving, by means of tools from the theory of concentration inequalities, non-asymptotic credible intervals for the Bayesian nonparametric estimator of $K_m^{(n)}$. This part of research is currently an open project.

Our second objective is to study certain non-equilibrium random partitions. More specifically it is related to the genealogical structure of the Kingman's coalescent and, in particular, to the problem of making Bayesian nonparametric (predictive) inference on such a structure. Let $L_n(t)$ be the number of non mutant lineages at time t back in a Kingman's coalescent tree of a sample of n genes. While this distribution is well-known from Kingman [9], what seems unknown is the distribution of the number $L_{l,n}(t)$ of non mutant lineages with frequency l at time t back in a Kingman's coalescent tree of a sample of n genes. Recently we derived the distribution of $L_{l,n}(t)$, as well as related conditional distributions. These conditional distributions may be interpreted as genuine posterior distributions. During our stay at BIRS we completed this project by investigating some large n asymptotic properties of $L_{l,n}(t)$. A different project, still related to the Kingman's coalescent, concerns with the problem of making Bayesian nonparametric (predictive) inference on the genealogical structure of the Kingman's coalescent. Specifically, we aim at deriving the conditional distribution of the number of lineages in a Kingman's coalescent tree of a sample of $n + m$, given a Kingman's coalescent tree of a sample of n genes. From a Bayesian nonparametric perspective, such a conditional distribution takes on the interpretation of the posterior distribution of the number of lineages, and its expected value provides the corresponding Bayesian nonparametric estimator. Analogue of the celebrated Good-Turing and Good-Toulmin estimators has been introduced in the framework of lineages. This is also completed during our visit to BIRS.

Participants

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Chapter 28

Gorenstein Homological Algebra (16rit690)

May 22 - 29, 2016

Organizer(s): Daniel Bravo (Universidad Austral de Chile), Sergio Estrada (University of Murcia), Alina Iacob (Georgia Southern University)

Overview of the Field

Homological algebra is at the root of modern techniques in many areas of mathematics including commutative and non commutative algebra, algebraic geometry, algebraic topology and representation theory. Not only that all these areas make use of the homological methods but homological algebra serves as a common language and this makes interactions between these areas possible and fruitful. A relative version of homological algebra is the area called Gorenstein homological algebra. This newer area started in the late 60s when Auslander introduced a class of finitely generated modules that have a complete resolution. Auslander used these modules to define the notion of the G-dimension of a finite module over a commutative noetherian local ring. Then Auslander and Bridger extended the definition to two sided noetherian rings (1969). The area really took off in the mid 90s, with the introduction of the Gorenstein (projective and injective) modules by Enochs and Jenda ([1]). Avramov, Buchweitz, Martsinkovsky, and Reiten proved that if the ring R is both right and left noetherian and if G is a finitely generated Gorenstein projective module, then Enochs' and Jenda's definition agrees with that of Auslander's and Bridger's of module of G-dimension zero. The Gorenstein flat modules were introduced by Enochs, Jenda and Torrecillas as another extension of Auslander's Gorenstein dimension.

The Gorenstein homological methods have proved to be very useful in characterizing various classes of rings. Also, methods and results from Gorenstein homological algebra have successfully been used in algebraic geometry, as well as in representation theory. But the main problem in using the Gorenstein homological methods is that they can only be applied when the corresponding Gorenstein resolutions exist. So the main open problems in this area concern identifying the type of rings over which Gorenstein homological algebra works or, more generally, other categories where these methods can be applied. Of course one hopes that this is the case for any ring. The existence of the Gorenstein resolutions are still open problems. And they have been studied intensively in recent years. These are also the problems we considered.

Recent Developments and Open Problems

As already mentioned, we considered some of the main open problems in Gorenstein homological algebra - the existence of the Gorenstein resolutions.

