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BIRS

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Five-day Workshop Reports

Chapter 1

Modern Applications of Complex Variables: Modeling, Theory and Computation (15w5052)

January 12 - 16, 2015

Organizer(s): Linda Cummings (New Jersey Institute of Technology), Stefan Llewellyn Smith (UCSD), Paul Martin (Colorado School of Mines), Bartosz Protas (McMaster University)

Overview of the Field

All professional mathematicians have taken an undergraduate class in complex analysis, covering analytic functions, singularities, residue calculus, conformal mapping, and so on. Indeed, one could argue that an early appreciation of the beauty of complex analysis, and of its power to solve a range of physically-arising problems, is a good indicator of a budding mathematician. Nonetheless, following a "golden age" after the Second World War, when complex variable theory (including integral transform methods) was used to solve a huge range of applied, mainly idealized, problems arising in fluid dynamics, elasticity and many other areas, complex analysis fell into disfavor among some applied mathematicians, who believed that all of the exciting discoveries had been made and that the subject had little new to offer. (Classic references from this period include the books of Noble, Sneddon, and Mittra & Lee [31, 37, 48].) In particular, the increasing prevalence of computers towards the end of the 20th century meant that many previously "in principle" calculations could now be done in practice: more complicated models could be quickly and cheaply solved numerically, obviating what some saw as the primary use for complex variable methods, the need for exact solutions to simplified models.

The range of speakers and topics gathered together for the 5-day workshop 15w5052 provided resounding evidence that the field of applied complex analysis is moving into a new era, burgeoning with new discoveries. The broad spectrum of subject areas they represented implies that complex analysis continues to play a central role in many areas of mathematics ranging from harmonic analysis to applied scientific computing. Particularly encouraging was the large number of early to mid-career mathematicians among the participants, whose work indicates that the field will be healthy for decades to come. The following sections of this report summarize the cutting edges of the subject that our Workshop considered, the specific outcomes of our Workshop, and the outlook for the future.

Recent Developments and Open Problems

The 21st century has brought a real renaissance in applied complex analysis, with a new generation of researchers using complex analysis in many different ways. Some of the developments that first led to the temporary decline in popularity of complex analysis as a key tool in applied mathematics are now responsible for its revival. For example, effective computational algorithms are being developed for finding conformal mappings [53] and for solving Riemann–Hilbert problems [40]. Researchers over the past decade or so have also cultivated an increasing appreciation of the role of complex singularities. There are physical examples (such as point vortices or dislocations) and there are mathematical examples: as Littlewood once remarked, the point of the Green's function is to make an infinity do positive work instead of being a disaster! In fact, the evolution and nature of the singularities of a system tell us much about the system itself. For example, the regularity and possible singularity formation in nonlinear PDE problems are conveniently studied by tackling the evolution of the singularities characterizing the extension of the solution to the complex plane [47]. Other good examples are Riemann–Hilbert problems, Painlevé transcendents, exponential asymptotics and Stokes lines (as arising in many different applications), conformal mapping applied to free boundary problems, and vortex dynamics – all of which made an appearance at our BIRS workshop. Again, modern computing power may be harnessed to great effect in studying these problems.

Progress has also been made recently on problems that were thought intractable. For example, within the past 10 years DeLillo et al. [15] and Crowdy [11] have made critical extensions to the classical Schwarz–Christoffel formula, extending its applicability from simply- to arbitrarily-connected domains. Such developments have paved the way for addressing a whole range of highly-relevant multiply-connected problems (for example in the manufacture of microstructured "holey" optical glass fibers). Moreover, new connections continue to be discovered between classical "complex variable" problems and other areas of mathematics: for example, the well-known Hele-Shaw free boundary problem is now known to be closely related to the theory of integrable systems, and to random matrix theory. The discovery of such connections stimulates new research and technology transfer between disciplines, and draws in a new set of researchers with different skill-sets.

Presentation Highlights

The theme of this workshop can be viewed as touching on three areas: new techniques in complex analysis, computational complex analysis, and modeling by means of complex analysis. The presentations covered all three areas. The workshop started with a biographical overview of the work and contributions of Alan Elcrat, by Tom DeLillo. Alan was a major figure in the field of applied and computational complex variable, and was one of the original organizers of the workshop. Most unfortunately, Alan passed away the week after our workshop proposal was accepted; his work was also honored in another workshop talk given by one of the organizers (Bartosz Protas).

Other presentations were divided into a number of groups, leading to an approximate path from more fundamental topics to applications. In addition, Elias Wegert gave a special evening lecture showing how complex functions can be usefully visualized using a variety of colored phase plots to illustrate different aspects of the function's behavior, in particular in the neighborhood of singularities. Apart from the valuable insight such plots afford, these plots are beautiful and mesmerizing, and examples feature in an annual calendar created by Prof. Wegert's group. His MATLAB-based Complex Function Explorer is freely available; further information may also be found in his book [51].

The other workshop presentations are summarized by theme below.

Conformal maps

Conformal mapping is very much a classical topic. Nevertheless, new results, new applications, and new computational methods continue to emerge.

Donald Marshall [29] spoke on recent work with Steffen Rohde on "conformal welding" and its application to planar graphs. His talk focused on computational aspects of this new application of conformal maps. Toby Driscoll described work with Everett Kropf on the creation of the Conformal Mapping Toolbox, an open-source, github-hosted project for the next generation of numerical conformal mapping software. Michael Booty described his joint work with Michael Siegel [5] showing how conformal mapping has been particularly useful in developing and validating a hybrid asymptotic-numerical method for solving problems of two-phase flow with soluble surfactant.

New developments in transform methods

There have been new developments with integral transforms. One that has attracted a lot of interest is the so-called "unified method" of Fokas, which is a method for solving linear boundary-value problems (and some nonlinear problems) [17, 14, 18]. Although realistic applications of this methodology are in their infancy, the organisers hoped that discussion at the workshop would be profitable: they were not disappointed.

Darren Crowdy described some of his recent work giving a new way of understanding the unified method, with applications to harmonic and biharmonic fields in complicated geometries, and involving boundary conditions of mixed type. Tony Davis presented his joint work with Darren Crowdy on the difficulties of using the Fokas method to study Stokes flow in an L-shaped channel; this is a canonical two-dimensional problem for the biharmonic equation. Bernard Deconinck (in joint work with Natalie Sheils) showed how the unified method could be combined with recent insights about interface problems [46] so as to derive fully explicit solutions of the time-dependent Schrödinger equation with piecewise constant potential.

Painlevé theory

The Painlevé equations are six nonlinear ordinary differential equations that have been the subject of much interest in the past forty years. They have arisen in a variety of physical applications, and they may be thought of as defining nonlinear special functions.

Peter Clarkson discussed special polynomials associated with rational solutions for the Painlevé equations and soliton equations [9]. He illustrated how these special polynomials arise in vortex dynamics [8]. Robert Buckingham presented joint work with Peter Miller in which rigorous asymptotic expressions for the large-degree behavior of rational Painlevé-II functions in the entire complex plane are derived [6]. Along the way, the Kametaka-Noda-Fukui-Hirano conjecture from 1986, concerning the pattern of zeros and poles, was confirmed.

Bengt Fornberg started his lecture by noting that the six Painlevé equations have a reputation of being numerically challenging. In particular, their extensive pole fields in the complex plane have often been perceived as "numerical mine fields". He then showed that, on the contrary, these pole fields provide excellent opportunities for fast and accurate numerical solutions across the complex plane. This was illustrated with several examples, from joint work with André Weidman [19, 20] and Jonah Reeger [44, 45]. Saleh Tanveer noted that, although there is much computation of Painlevé solutions, there are no methods to rigorously determine global error bounds. With Ali Adali, he used a recently developed method that was used to prove the Dubrovin conjecture [10] to determine approximate analytical expression for tritronquée solution for Painlevé I with rigorous bounds.