We recall that the Gorenstein (projective, injective, flat) modules are defined in terms of totally acyclic complexes. We recall that an acyclic complex P of projective R-modules (R is an arbitrary ring) is called totally acyclic if the complex $Hom(P, Q)$ is still exact for any projective module Q. A totally acyclic complexes of injective modules is defined dually. And an exact complex F of flat left R-modules is said to be F-totally acyclic if $I \otimes F$ is exact for any injective right R-module I. A module M is Gorenstein injective if and only if it is a cycle of a totally acyclic complex of injective modules. Dually, a module G is Gorenstein projective if it is a cycle of a totally acyclic complex of projective modules. And a Gorenstein flat module is a cycle of an F-totally acyclic complex of flat modules.
A Gorenstein ring (in the sense of Iwanaga, [5] and [6]) is a two sided noetherian ring R that has finite self injective dimension on both sides. Over such a ring the exact complexes of projective (injective, flat) modules have some very nice homological properties. More precisely, over a Gorenstein ring every acyclic complex of projective (injective) modules is totally acyclic. And every acyclic complexes of flat modules is F-totally acyclic over any Gorenstein ring. So over such a ring the class of Gorenstein projective (injective, flat) modules coincides with that of the cycles of acyclic complexes of projectives (injective, flat modules respectively).

It is a natural question to consider whether or not these conditions actually characterize Gorenstein rings, or more generally whether or not it is possible to characterize Gorenstein rings in terms of acyclic complexes of (Gorenstein) injectives, (Gorenstein) projectives and (Gorenstein) flats. We considered this question and we gave several characterizations of the rings with this property.

Presentation Highlights

Since this was a Research in Teams workshop for three people, there were no formal presentations.

Scientific Progress Made

The major contribution made at this RIT was the completion of a paper that has been in progress for a while. This paper is submitted now.
The participants also worked on a second paper that is still in progress.

Participants Estrada and Iacob are two of the three authors of the paper "Totally acyclic complexes" ([2]). As noted above the main problem considered here is : What is the most general type of ring R with the property that the Gorenstein projective (injective, flat respectively) are simply the cycles of exact complexes of projective (injective, flat respectively) modules?
It is worth mentioning that over such rings the class of Gorenstein injective modules is both covering and enveloping, and the class of Gorenstein flat modules is preenveloping. This guarantees the existence of the minimal Gorenstein injective left and right resolutions, and also, the existence of the right Gorenstein flat resolutions.

We proved first ([2], Proposition 3) that, over any ring R, an acyclic complex of projective modules is totally acyclic if and only if the cycles of every acyclic complex of Gorenstein projective modules are Gorenstein projective. The dual result for injective and Gorenstein injective modules also holds over any ring R ([2], Proposition 4). And, when R is a GF-closed ring, the analogue result for flat/Gorenstein flat modules is also true (Proposition 5). Then we showed (Theorem 2) that over a left noetherian ring R, a third equivalent condition can be added to those in Proposition 3, more precisely, we proved that the following are equivalent: 1. Every acyclic complex of injective modules is totally acyclic. 2. The cycles of every acyclic complex of Gorenstein injective modules are Gorenstein injective. 3. Every complex of Gorenstein injective modules is dg-Gorenstein injective.

Theorem 3 shows that the analogue result for complexes of flat and Gorenstein flat modules holds over any left coherent ring R. We prove (Corollary 1) that, over a commutative noetherian ring R, the equivalent statements in Theorem 3 hold if and only if the ring is Gorenstein. We also prove (Theorem 4) that when moreover R is left coherent and right n-perfect (that is, every flat right R-module has finite projective dimension $\leq n$) then statements 1, 2, 3 in Theorem 2 are also equivalent to the following: 4. Every acyclic complex of projective right R-modules is totally acyclic. 5. Every acyclic complex of Gorenstein projective right R-modules is in $\widetilde{\mathcal{GP}}$. 6. Every complex

of Gorenstein projective right R-modules is dg-Gorenstein projective.
Corollary 2 shows that when R is commutative noetherian of finite Krull dimension, the equivalent conditions (1)-(6) from Theorem 4 are also equivalent to those in Theorem 3 and hold if and only if R is an Iwanaga-Gorenstein ring. Our Corollary 2 improves on results by Iyengar and Krause ([4]) and by Murfet and Salarian ([7]). Iyengar and Krause proved that for a commutative noetherian ring R with a dualizing complex, the class of acyclic complexes of injectives coincides with that of totally acyclic complexes of injectives if and only if R is Gorenstein. Then Murfet and Salarian removed the dualizing complex hypothesis and characterized Gorenstein rings in terms of totally acyclic complexes of projectives. We are adding more equivalent characterizations, still under the assumption that R is commutative noetherain of finite Krull dimension.