Riemann–Hilbert problems

As their name suggests, Riemann–Hilbert problems are classical. The basic problems are linear and scalar, but there are matrix and nonlinear versions.

Sheehan Olver reviewed several classical problems that can be reduced to Riemann–Hilbert problems, falling into three categories: integral representations, differential equations and inverse spectral problems. In all three cases, applying numerics to the Riemann–Hilbert problem allows for efficient approximation, that is uniformly accurate in the complex plane [40, 41, 42]. A particularly exciting aspect of these advances is the development of computational methods which achieve spectral accuracy, which is nontrivial given that the integral operators involved in the Riemann–Hilbert problem are singular [50]. Vladimir Mityushev solved a Riemann–Hilbert problem for circular multiply connected domains, and then used his results to explicitly write the effective conductivity tensor for regular doubly periodic arrays of cylinders [33, 34]. He then discussed some analogous problems for random arrays [32].

Alexander Minakov reformulated the Cauchy problem for the Camassa–Holm equation in terms of a vector Riemann–Hilbert problem so as to study the asymptotic behavior of the solution of the initial-value problem as $t \to \infty$, thus extending previous work [25]. Elias Wegert gave a survey of nonlinear Riemann–Hilbert problems with an emphasis on geometric aspects. In particular he addressed the existence and uniqueness of solutions for different classes of nonlinear boundary value problems and characterized solutions by extremal properties. He also discussed circle-packing problems [52] and the development of numerical methods.

Partial differential equations

Complex variables arise in various ways in the context of methods for solving partial differential equations.

Seung-Yeop Lee described his recent work with Roman Riser on the behavior of the two-dimensional Coulomb gas system, using large-degree asymptotic expansions of shifted Hermite polynomials [27]. Tom Trogdon presented joint work with Gino Biondini [4] on the Gibbs phenomenon for dispersive partial differential equations. They establish sufficient conditions for the classical smoothness of the solutions of linear dispersive equations for positive times, and they derive an oscillatory and computable short-time asymptotic expansion of the solution. Christopher Green presented his joint work with Jonathan Marshall [23] on constructing Green's function for the Laplace–Beltrami operator on a toroidal surface. It is written in terms of a single complex variable using two special functions: the Schottky–Klein prime function associated with an annulus, and the dilogarithm function.

John King described some model nonlinear parabolic problems, focussing on the implications of the nature of the complex singularities for real-line behavior such as blow up. This research direction is related to the question of global in time regularity of smooth solutions of important equations of mathematical physics, such as the 3D Navier-Stokes and Euler systems, a problem which still remains open [47]. Alexander Odesskii presented a simple construction of integrable Whitham type hierarchies [39]. André Weideman presented his work with Nick Hale on contour integral methods for the integration of elliptic partial differential equations on cylindrical domains. Such methods have been developed by I. P. Gavrilyuk and co-workers, but here the emphasis was placed on practical numerical considerations.

Applications: fluid mechanics

Conformal mapping, and related techniques of complex analysis, have long been applied to classical two-dimensional free boundary fluid dynamical problems such as the Hele-Shaw problem. But there are many other areas of fluid mechanics where the use and development of analytic function theory has been productive.

Chris Howls (in joint work with Jonathan Stone and Rod Self) explained how complex ray theory can be used to determine "cones of silence" in aeroacoustic applications [49]. Jon Chapman (with Chris Lustri, Phil Trinh and Jean-Marc VandenBroeck) considered free-surface potential flow in the limit of small Froude number. He explained how, in that limit, the surface waves are exponentially small, and arise via Stokes' phenomenon [28]. Jean-Marc Vanden-Broeck used numerical methods based on complex variables and series representations to solve a variety of two-dimensional potential free-surface flows, with special attention devoted to flows where the free surfaces intersect rigid surfaces.

Mike Siegel presented his work with David Ambrose in which they showed that a truncated system of equations for water waves that forms the basis of a widely-used numerical method is ill-posed. Their demonstration is based in an essential way on complex analysis. Chris Rycroft discussed his work with Martin Bazant on interfacial dynamics of dissolving objects in fluid flow. Their numerical method tracks the evolution of the object boundary in terms of a time-dependent Laurent series. Kostya Kornev (in joint work with Mars Alimov) introduced the problem of the capillary rise of a meniscus on substrates with complicated shapes [2]. He employed Chaplygin's hodograph transformation, and discovered that the contact line may form singularities even if the fiber has a smooth profile.

Bartosz Protas spoke about an unfinished project, studying vortex stability with Alan Elcrat. Their main focus was on a general approach to analyze the stability of inviscid flows with finite-area vortices [16], using methods of complex analysis. In order to handle the stability problem, new techniques of more widespread utility were devised to shape-differentiate singular contour integrals. Takashi Sakajo, in joint work with Rhodri Nelson and Tomoo Yokoyama, described the entrapment of force enhancing vortex equilibria in the vicinity of a Kasper wing [36].

Robb McDonald (work with A. Khalid, Jean-Marc Vanden-Broeck, Mark Mineev-Weinstein and Giovani Vasconcelos) considered unsteady propagating elliptical bubbles in an unbounded Hele-Shaw cell in the case of zero surface tension [24]. Numerical simulations demonstrate the important role played by singularities of the Schwarz function of the bubble boundary in determining the evolution of the bubble [30]. Scott McCue described his studies (with Bennett Gardiner, Michael Dallaston and Timothy Moroney) of the effect of a kinetic undercooling condition on the interface of an evolving bubble in a Hele-Shaw cell [21].

Other applications

Complex variable methods have a long history in other areas of mechanics, especially in the theory of elasticity. Indeed, it was these applications that motivated deep studies of singular integral equations over contours.

Anna Zemlyanova presented some of her work on systems of singular integro-differential equations in the context of a new model of fracture with a curvature-dependent surface tension [54]. The regularization and numerical solution of these systems was addressed and numerical examples were presented. Yuri Antipov discussed his recent work on singular integral equations on a segment with two fixed singularities [3]. The problem is reduced to a vector Riemann–Hilbert problem on the real axis with a piecewise constant matrix coefficient. As an application, the three-dimensional Dirichlet problem for the Helmholtz equation in the exterior of an infinite cone whose cross-section is a circular sector was solved.

Sonia Mogilevskaya used the Cauchy–Pompeiu formula to reduce integrals over an element arising in the three-dimensional boundary element method to integrals around the boundary of such an element, integrals that can often be evaluated analytically. This leads to fast and efficient algorithms [35, 43]. Andrew Norris gave a lecture on acoustic transparency and causality. It is well known that causality implies that the far-field pattern is analytic in the upper half of the complex ω-plane (ω is the frequency) together with other less well known properties. Implications of these general properties were explored, especially in the context of cloaking [38].

Yuri Godin presented his recent work on exact calculations of effective properties of periodic tubular structures.

His approach is based on the construction of a quasiperiodic harmonic potential in the form of the Weierstrass zeta-function and its derivatives [22] Nick Trefethen presented joint work with Jon Chapman and Dave Hewett on a two-dimensional mathematical model of a Faraday cage. He showed numerical simulations, a theorem proved by conformal mapping, and a continuous model derived by multiple scales analysis.

Scientific Progress Made

With plenty of time available for informal interactions during the breaks and after the sessions, there were many opportunities for the workshop participants to engage in discussion aimed at addressing some of the open problems. While making measurable progress with such problems typically requires much longer time, below we highlight a few topics where some advances have been made.