In the second part of the paper we focus on two sided noetherian rings that satisfy the Auslander condition. We prove (Theorem 7) that for such a ring R that also has finite finitistic flat dimension, every complex of injective (left and respectively right) R-modules is totally acyclic if and only if R is an Iwanaga-Gorenstein ring.
The paper [2] was submitted for publication shortly after the workshop. A preprint is available on arXiv.

Further progress in studying the existence of the Gorenstein flat precovers was made by the three participants in the workshop at BIRS (Bravo, Estrada, Iacob), and a paper on these new results is being prepared ([3]). We consider a Grothendieck closed symmetrical monoidal category $(\mathcal{C}, -\otimes -, [-,-])$. This important class of categories includes, among others, categories of modules over a commutative ring, sheaves of $\mathcal{O}_X$-modules, quasi-coherent sheaves of $\mathcal{O}_X$-modules, comodules over a flat Hopf algebroid, and categories of representations by modules of arbitrary quivers. An object X of $\mathcal{C}$ is flat if the functor $X \otimes -$ is exact. Given a class of objects $\mathcal{A}$, let $\widetilde{\mathcal{F}}$ denote the class of all acyclic complexes of flat objects F such that $A \otimes F$ is still acyclic, for any $A \in \mathcal{A}$. We show that the class $\widetilde{\mathcal{F}}$ is covering. Moreover, if $\mathcal{C}$ has a flat generator then $(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}^{\perp})$ is a hereditary perfect cotorsion pair and there is an induced cofibrantly generated abelian model category structure with $\widetilde{\mathcal{F}}$ as the class of cofibrant objects. These results are in [3] which is still in progress.

The three participants also considered duality pairs and used the connections between duality pairs and cotorsion pairs in order to prove the existence of precovers/preenvelopes with respect to some classes of modules: FP_n-injective and flat modules, Gorenstein AC-injective and flat modules, Ding-injective modules. This is still work in progress.

Outcome of the Meeting

We expect that the paper [2] submitted shortly after the workshop will be accepted in 2016.
The three participants in the workshop plan to finalize our joint work on the existence of the Gorenstein flat precovers, and submit it as soon as possible.

The fact that we could meet and discuss our ideas was very valuable. We thank BIRS for hospitality and for giving us the opportunity to advance our work in such a pleasant setting.

Participants

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Chapter 29

Dynamics in Applied Functional Differential Equations (16rit675)

July 24 - 31, 2016

Organizer(s): Elena Braverman (University of Calgary, Canada), Karel Hasik (Silesian University in Opava, Czech Republic), Anatoli Ivanov (Pennsylvania State University, USA), Sergei Trofimchuk (University of Talca, Chile)

Overview of the Field

The significance of Functional Differential Equations (FDEs), a large part of the broad field of Differential Equations, has grown enormously in the past two decades or so. They play a unique role and are of irreplaceable value in modelling real-world phenomena with aftereffects. The time delay effects are intrinsic features of numerous mechanisms of natural phenomena around us, and their mathematical models of functioning naturally lead to delay or functional differential equations; see, for example, relevant references in the monographs [2, 4, 8, 12]. FDEs are complex mathematical objects that provide in part a new insight into dynamical properties of differential equations. For example, simple FDEs can exhibit complicated dynamical behaviors which do not exist for corresponding ordinary differential equations. Many important theoretical issues for FDEs are well developed and described in classical monographs, such as [1, 5, 7]. Rapid and expansive growth of scholar activities in research and applications of FDEs in the past decade or so has led to new theoretical advances in the field as well as to their broad applicability in diverse areas. Particularly impressive has been the use of FDEs in applications. This is reflected in several recent monographs emphasizing applied aspects of FDEs together with relevant theoretical issues behind them [2, 4, 8, 12, 13].