One of the recurrent themes discussed during the workshop was the development of numerical techniques for the solution of the Riemann–Hilbert and related problems. Of particular interest are methods which achieve "spectral" accuracy (i.e., characterized by the approximation error vanishing exponentially with the refinement of the discretization). Such accuracy is however difficult to achieve given the singularity of the operators involved. Many of the discussions revolved around the presentation of Olver which featured results from his forthcoming monograph [50]. One question discussed at length was the extent to which such highly-accurate numerical methods can be generalized to other problems with a similar structure, such as various singular integral equations arising in the solution of elliptic boundary-value problems (these questions arose also in a number of other presentations).

A closely related topic which received a lot of attention during the workshop was development of software which can be used to solve such problems. An emerging direction in scientific computing, combining numerical and symbolic calculations in a hybrid approach, is represented by `Chebfun` [7], and two key members of the `Chebfun` development team, Toby Driscoll and Nick Trefethen, were present among the workshop participants. A number of discussions concerned how various singular integral operators can be efficiently implemented in `Chebfun`, which will greatly simplify the numerical solution of several problems at the heart of computational complex analysis.

On the more applied side, Llewellyn Smith and Protas were able to draw some interesting and promising connections between the recent work of Elcrat & Protas on the stability of Hill's vortex based on the shape-differential formulation and Llewellyn Smith's earlier studies of the same problem using asymptotic methods [26]. This connection may ultimately lead to a resolution of an open problem in theoretical fluid mechanics.

Outcome of the Meeting

It was evident from the atmosphere and discussions at the meeting that complex analysis in the 21st century is very much alive, vibrant with new theoretical developments, new computational algorithms, new applications, and new challenges, and crucially, with a new generation of mathematicians and physicists eager to address them (e.g. Tom Trogdon, Anna Zemlyanova, Chris Ryland, Chris Green, not to mention the graduate students of many participants who co-authored much of the work presented). The workshop brought together expert participants from different disciplines within Mathematics, Engineering and Physics, with their different perspectives contributing to the session discussions.

A specific session was held on the Wednesday evening (following the entertaining lecture by Elias Wegert on visualization of complex functions) to discuss aspects of the work presented thus far, as well as ideas for future collaborative opportunities. Several options were explored, including an application for a long program at IPAM (the Institute for Pure and Applied Mathematics at UCLA) and a thematic program at the Isaac Newton Institute in Cambridge, UK. Both ideas were received with enthusiasm, and the organizers are currently in discussions with the Director of IPAM, Russ Caflisch, to determine the feasibility of such a program. Two organizers (Protas and Cummings) also recently (March 2015) participated in an Applied and Computational Complex Analysis workshop held at Imperial College, London, organized by BIRS participants Darren Crowdy and Takashi Sakajo. From the discussions at that workshop it was evident that the BIRS workshop is already acknowledged as a landmark event in the recent development of complex analysis. At that workshop, the idea of an Isaac Newton program was discussed further. Crowdy may be willing to act as a local proposer and organizer of such a program.

It was apparent during the many discussion that all participants, and particularly the younger generation, found the BIRS workshop an extremely valuable experience. There are few opportunities to assemble such a critical mass of minds, with different but related perspectives, to use as a sounding-board. In addition, several collaborations were cemented or forged during the workshop, some of which were summarized in §1 above. Again, the younger participants in particular were able to meet for the first time with many researchers whose papers they had read, and took the opportunity to develop connections in informal discussions (the organizers themselves had several such discussions with younger participants; for example, Chris Ryland's talk drew on early work by Cummings & Kornev [13], and Cummings was able to direct Ryland to other work of relevance to his research [12]).

It was also apparent that many participants are engaged in solving physically-relevant problems which impact several other disciplines. Therefore we anticipate that the mathematical techniques that were discussed should ultimately become part of the arsenal deployed by scientists and engineers in solving many real problems in engineering, biological and medical sciences.

Participants

Antipov, Yuri (Louisiana State University)
Booty, Michael (New Jersey Institute of Technology)
Buckingham, Robert (University of Cincinnati)
Chapman, Jon (Oxford University)
Clarkson, Peter (University of Kent)
Crowdy, Darren (Imperial College London)
Cummings, Linda (New Jersey Institute of Technology)
Davis, Tony (UCSD)
Deconinck, Bernard (University of Washington)
DeLillo, Thomas (Wichita State University)
Driscoll, Tobin (University of Delaware)
Fornberg, Bengt (University of Colorado, Boulder, USA)
Godin, Yuri (University of North Carolina at Charlotte)
Green, Christopher (UCSD)
Howls, Chris (University of Southampton)
King, John R. (University of Nottingham)
Kornev, Konstantin (Clemson University)
Lee, Seung-Yeop (University of South Florida)
Llewellyn Smith, Stefan (University of California, San Diego)
Marshall, Donald (University of Washington)
Martin, Paul (Colorado School of Mines)
McCue, Scott (Queensland University of Technology)
McDonald, Robb (University College London)
Minakov, Alexander (Czech Technical University in Prague)
Mityushev, Vladimir (Pedagogical University of Krakow)
Mogilevskaya, Sofia (University of Minnesota)
Norris, Andrew (Rutgers University)
Odesskii, Alexander (Brock University)
Olver, Sheehan (The University of Sydney)
Protas, Bartosz (McMaster University)
Rycroft, Chris (Harvard University)
Sakajo, Takashi (Kyoto University)
Siegel, Mike (New Jersey Institute of Technology)
Tanveer, Saleh (Ohio State University)
Trefethen, Lloyd N. (University of Oxford)
Trogdon, Tom (New York University)
Van-den-Broeck, Jean-Marc (University College London)
Wegert, Elias (Technische UniversitŁt Bergakademie Freiberg)
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Chapter 2

Random Dynamical Systems and Multiplicative Ergodic Theorems Workshop (15w5059)

January 18-23, 2015

Organizer(s): Beniamin Goldys (University of Sydney), Cecilia González-Tokman (University of New South Wales), Anthony Quas (University of Victoria)

Overview of the Field

The conference was intended to broadly address recent developments in random dynamical systems. Modern research in random dynamical systems comes principally from two directions: stochastic partial differential equations and autonomous (non-random) dynamical systems. Both strands, which have traditionally worked largely in isolation from each other, were well represented at the workshop. We estimate that few if any of the participants (including the organizers) had previously met more than 50% of the other participants.

Some of the central themes that emerged during the workshop were as follows:

Synchronization: This is when trajectories of random dynamical systems subjected to the same randomness, but starting from different initial configurations converge in time to a single (random) solution. This appeared in at least 5 talks over the course of the workshop. Crudely speaking, in order to see synchronization, one needs two ingredients: local contraction (negative Lyapunov exponents) so that nearby points approach each other; along with a global irreducibility condition.

Derandomization: By building the realization of the randomness process into the current state of the system, random dynamical systems can often be represented as deterministic dynamical systems on enlarged state spaces. This is a point of entry for the application of the deterministic dynamical systems theory to these questions.

SPDEs as limits of deterministic dynamical systems: This is in a sense the converse of derandomization. Given a fast-slow system (where one set of variables evolves at a rapid rate and a second set evolves slowly, with mutual influence between the two), under suitable conditions, the influence of the fast variables on the slow variables may be averaged, so that the slow variables (in the limit where the ratio of the rates of evolution diverges to infinity) evolve according to an explicit stochastic partial differential equation.

This draws on a large body of work over several decades by a number of authors studying what can be described as dynamical limit theorems and related work. For these results, one is studying the behaviour of the sequence of partial sums of an 'observable' (a function) evaluated at regular intervals along the orbit of a dynamical system. If the dynamical system is sufficiently hyperbolic and the function is sufficiently smooth, one has a decay of correlation between the summands, so that it may not be too surprising that the distribution of the partial sum up to time N is approximately normal for sufficiently large N. This statement is known as a dynamical central limit theorem.

More refined questions (dynamical invariance principles) study versions of the graph of the interpolated partial sums with suitable scalings of both time and displacement. In this case, under suitable hypotheses, one shows that the scaled partial sum trajectory converges in distribution to a Brownian motion.

Multiplicative ergodic theorems: In the dynamical systems literature, for many years the focus was restricted to dynamical systems where the noise was independent and identically distributed. In practice, this means that one has a family of maps, and at each stage a map is randomly selected and applied to give a new point of the orbit. The independence in the map selection made it possible to describe the evolution of the random dynamical system using annealed evolution operators, in which one computes the average distribution of the random system as the state evolves. The independence can be shown to ensure that the distribution at the $(n+1)$st time step is a function of the distribution at the nth time step. This is no longer true if the maps are not applied in an i.i.d. manner.

Since the assumption of i.i.d. random maps is not reasonable in many physical situations (e.g. where a dynamical system is forced by a slowly moving much more massive dynamical system), there was a substantial gap between the theory and desired applications. One way to deal with this issue that has seen a great deal of development in the last 5–10 years is the use of the Oseledets multiplicative ergodic theorem (MET) to understand non-i.i.d.forcing of dynamical systems. An informal statement of the MET is that one has a base dynamical system and for each point of this system, one has a matrix. As one iterates the base dynamical system, one takes the product of the matrices along the orbit. This structure (a multiplicative cocycle) occurs widely in dynamical systems. The MET may be loosely seen as an analogue of the Jordan normal form, where the space on which the matrices act ($\mathbb{R}^d$) is decomposed into a direct sum of vector subspaces, each expanding or contracting at a different rate under the iterated matrix products.

Recent generalizations of the MET replace matrices by operators acting on Banach spaces. Here, rather than considering the annealed system, one uses the MET to prove structural properties of the quenched evolution operators (that is where one studies one realization of the randomness at a time).

Bifurcations of random dynamical systems: It has been known for a long time that the addition of (additive) noise can destroy bifurcations. A simple mechanism for this is that noise allows a random dynamical system to move from one fixed point to another. This mechanism can be curtailed or prevented if instead of adding unbounded noise, the noise is constrained to lie in a bounded region.

Recent Developments and Open Problems

The workshop had an open problem session early in the week, where all participants were invited to present a problem or topic for discussion. There were presentations, as follows.

1. Thomas Kaijser:

   In work from the 1970's and 80's, Kaijser showed that i.i.d. random dynamical systems with the property that the maps are bi-Lipschitz (condition R), and satisfy a minimality condition allowing any initial point to be moved arbitrarily close to a desired target point in a finite number of steps (condition M) that satisfy an additional contraction property (condition B or G) have the property that they satisfy synchronization. As a result, there is a unique invariant measure. Kaijser's question was regarding the necessity of the contraction condition.

2. Ian Melbourne:

   Abstract: This conjecture was presented at the open question session on Monday. A successful resolution would extend the scope of the theory presented in the talk "Stochastic limits for deterministic fast-slow systems" given on Thursday.

Moment estimates Let $T : X \to X$ be a measure preserving transformation on a probability space (X, μ). Let P denote the transfer operator, so $\int Pv\, w\, d\mu = \int v\, w \circ T\, d\mu$ for all $v \in L^1$, $w \in L^\infty$.

Suppose that v is a mean zero L^p observable, $p \geq 2$. If $\{v \circ T^j,\ j \geq 0\}$ is a sequence of martingale differences, then it is standard that

$$\Big\|\sum_{j=0}^{n-1} v \circ T^j\Big\|_p \leq Cn^{1/2}, \tag{2.0.1}$$

for some constant C (depending on v). More generally, if $v = m + \chi \circ T - \chi$ where $m, \chi \in L^p$ and $\{m \circ T^j,\ j \geq 0\}$ is a sequence of martingale differences, then (2.0.1) still holds.

For large classes of dynamical systems, sufficiently regular observables v admit a decomposition of the form

$$v = m + \chi \circ T - \chi, \quad m, \chi \in L^p, \quad Pm = 0. \tag{2.0.2}$$

The last condition means that $E(m|T^{-1}\mathcal{M}) = 0$ where $\mathcal{M}$ denotes the underlying σ-algebra. It follows that $\{m \circ T^j,\ j \geq 0\}$ is a sequence of "reverse" martingale differences. Passing to the natural extension of T, it is easy to show that (2.0.1) still holds.

In the situation of (2.0.1), it is often assumed that v lies in L^∞ even though m and χ are only L^p. In such situations, $\| \sum_{j=0}^{n-1} v \circ T^j \|_q$ is well-defined for all q and it is natural to ask for which q is it the case that $\sqrt{n}$ growth holds. Using an inequality of Rio [8], it was shown in [5] that $\sqrt{n}$ growth holds for $q = 2p$ and that this is optimal. Moreover, it suffices that $p \geq 1$. To summarise,

Proposition [5]. Suppose that $v \in L^\infty$ has mean zero and that the decomposition (2.0.2) holds for some $p \geq 1$. Then there is a constant $C = C_v > 0$ such that

$$\Big\|\sum_{j=0}^{n-1} v \circ T^j\Big\|_{2p} \leq Cn^{1/2}, \quad \text{for all } n \geq 1.$$

Iterated moment estimates

Now let $v_1, v_2 : X \to \mathbb{R}$ be a pair of mean zero L^∞ observables admitting a decomposition of the form (2.0.2). We are interested in the moments of the iterated sum $I_n = \sum_{0 \leq i < j < n} v_1 \circ T^i\, v_2 \circ T^j$. It is straightforward to show that $\|I_n\|_{p/2} \leq Cn$ provided $p \geq 4$, and using Rio's inequality it is shown in [3, Proposition 7.1] that $\|I_n\|_{2p/3} \leq Cn$ provided $p \geq 3$. It seems unlikely that this is optimal.

Conjecture [3]. Suppose that $v_1, v_2 \in L^\infty$ have mean zero and satisfy the decomposition (2.0.2) for some $p \geq 2$. Then there is a constant $C = C_{v_1,v_2} > 0$ such that

$$\Big\|\sum_{0 \leq i < j < n} v_1 \circ T^i\, v_2 \circ T^j\Big\|_p \leq Cn, \quad \text{for all } n \geq 1.$$

A successful resolution of this conjecture would extend the scope of the results on homogenization in [3]. In [3, Corollaries 9.2 and 9.4], it would suffice that $p > 3$, whereas currently we require $p > 9/2$. In the situation of intermittent maps ([3, Example 10.3]), the results currently valid for $\alpha < \frac{2}{11}$ would be valid for $\alpha < \frac{1}{4}$.