Recent Developments and Open Problems

The principal goal of this project is to initiate and develop a long term collaborative research agenda between the four participants of the workshop. Such collaborative plans also imply a subsequent inclusion into the research program of graduate and postgraduate students at their home universities, as well as colleagues and collaborators working in adjacent areas of research in the field.

There is a range of important topics and open problems in the field, many of which are within the common interests of this group, and which the participants would like to consider for joint research as parts of this program. In very broad terms this project deals with the qualitative and numerical analyses of several classes of FDEs appearing in various problems of an applied nature. The research work will be extended to specific classes of differential delay equations, difference equations and discrete maps (in particular, to discretizations of continuous time FDEs), and FDEs subject to random perturbations. It will be aimed at studies of global dynamical properties, such as global asymptotic stability, periodicity, complex (chaotic) behaviors, and persistence of the dynamics under various perturbations.

The equations under consideration appear as mathematical models of numerous phenomena in physics, biology, economics, physiology, life sciences, and other fields. They also come from related problems of applied mathematics, such as wave processes described by hyperbolic partial differential equations. Extensive numerical simulations are expected to provide guidelines and directions for subsequent theoretical justification of observed dynamics. They will also serve as a means of verification between the theoretical results predicting the dynamics and available observed data for the corresponding real world phenomena.

The following general directions of research are included into collaborative plans between members of the team:

(i) Existence, stability, and shape of periodic solutions for functional differential equations and difference equations; associated non-autonomous dynamical systems, existence and properties of their global attractors;

(ii) Discretizations of FDEs, difference equations, discrete maps and their properties (global stability of equilibria, periodic solutions and their properties, chaotic behavior, general attractors);

(iii) Problems of optimization and optimal control for functional differential equations; Stabilization of dynamical equilibria, relevance to applications;

(iv) Deterministic and stochastic functional differential equations: global stability and existence of periodic solutions, persistence of dynamics under small stochastic perturbations, global dynamics;

(v) Applications to modeling real life phenomena in applied sciences (biology, physics, physiology), life sciences (economics, medicine, environmental problems), other areas of mathematics (e.g., boundary value problems for partial differential equations).

We have discussed those topics in general settings during the meeting and made an initial research progress on some of them.

Scientific Progress Made

One of the principal mathematical objects of the research exploration during the meeting was a cyclic system of delay differential equations of the form

$$\begin{array}{rcl} x_1'(t)+\lambda_1 x_1(t) &=& f_1(x_2(t-\tau_2)) \\ x_2'(t)+\lambda_2 x_2(t) &=& f_2(x_3(t-\tau_3)) \\ \dots & \dots & \dots \\ x_{n-1}'(t)+\lambda_{n-1} x_{n-1}(t) &=& f_{n-1}(x_n(t-\tau_n)) \\ x_n'(t)+\lambda_n x_n(t) &=& f_n(x_1(t-\tau_1)), \end{array} \tag{29.0.1}$$

where $\lambda_k \geq 0$ and $f_k \in C(\mathbf{R},\mathbf{R})$. In addition, the nonlinearities f_k satisfy a sign condition $sign\,\{x \cdot f_k(x)\} = \sigma_k \in \{-1;+1\}$, allowing for system (29.0.1) to have zero as the only constant solution. In addition, we have assumed the eventual negative feedback in the system which can be expressed as $\sigma_1\sigma_2\dots\sigma_n = -1$.

System (29.0.1) has numerous applications in modelling various real world phenomena. Just to mention a few, it was proposed as a mathematical model of a protein synthesis process where natural physiological delays are taken into account [3, 9]. Its two-dimensional version $n = 2$ was used as a model of intracellular circadian rhythm generator [10]. For other applications such as models of neural networks see e.g. [6, 13] and further references therein. Its one-dimensional case $n = 1$ is comprehensively studied in numerous publications, many of which are summarized as parts of several monographs, see e.g. [1, 2, 5, 8, 11, 12]. Those monographs also offer an extensive list of references to other applications. Though some theoretical results on system (29.0.1) are available (in particular, for the one-dimensional case $n = 1$), there are many theoretical and applied questions and open problems that still remain unsolved (especially for the higher dimensional case $n \geq 2$).