3. Julian Newman:

Suppose we have a standard measurable space (X, Σ), a probability space $(I, \mathcal{I}, \nu)$ and an I-indexed family $(f_\alpha)_{\alpha \in I}$ of functions $f_\alpha : X \to X$ such that the map $(\alpha, x) \mapsto f_\alpha(x)$ is $(\mathcal{I} \otimes \Sigma, \Sigma)$-measurable. Let ρ be a probability measure on X which is stationary with respect to the transition kernel

$$P(x, A) \;=\; \nu(\alpha \in I \,:\, f_\alpha(x) \in A).$$

For each $n \in \mathbb{N}$, let Q_n be the image measure of $\nu^{\otimes n}$ under the measure-valued map $(\alpha_1, \ldots, \alpha_n) \mapsto (f_{\alpha_n} \circ \ldots \circ f_{\alpha_1})_* \rho$. It can be shown that there exists a probability measure Q on the space of probability measures on X such that, given any separable metrisable topology on X generating Σ, Q_n converges to Q in the narrow topology of the narrow topology of X. One can then show that the probability measure $\rho^{(2)}$ on $X \times X$ given by

$$\rho^{(2)}(B) \;=\; \int_{Pr(X)} \mu \otimes \mu(B)\, Q(d\mu)$$

is stationary with respect to the two-point transition kernel

$$P^{(2)}(x, y, B) \;=\; \nu(\alpha \in I \,:\, (f_\alpha(x), f_\alpha(y)) \in B).$$

Now it is well-known that for deterministic dynamical systems, an invariant measure μ is weakly mixing if and only if $\mu \otimes \mu$ is ergodic with respect to the two-point motion. Our question is: if $\rho^{(2)}$ is ergodic with respect to the transition kernel $P^{(2)}$, does it follow that ρ is weakly mixing with respect to the transition kernel P?

4. Charlene Kalle asked a question about the invariant measure of a dynamical system arising in the study of some hybrid continued fraction algorithms. This system can be seen as a random combination of the regular continued fraction transformation (or Gauss map) and the backwards continued fraction transformation (or Rényi map). Specifically the question asks whether the absolutely continuous invariant measure of this system has a piecewise analytic invariant density.

5. Andy Hammerlindl:

   Convergence of the spectrum for skew products. Is it true that for certain skew products over expanding maps the spectrum of the transfer operator converges in some sense as the skew product converges to a direct product?

Besides the formal open problem session, several talks included open problems and conjectures.

6. In their joint talk, Gess and Scheutzow posed the following problem: does an SDE of gradient type with additive noise which admits an invariant probability measure always have a negative Lyapunov exponent (or does it at least satisfy the slightly weaker form of local asymptotic stability introduced in the talk) ? (The slides are available online at the workshop website.) Another open problem which came up during the discussion of the talk was: does the Lorenz system (which is not of gradient type) with additive noise have a positive Lyapunov exponent for small noise intensity and a negative one for large noise intensity? Simulations suggest that the answer is 'yes'.

7. Z. Brzezniak asked whether there exists conditions on the noise implying that the invariant measure for the 2-d stochastic Navier Stokes equations in unbounded domains satisfying the Poincaré inequality is unique (even in the additive noise case).

8. M. Nicol has posed the following question. Suppose $T : X \to X$ is an ergodic transformation of (X, μ).

   Borel-Cantelli Properties: Given a sequence of sets (A_n) (balls, rectangles,...) such that $A_j \in X$ and $\sum_j \mu(A_j) = \infty$, does $T^n(x) \in A_n$ infinitely often (i. o) for μ a. e. $x \in X$? If so, is there a quantitative rate?

   We let

$$S_n(x) = \sum_{j=0}^{n-1} 1_{A_j}(T^j x)$$

$$E_n = \sum_{j=0}^{n-1} \mu(A_n)$$

We say a sequence of sets (A_n) satisfies the Strong Borel-Cantelli (SBC) property if $\sum_{j=0}^{\infty} \mu(A_j) = \infty$ and

$$\lim_{n\to\infty} \frac{S_n(x)}{E_n} = 1$$

for μ a. e. $x \in X$.

A sequence of sets (A_n) satisfies the Borel-Cantelli (BC) property if $S_n(x)$ is unbounded for μ a. e. $x \in X$.

Question: Are there examples of ergodic dynamical systems and sequences of balls or rectangles which are BC but not SBC? What conditions entail BC if and only if SBC?

9. Lian and Lu posed the problem of proving a multiplicative ergodic theorem, which is valid in non-separable Banach spaces. This would be an extension of their recent monograph [4] in which they proved an MET where instead of a matrix for each point of the base system, one has an operator on a Banach space. This would likely also extend work of González-Tokman and Quas [2]. Of special interest in applications is the situation where the Banach space is L^∞, as it is important for PDE applications.

Presentation Highlights

The workshop included 20 talks overviewing recent progress and challenges around theoretical, applied and numerical sides of the subjects of random dynamical systems and multiplicative ergodic theorems.

Kening Lu (Brigham Young University) gave a talk on Entropy, Chaos and weak Horseshoes for Infinite Dimensional Random Dynamical Systems. In particular, he gave an answer to a problem on characterizing the chaotic behavior of orbits topologically or geometrically in the presence of only positive entropy for infinite dimensional dynamical systems. He showed that if a random dynamical system has a compact random invariant set such as random attractor with positive topological entropy, then the system is chaotic and has a weak horseshoe. As a corollary, he presented the same conclusion for a deterministic dynamical system with a compact invariant set of positive topological entropy. The chaotic behavior here is due to the positive entropy, not the randomness of the system. This is a joint work with Wen Huang.

Ian Melbourne (University of Warwick) talked about stochastic limits for deterministic fast-slow systems of the form

$$\dot{x} = a(x, y) + \varepsilon^{-1} b(x, y), \dot{y} = \varepsilon^{-2} g(y),$$

where it is assumed that b averages to zero under the fast flow generated by g. Here $x \in \mathbb{R}^d$ and y lies in a compact manifold. He presented conditions under which solutions to the slow equations converge to solutions of a d-dimensional stochastic differential equation as $\varepsilon \to 0$. The limiting SDE is given explicitly. The underlying theory applies when the fast flow is Anosov or Axiom A, as well as to a large class of nonuniformly hyperbolic fast flows (including the one defined by the well-known Lorenz equations), and the main results do not require any mixing assumptions on the fast flow. This is joint work with David Kelly and combines methods from smooth ergodic theory with methods from rough path theory.

Gary Froyland (University of New South Wales) talked about existence, stability, and applications of Oseledets splittings for semi-invertible linear cocycles. He reported on a program of work to establish existence and stability results for linear cocycles in the semi-invertible situation - where the driving mechanism is invertible, but the linear actions may be non-injective - and to create numerical methods to apply to real-world models and data. The "existence of Oseledets splitting" results provide a stronger multiplicative ergodic theorem than the "classical" theorems, which only guarantee the existence of measurable Oseledets filtrations. The stability results concern continuity properties of the Lyapunov exponents and their corresponding splitting elements when the linear actions are subjected to a variety of perturbations. The applied motivations for this work are the detection and tracking of so-called coherent structures in time-dependent dynamical systems, and I will also report on the application of these constructions to fluid flow in the ocean and atmosphere. This is joint work with Cecilia González Tokman, Christian Horenkamp, Simon Lloyd, Adam Monahan, Anthony Quas, Vincent Rossi, Naratip Santitissadeekorn, Alex Sen Gupta, and Erik van Sebille.

Matthew Nicol (University of Houston) presented recent results on annealed and quenched limit theorems for random expanding dynamical systems. In particular he discussed results on annealed and quenched versions of a central limit theorem, a large deviation principle, a local limit theorem, and Erdös- Renyi type limit laws. This is joint work with Romain Aimino (Aix Marseille Universite) and Sandro Vaienti (CPT Luminy).

Chris Bose (University of Victoria) presented asymptotics for random intermittent maps in the context of expanding interval maps with a neutral fixed point. These are examples of nonuniformly hyperbolic systems which are frequently studied for their potential to give interesting statistical behaviour such as sub-exponential decay of correlation, intermittency or so-called anomalous diffusion (different terms that amount to essentially the same thing: slow relaxation to equilibrium). A random map (skew product with a Bernoulli shift) constructed from a family of such nonuniformly hyperbolic maps undoubtedly inherits some of these intermittency features, but exactly how they combine may not be immediately obvious. He showed, among other results, that the rate of correlation decay is completely determined by the 'least nonuniformly hyperbolic' map in the family, no matter how infrequently the map is chosen in the randomization. This talk reports on joint work with Wael Bahsoun and Yuejiao Duan, University of Loughborough, UK.

Andrew Török (University of Houston) presented an almost sure invariance principle for sequential and non-stationary dynamical systems. This strong form of approximation by Brownian motion was shown to hold in various examples, including observations on sequential expanding maps, perturbed dynamical systems, non-stationary sequences of functions on hyperbolic systems as well as applications to the shrinking target problem in expanding systems. (Authors: Nicolai Haydn (USC), Matthew Nicol (UH), Andrew Török (UH), Sandro Vaienti (Marseille).)

Ian Morris (Surrey) talked about the transfer operator for the binary Euclidean algorithm. This algorithm is a modification of the classical Euclidean algorithm which replaces division by an arbitrary integer with division by powers of two only. Statistical properties of the classical Euclidean algorithm – such as the average number of steps required to process a pair of integers both of which are less than N – can be studied via the thermodynamic formalism of the Gauss map acting on the unit interval. To investigate similar properties for the binary Euclidean algorithm one must instead study the thermodynamic formalism of an IID random dynamical system on the interval. He described a recent result on the transfer operator of the binary Euclidean algorithm which can be applied to resolve conjectures of R.P. Brent, B. Valleé and D.E. Knuth.

Dalia Terhesiu (University of Vienna) presented a renewal scheme for non uniformly hyperbolic flows. In recent work, I. Melbourne and D. Terhesiu, 2014 obtain optimal results for the asymptotic of the correlation function associated with both finite and infinite measure preserving suspension semiflows over Gibbs Markov maps. The involved observables are supported on a thickened Poincaré section. In more recent work with H. Bruin, a different renewal scheme for suspension flows over non uniformly hyperbolic maps is investigated by inducing to a well chosen region Y of the same dimension as the manifold (on which the flow is defined). By forcing expansion on the flow direction, they can ensure that the induced version of the flow is a hyperbolic map F. Combined with the type of renewal equation established in Melbourne and Terhesiu, 2014 and several abstract assumptions on the hyperbolic map F (and thus on the underlying map of the suspension flow), this scheme is used to estimate the correlation function of observables supported on the whole region Y.

Jairo Bochi (Pontificia Universidad Católica de Chile) talked about optimization of Lyapunov exponents of matrix cocycles. The main result presented says that if a 2x2 one-step cocycle has certain hyperbolicity properties (namely, there exist strictly invariant cones whose images do not overlap) then the Lyapunov-optimizing measures have zero entropy. The proof has two steps: first, a generalization of the Barabanov norm (similar to Mañé lemma) and second, a study of geometrical constraints between the invariant directions.

Christoph Kawan (New York University) talked about entropy for control problems and random escape rates. Namely, this talk discussed the control-theoretic problem to determine the smallest rate of information in a feedback loop above which a control system can solve a given control task. Such minimal data rates can be described by quantities that resemble topological entropy. For the control problem to render a given subset of the state space invariant, the associated entropy is related to escape rates of random dynamical systems which arise by putting shift-invariant measures on the space of admissible control functions. Here the multiplicative ergodic theorem comes into play, which allows to estimate the escape rates in terms of Lyapunov exponents.

Martin Rasmussen (Imperial College) talked about bifurcations of random dynamical systems. Despite its importance for applications, relatively little progress has been made towards the development of a bifurcation theory for random dynamical systems. In this talk, it was demonstrated that adding noise to a deterministic mapping with a pitchfork bifurcation does not destroy the bifurcation, but leads to two different types of bifurcations. The first bifurcation is characterized by a breakdown of uniform attraction, while the second bifurcation can be described topologically. None of these bifurcations correspond to a change of sign of the Lyapunov exponents, but it was explained that these bifurcations can be characterized by qualitative changes in the dichotomy spectrum and collisions of attractor-repeller pairs. This is joint work with M. Callaway, T.S. Doan, J.S.W Lamb (Imperial College) and C.S. Rodrigues (MPI Leipzig).

Anna Cherubini (University of Salento) discussed attractors for nonautonomous random dynamical systems with an application to stochastic resonance. She considered random dynamical systems with nonautonomous deterministic forcing and provide existence results for nonautonomous random attractors. In particular, she proved the existence of an attracting random periodic orbit for a class of one-dimensional random dynamical systems with a time-periodic forcing, generalising results obtained by Hans Crauel and Franco Flandoli. As an application, she discussed a standard model for the stochastic resonance, given by the one-dimensional 'overdamped' approximation of the stochastic Duffing oscillator. (Joint work with J.S.W. Lamb, M. Rasmussen and Y. Sato.)

Zdzislaw Brzezniak (York University) talked about invariant measures stochastic Navier-Stokes equations in unbounded domains via bw-Feller property. He described a general result on the existence of invariant measures for Markov processes having the so-called bw-Feller property and showed how this can be applied to stochastic Navier-Stokes equations in unbounded domains. This talk is based on joint works with M. Ondrejat and Ela Motyl. The results presented are in some sense generalisations of related results for stochastic nonlinear beam and wave equations (where a Pritchard-Zabczyk trick playes an essential rôle) obtained in a joint woth with M. Ondrejat and J. Seidler.

Benjamin Gess (University of Chicago) and Michael Scheutzow (Technische Universität Berlin) gave a joint presentation on recent results about synchronization by noise. They introduced sufficient conditions under which weak random attractors for random dynamical systems consist of single random points. These conditions focus on SDE with additive noise, for which they are also essentially necessary. In addition, they identified sufficient conditions for the existence of a minimal weak point random attractor consisting of a single random point.

As a model example, they proved synchronization by noise for an SDE with drift given by a (multidimensional) double-well potential and additive noise. While similar results are well-known in one dimension, these make essential use of monotonicity, which is is not available in higher dimensions. Another key simplifying assumption, that of convexity of potential, is also unavailable in this problem.

This work raises a number of interesting questions because the mechanism for synchronization exhibited in their work has extremely low (but still positive!) probability at small values of the noise parameter. Simulations indicate that the mechanism for synchronization that they study (while sufficient to show that synchronization will eventually take place) is different from the one that the system uses in practice. A number of very interesting questions were raised about the true synchronization mechanism, its time to occur. Tantalizingly, this suggests that in this example, the time taken for weak synchronization (any two fixed points are close w.v.h.p.) is much lower than the time taken for strong synchronization (any compact region is contracted to arbitrarily small diameter w.v.h.p.). This is joint work with Franco Flandoli.

Peter Imkeller (HU Berlin) talked about the dynamics of the Chafee-Infante equation with Lévy noise. Dynamical systems of the reaction-diffusion type with small noise have been instrumental to explain basic features of the dynamics of paleo-climate data. For instance, a spectral analysis of Greenland ice time series performed at the end of the 1990s representing average temperatures during the last ice age suggest an $\alpha-$stable noise component with an $\alpha \sim$ 1:75: The model of the time series consisted of a dynamical system perturbed by α-stable noise, and he introduced an efficient testing method for the best fitting α. A class of reaction-diffusion equations with additive α-stable Lévy noise (a stochastic perturbation of the Chafee-Infante equation) was introduced, in order as a generalization of the solution of this model selection problem. He described a study of exit and transition between meta-stable states of their solutions. (Joint work with A. Debussche, J. Gairing, C. Hein, M. Högele, I. Pavlyukevich)

Szymon Peszat (Institute of Mathematics, Jagiellonian University and Institute of Mathematics, Polish Academy of Sciences) spoke about time regularity of solutions to SPDEs. When driven by a Wiener process, this regularity can be studied using either Kolmogorov criterion, Kotelenez theorem or Da Prato-Kwapień-Zabczyk factorization. It turns out that very often the solution is continuous is a given state space E even if the noise takes values in a bigger space $U \hookleftarrow E$. If the noise is of jump type and does not take values in the space space then typically the solution is not càdlàg. In fact during the talk different concepts of càdlàg property were discussed. A special emphasis was put on infinite systems of linear equations driven by independent Lévy processes. The talk was based on the papers [6, 7, 1].