Our principal approach is to derive some of the basic properties of solutions of system (29.0.1) from those of the corresponding linear system

$$\begin{aligned} x_1'(t) + \lambda_1 x_1(t) &= a_1\, x_2(t-\tau_2) \\ x_2'(t) + \lambda_2 x_2(t) &= a_2\, x_3(t-\tau_3) \\ \dots \quad & \dots \quad \dots \\ x_{n-1}'(t) + \lambda_{n-1} x_{n-1}(t) &= a_{n-1}\, x_n(t-\tau_n) \\ x_n'(t) + \lambda_n x_n(t) &= a_n\, x_1(t-\tau_1), \end{aligned} \tag{29.0.2}$$

where $a_k := f_k'(0) \neq 0$. Many properties of solutions of system (29.0.2) are well studied and understood due to its linear nature. Those properties include the stability and instability of the zero solution, the oscillating nature of solutions, and the existence of monotone (non-oscillating) solutions. Associated with the linear system (29.0.2) is the so-called characteristic equation

$$(z+\lambda_1)(z+\lambda_2)\dots(z+\lambda_n) = a\,\exp\{-\tau\, z\}, \quad a = a_1 a_2 \dots a_n. \tag{29.0.3}$$

The location of zeros in the complex plane of the characteristic equation (29.0.3) completely determines multiple properties of the solutions of the linear system (29.0.2). We have shown that they also determine some of the properties for the nonlinear system (29.0.1). In particular we have shown the following:

- All solutions to system (29.0.1) oscillate if and only if the characteristic equation (29.0.3) does not have any real solutions;
- System (29.0.1) has a monotone (non-oscillating) solution if and only if the characteristic equation (29.0.3) has a real solution;
- There exist parameter values a_0 and a_1 such that all solutions to system (29.0.1) oscillate when $a > a_0$, and the zero solution is unstable when $a > a_1$. Both options $a_0 < a_1$ and $a_0 > a_1$ are possible in higher dimension $n \geq 3$ only.

One of the open problems that stands out in this direction of research for a number of years is the existence of the so-called slowly oscillating periodic solutions in a general setting. We conjecture that such periodic solutions exist for our system (29.0.1) when $a > \max\{a_0, a_1\}$ and at least one of the functions f_k is bounded from one side.

Outcome of the Meeting

A program of joint research is initiated and currently being further developed into a comprehensive research direction in this particular area of modern nonlinear dynamics. It includes qualitative analyses of autonomous systems of nonlinear delay differential equations modeling various real world phenomena. Besides the joint research efforts by the entire group as described above additional initial steps of further research exploration were undertaken between participants within the group on the following topics:

- Optimal control and optimization in a delay system modeling an economic production cycle (Ivanov and Trofimchuk);
- Existence and uniqueness of travelling wave solutions in partial differential equations models with delay (Hasik and Trofimchuk)
- Oscillation and stability of solutions in discrete systems with delay (Braverman and Ivanov)

We have made a substantial progress in our investigation of system (29.0.1) during the meeting itself. We have studied the corresponding characteristic equation (29.0.3) and related its properties to the solutions of the nonlinear system. Preparation of a typescript of a joint paper of the four co-authors is currently in progress. We expect to finish writing the typescript and to prepare it for submission by the end of this year.

Participants

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Chapter 30

New Examples with Almost Nonnegative Curvature (16rit692)

July 31 - August 7, 2016

Organizer(s): Catherine Searle (Wichita State University), Pedro Solórzano (CONACyT-UNAM), Frederick Wilhelm (University of California Riverside)

Overview of the Field

One of the great unsolved problems of Riemannian geometry is to determine the structure of collapse with a lower curvature bound. An apparently simpler, but still intractable problem, is to determine which closed manifolds collapse to a point with a lower curvature bound. Such manifolds are called almost nonnegatively curved. While many examples of such manifolds are known, few topological obstructions to this class exist and we are far from obtaining a complete classification. Our project is to construct new examples with almost nonnegative curvature, with an eye towards achieving a better understanding of this important class of manifolds.

Recent Developments and Open Problems

For our Research in Teams meeting, we proposed to study the following two conjectures. The first generalizes a result of Dyatlov from [4].