Barbara Gentz (University of Bielefeld) talked about the effect of noise on mixed-mode oscillations. Many neuronal systems and models display so-called mixed-mode oscillations (MMOs) consisting of small-amplitude oscillations alternating with large-amplitude oscillations. Different mechanisms have been identified which may cause this type of behaviour. In her talk, she focused on MMOs in a slow-fast dynamical system with one fast and two slow variables, containing a folded-node singularity. The main question she addressed whas whether and how noise may change the dynamics.

After outlining a general approach to stochastic slow-fast systems, she discussed how to apply this method to the model system, showing the existence of a critical noise intensity beyond which the small-amplitude oscillations become hidden by noise. Furthermore, she showed that in the presence of noise sample paths are likely to jump away from so-called canard solutions earlier than the corresponding deterministic orbits. This early-jump mechanism can drastically change the mixed-mode patterns, even for rather small noise intensities. The methods used to derive the results range from deterministic bifurcation theory and averaging to martingale techniques and estimates on Markov transition kernels. Joint work with Nils Berglund (Université dOrléans) and Christian Kuehn (TU Wien).

Francesco Ginelli (University of Aberdeen) talked about characterizing dynamics with covariant Lyapunov vectors (CLVs), or Oseledets splittings, which have an important tool for characterizing chaotic dynamics in high dimensional systems. CLVs define an intrinsic, non orthogonal basis at each point in phase space which is equivariant with the dynamics. He presented details of the dynamical algorithm he introduced to efficiently compute CLV's. He also discussed its numerical performance and compared it with other algorithms presented in the literature. He presented selected applications of CLV's to characterize the collective dynamics of globally coupled systems, to quantify the degree of hyperbolicity, and to evaluate the number of effective degrees of freedom in chaotic, spatially extended dissipative systems such as the Kuramoto-Sivashinsky equation.

There were also two student talks: Julian Newman (Imperial College) presented necessary and sufficient conditions for stable synchronisation in random dynamical systems, and Joseph Horan (University of Victoria) talked about triangularizability in the Multiplicative Ergodic Theorem.

Scientific Progress Made and Comments from Participants

- Dalia Terhesiu has discussed research visits of two workshop participants to Vienna. In March, 13 or April 15 2015 Ian Morris will give a talk in the Budapest-Wien dynamical seminar. Gary Froyland has been invited to give a talk in the weekly dynamical seminar organized at Vienna University in June/July 2015.

- Terhesiu and Melboune established some research steps on the topic of mixing for the periodic geodesic flow.

- Brzezniak started a collaboration with Chojnowska-Michalik about the generalisations of the results presented by Imkeller to the 2-d domains (for instance a sphere S^2).

- Brzezniak discussed with Peszat a possibility of restarting their work (joint with R Tribe) about the 2-d Anderson model with space time white noise.

- Scheutzow and Brzezniak discussed briefly their ongoing project on stabilization by noise for linear SPDEs in Hilbert spaces.

- Brzezniak and González-Tokman discussed about prospects of extending results on multiplicative ergodic theorems and Oseledets splittings for non-invertible maps to the case of SPDEs.

- Horan mentioned: "talking to Ian Morris helped guide me to some previously unknown literature on the subject of my research. This was extremely invaluable, for both motivating and placing my research in the field. As well, the wide variety of talks helped introduce me to other areas of dynamical systems, which helps me to get a better picture of what's out there and what I might wish to research".

- Charlene Kalle is using the techniques presented by Matt Nicol, to see if she can prove CLT, large deviations and dynamical Borel-Cantelli lemmas for a random composition of Gauss like maps. Nicol mentions: "It will need some modification of our proofs as the partition on which the maps are piecewise C^2 is infinite. However recent work of Inoue proofs a spectral gap so some of the techniques I spoke about (my joint work with Aimino and Vaienti) will almost certainly work".

- Scheutzow states: "I had fruitful scientific discussions with Martin Rasmussen, Maximilian Engel and Julian Newman about questions related to synchronization by noise and Hopf bifurcations. I agreed with Martin on a continued exchange of ideas in the future (I invited him to come to Berlin for a few days and he invited me to come to Imperial College for a few days later this year). I had discussions with Kening Lu on possible generalizations of the concept of entropy. I had an interesting discussion with Anthony Quas about the order of magnitude of the time until synchronization takes place when the noise intensity of an sde tends to 0. I discussed with Hans Crauel and Benjamin Gess about invariant measures for RDS which are not of white noise type (it is possible that this will initiate a collaboration resulting in a joint paper)."
- Bose writes: "As someone who works mostly on the theory side, I found the excellent presentations on applications to be really inspiring. Specific high points for me:
  - Matt Nicol's talk. I was really intrigued by his problem on chilled limit theorems and the technical obstructions that he identified. We talked briefly about this. It was something I was not aware of prior to this conference. Matt made it very clear what the problem is, and what needs to be done.
  - Ian Melbourne's presentation on fast-slow systems. As always, a masterful and inspiring presentation. Although I've dabbled in random maps for a number of years, I was not aware of the nice applications that are going on out there. Ian's talk bridged the gap between theoretical results and implications for application.
  - Dalia Terhesiu's talk, and a short discussion we had afterward. The kinds of problems she reported on are very close to things I have been thinking about. Dalia will be in Loughborough later this spring, at a workshop organized by my co-author Bahsoun and we were kind of hoping to pick up the conversation again, through Bahsoun if I cannot be there in person."
- Julian Newman and Anthony Quas solved Thomas Kaijser's open problem 1, to decide whether there exists a bi-Lipschitz iterated function system on a compact metric space with uniformly transitive dynamics and yet with more than one stationary probability measure. (They found, independently, that such systems do exist.)
- Newman also presented a problem during his talk, which Anthony Quas then solved. The question was to prove or disprove a formula for the size of the random attractor of the RDS generated by a circle map of degree one perturbed by conjugation with a random rotation selected with uniform distribution.
- Gottwald stated: "I thoroughly enjoyed it and learned a lot from the talks and the many discussions I had".
- Blumenthal, González-Tokman and Quas discussed various proof strategies and techniques for establishing multiplicative ergodic theorems on Banach spaces. Such discussions are likely to lead to future collaborations connected to Lian and Lu's problem 9.
- Török invited González-Tokman to give a seminar at the University of Houston in February.
- Nicol and González-Tokman discussed the conditions for the quenched CLT presented in Nicol's talk. The discussion is likely to continue during González Tokman's visit to the University of Houston.
- Froyland and González-Tokman continued their ongoing discussion on finding more efficient ways of using Oseledets splitting information to identify coherent structures from models and data coming from geophysical flows.
- Doan and González-Tokman talked about various open problems regarding Lyapunov exponents for operator cocycles on Banach spaces. This discussion will likely develop into collaborative work.

- Kalle and González-Tokman initiated a discussion on metastability questions, and about the possibility of using more probabilistic tools in the deterministic setting.

- Before the workshop started, some people indicated to us that they would be interested in watching the talks online. Alexandra Neamtu (Jena) commented afterwards: "It was a great workshop and I am very happy that I could watch the talks and learn a lot."