Conjecture 1 *Let M be a closed Riemannian G–manifold so that $(M/G, \mathrm{dist}_{\mathrm{orb}})$ has the property that for the image of any stratum, S/G, in the quotient space, M/G, and any $x \in S/G$, $\Sigma_x^{\perp} S/G$ is a join of circles or constant curvature spheres. Then M/G is homeomorphic to a smooth manifold $\bar{M}$ that admits a sequence $\{g_i\}$ smooth Riemannian metrics with curvature $\geq k - \left|O\left(\frac{1}{i}\right)\right|$ so that*

$$\left(\bar{M}, g_i\right) \xrightarrow{\mathrm{GH}} (M/G, \mathrm{dist}_{\mathrm{orb}})\,.$$

Although it is possible that this result could be used for constructing new examples of Riemannian manifolds with positive sectional curvature, it is clear that the number of possible constructions is severely limited by Wilking's connectivity principle [10] and the strong restriction imposed on the space of directions by Conjecture 1. Rather we proposed to use a parametrized version of Theorem 1 below to prove the following.

Conjecture 2 *Let bP_{n+1} be the cyclic group of n-dimensional exotic spheres that bound parallelizable $(n+1)$-dimensional manifolds. For all n there is a generator of bP_{n+1} that admits a family of almost nonnegatively curved metrics. Moreover, when $n \equiv 3 \bmod 4$, the family is invariant under a cohomogeneity four $O(n)$-action.*

The group bP_{n+1} is trivial if n is even and has order 1 or 2 for $n \equiv 1 \bmod 4$ [7]. However, the order of bP_{n+1} grows faster than exponentially in n for $n \equiv 3 \bmod 4$. When $n \equiv 1 \bmod 4$, the generator of bP_{n+1} is a Kervaire sphere (see [6]). It is known to admit a cohomogenity one action. Hence it admits almost nonnegative curvature by work of Schwachhöfer and Tuschmann [9].

Our plan to prove Conjecture 2 begins with Brieskorn's observation ([3]) that in dimensions $n \equiv 3 \mod 4$ all elements of bP_{n+1} can be realized by the varieties, $\mathrm{Br}(6k-1, 3, 2, \ldots, 2)$, described by the following set of complex equations:

$$\begin{aligned} u^{6k-1} + v^3 + z_1^2 + \cdots + z_n^2 &= 0, \\ |u|^2 + |v|^2 + |z_1|^2 + \cdots + |z_n|^2 &= 1. \end{aligned} \tag{30.0.1}$$

We have already shown that:

Theorem 30.0.1 *Except when $k = 1$, Br$(6k-1, 3, 2, \ldots, 2)$ does not admit a family of metrics that are invariant under the $(S^1 \times O(n))$-action that leaves Display 30.0.1 invariant.*

When $k = 1$, the quotient, $Br(5, 3, 2, ..., 2)/O(n)$ is homeomorphic to D^4, with singular stratum ∂D^4 containing $K(3, 5)$, the $(3, 5)$–knot. (see [2], [5]).

We observe, by using Theorem B in [8], that in order to prove Conjecture 2, it is sufficient to show that D^4 admits a family of almost non-negatively curved Alexandrov metrics which has the following characteristics:

1. For $x \in K(3, 5) \subset D^4$, the space of unit normal directions is isometric to the spherical suspension of the constant curvature 4 sphere.
2. For $x \in \partial D^4 \setminus K(3, 5)$ the space of directions is isometric to the constant curvature 1 hemisphere.
3. The interior of D^4 is a smooth Riemannian manifold.

Outcome of the Meeting

As Conjecture 1 is part of our plan to prove Conjecture 2, we focused most of our attention at the workshop on obtaining the proof of Conjecture 1. Prior to the meeting most of the details of the proof of Conjecture 1 were written down or had at least been discussed in one way or another. Although our pre-conference efforts were productive, the results were far from publishable form, as different parts of the project had been tackled by different subsets of our group, and the arguments did not hang together well.

We exploited the luxury of being physically together at BANFF to devise a coherent organizational plan. This had at least two positive and unanticipated benefits, which we outline here below.

First, it became clear that it would be considerably easier to approach the proof of Conjecture 1 in the context of certain abstract Alexandrov spaces that we have tentatively decided to call "quotient-like". This led to a long discussion of just what analytic properties characterize those Alexandrov spaces that are quotients of Riemannian manifolds. It seems that we are close to a complete understanding of this characterization, which would be exciting in its own right; in particular, it could lead to further applications of the lifting theorems of [8].

Second, in the quest for this characterization, we were led to the observation that methods of [8] can be applied to infinitely many of the spaces in [1], thus leading us to new and unanticipated examples with almost nonnegative curvature.

In summary, we feel that the week spent at Banff International Research Station during the Research in Teams workshop substantially improved our understanding of the general context of the problems under consideration and we feel that we made significant progress towards solving Conjectures 1 and 2.

Participants

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Chapter 31

Bivariate Orthogonal Polynomials and Eigenvalues of Hankel Matrices (16rit684)

August 28 - September 4, 2016

Organizer(s): Mourad E. H. Ismail (University of Central Florida)

Participants

The participants in this research team were:

1. Mourad E. H. Ismail, University of Central Florida, Orlando, Florida, USA.

2. Zeinab Mansour, King Saud University, Riyadh, Saudi Arabia

3. Ruiming Zhang, Northwest A&F University Yangling, Shaanxi P. R. China.

Both Ismail and Zhang are males and are Canadian citizens. Mansour is a female and is a young researcher. We all stayed at Corbett Hall most of the time and met periodically for discussion.

Background: The original proposal was to study bivariate orthogonal polynomials and eigenvalues of Hankel matrices. Before we came to Banff we made some progress on both topics. It became clear very quickly that we need to concentrate on one of the two topics so we decided to concentrate on the bivariate special functions and related topics. Although there have many many systems of orthogonal polynomials in two or more variables, the class of polynomial we are considering is only few years old, [5], [6], [7], [8], [9], and one of the early examples goes back to Ito, [10]. Contrast this with other types as in [2]. Ito's example is now central to a whole class of polynomial.

Our investigations also led us to consider several special functions in two real variables. A related problem which arose is a q-analogue of the Lidstone series and polynomial expansions [3]. This will lead to a conceptual approach to identities in the the theory of basic hypergeomtric functions and will lead to new identities. This corresponds to expansion of functions around two points. Ismail and his collaborators extensively studied the q-Taylor series and had a lot of success. There is an ongoing book project between Ismail and Dennis Stanton to develop the whole theory of q-special functions using q-Taylor series and Askey-WIlson operators. As such one would expect the q-Lidstone theory to be as rich and the hope is that the few stumbling blocks in the Ismail-Stanton approach to q-series will be resolved using q-Lidstone series.

Progress Report: The participants were engaged in the following projects.

1. Ismail and Zhang completed a 45 page survey article on bivariate orthogonal polynomials. This work started earlier but was completed while the authors were in residence at BIRS. It acknowledges the support of BIRS. This covers the recent progress in the subject and is expected to influence the development of this evolving subject. Ismail is involved in the Digital Library Project by NIST (National Institute of Science and Technology), which is part of the department of Energy. The NIST library will only list formulas so the paper will be a companion to the orthogonal polynomials in several variables material in the NIST project.

2. Ismail and Zhang spent the week before coming to BIRS visiting the university of Calgary, supported by their own research grants, and discussed the topic of inequalities and complete monotonicity of special functions with Peter Zvengrowski. This initiated another research project on special functions of two variables. They started a paper on the subject and typed about 15 pages of it. This work is still going strong and we expect the final product, which will acknowledge support from BIRS, to be about 30 pages long. In a way this is a contribution to the theory of monotonicity of the real, imaginary or modulus of the gamma function initiated by Ahern and Rudin in [1]. A note worthy reference is [11].

3. Ismail and Mansour started the q-Lidstone theory. During the stay they formulated how the expansion will look like and obtained preliminary results. The key step of expanding the Cauchy kernel in a certain Lidstone series has been achieved. Many technical details are left but we are positive that they will be carried out successfully. The finished product will acknowledge support from BIRS.

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