Outcome of the Meeting

BIRS provided a wonderful atmosphere for research and scientific collaboration. New research connections have been established among the communities of random, non-autonomous and deterministic dynamical systems. Several scientific collaborations have been initiated and progressed during the workshop week.

The scientific talks presented at the workshop will remain available for participants and other interested scientists at the workshop website **http://www.birs.ca/events/2015/5-day-workshops/15w5059**. During the workshop, talks were recorded using BIRS video facilities. Furthermore, slides of most talks – all except those presented at the board or with the document camera – are available at the workshop website.

Participants

Blumenthal, Alex (New York University)
Bochi, Jairo (Pontificia Universidad Catlica de Chile)
Bose, Chris (University of Victoria)
Brzezniak, Zdzislaw (York University)
Cherubini, Anna Maria (University of Salento, Italy)
Chojnowska-Michalik, Anna (University of Lodz)
Crauel, Hans (Johann Wolfgang Goethe-UniversitŁt Frankfurt)
Doan, Thai Son (Imperial College London)
Engel, Maximilian (Imperial College London)
Froyland, Gary (University of New South Wales)
Gentz, Barbara (University of Bielefeld)
Gess, Benjamin (University of Chicago)
Ginelli, Francesco (University of Aberdeen)
Gonzlez Tokman, Cecilia (The University of Queensland)
Gottwald, Georg (The University of Sydney)
Hammerlindl, Andy (University of Sydney)
Homburg, Ale Jan (University of Amsterdam)
Horan, Joseph (University of Victoria)
Imkeller, Peter (Humboldt University)
Kaijser, Thomas (Linkping University)
Kalle, Charlene (Leiden University)
Kawan, Christoph (New York University)
Lian, Zeng (Loughborough University)
Lu, Kening (Brigham Young University, USA)
Melbourne, Ian (University of Warwick)
Morris, Ian (University of Surrey)
Newman, Julian (Imperial College London)
Nicol, Matthew (University of Houston)
Peszat, Szymon (Jagiellonian University)
Quas, Anthony (University of Victoria)
Rasmussen, Martin (Imperial College London)
Scheutzow, Michael (Technische Universitat Berlin)
Terhesiu, Dalia (University of Vienna)

Török, Andrew (University of Houston)

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Chapter 3

Mathematics of Communications: Sequences, Codes and Designs (15w5139)

January 25 - 30, 2015

Organizer(s): Shamgar Gurevich (University of Wisconsin-Madison), Jonathan Jedwab (Simon Fraser University), Dieter Jungnickel (Universität Augsburg), Vladimir Tonchev (Michigan Technological University)

Overview

Modern society depends crucially on the ability to store and transmit large amounts of digital information at high speed. Satellite communication, movies on demand, portable music players, flash drives, and cellphones all rely on the mathematical theory of coding to ensure that the original images, speech, music, or data can be recovered perfectly even if mistakes are introduced during storage or transmission [2], [27]. As coding theory has developed over the last 65 years, deep connections with the theory of combinatorial designs [1], [3], [7] and with sequences [18] [20], have been discovered. Emerging applications continually lead to new problems of codes, designs and sequences; conversely, new theoretical developments in these areas enable novel applications [8].

The workshop brought together representatives of the applied and theoretical communities that study the mathematics of communications, working in Mathematics, Computer Science, and Engineering departments, in order to promote new linkages and collaborations. Among the participants were four graduate students and two postdoctoral fellows. Five speakers gave extended expository lectures, accessible to all participants, with an emphasis on methods, approaches, and open questions. Nineteen speakers gave contributed talks on a range of theoretical and practical topics. A panel discussion gave participants an opportunity to reflect on the entire workshop and to assess future research directions. Throughout these events, as well as in numerous individual interactions, participants exchanged information and ideas about both theoretical and practical aspects, and identified new connections between the principal objects of study.

Presentations

Codes and Designs

Tuvi Etzion opened the workshop with a wide-ranging expository talk illustrating many of the deep connections between coding theory and design theory. His examples included classical connections between perfect codes, Steiner systems, maximum distance separable (MDS) codes, and projective geometries, as well as modern applications of codes and designs in write-once memory [32], network coding [31], and distributed storage. Etzion emphasized throughout that, despite considerable recent progress, major open problems remain.

Coding Theory and Patent Law Jim Davis gave a fascinating account of his role as a testifying expert in 2012, in one of the more than fifty patent lawsuits fought in multiple jurisdictions between Apple and Samsung over third generation wireless technology. The disputed patent [23] describes a method for transmitting multiple services simultaneously and correctly, and is essential to an international standard that ensures wireless devices can interoperate. The technical heart of the patent centres on a specific subcode of the second order Reed-Muller code of length 32. Davis described how academic questions of coding theory intersected with patent law, against a backdrop of intense global competition in the mobile communications market. The dispute culminated in 2013 in President Obama's overturning of an International Trade Commission ban on the import of certain models of Apple products into the U.S., which was the first time a U.S. President had vetoed such a ban in more than 25 years [29].

Sequences Maximal linear recursive sequences (m-sequences) are used extensively in digital communications and remote sensing because of their favorable correlation properties [17]. Excluding trivial cases, the cross-correlations of a pair of m-sequences must take at least three distinct values. An equivalent formulation is that the dual of a cyclic error-correcting code with two primitive zeroes must have at least three nonzero weights. Until recently, only ten infinite families of m-sequence pairs attaining the minimum number (three) of distinct values were known. Daniel Katz (with P. Langevin) established the existence of an eleventh such family [21], and so proved a 2001 conjecture due to Dobbertin, Helleseth, Kumar, and Martinsen [11]. Katz's talk was the first public lecture describing this result, and Tor Helleseth was present as one of the workshop participants. In his talk, delivered very effectively on a chalkboard, Katz gave a careful overview of the study of m-sequences before outlining the proof of the conjecture involving trilinear forms, enumeration of points on curves via multiplicative character sums, and divisibility properties of Gauss sums.

A linear feedback shift register (LFSR) is a physical device for generating sequences over a finite field, including m-sequences. A transformation shift register is a generalization of an LFSR that confers practical advantages when used in a stream cipher. Whereas the number of irreducible LFSRs over a finite field is well known, the number of irreducible transformation shift registers in general is not. Daniel Panario's talk examined this counting question for irreducible transformation shift registers, giving an asymptotic formula for some special cases using classical results due to Cohen [6], and a new proof of Ram's exact formula for order two using Ahmadi's recent generalization of a theorem due to Carlitz.

Difference sets correspond to sequences or arrays with constant out-of-phase periodic autocorrelation. They are often studied by applying characters to a group ring equation, resulting in a set of Weil numbers that must satisfy certain mutual properties [25] [35]. Bernhard Schmidt considered the contrary question: when does a single Weil number yield a solution of a group ring equation? This not only gives immediate nonexistence results for relative difference sets, but allows progress to be made in problems involving unique differences in cyclic groups.

Network Coding In multicast network communications, data is sent to several receivers at the same time. Network coding permits multiple sources to transmit simultaneously to multiple receivers, by allowing each intermediate network node to re-encode information via linear combination of its inputs. This process is highly sensitive to errors, because a single corrupted message can affect the entire network via successive linear combinations with other messages. For this reason, effective error control is a crucial requirement in network coding [34]. In her expository lecture, Emina Soljanin of Bell Labs gave a broad survey of the main ideas from information theory, algebra, and combinatorics. She then focussed on the combinatorial framework, showing how practical questions of network coding lead to fundamental open problems involving arcs in projective spaces.

Two very important classes of codes now used in network coding are the rank metric codes introduced by Gabidulin in 1985 [12] and the closely related subspace codes. A rank metric code consists of $n \times n$ matrices over $\mathbb{F}_q$ with the distance function $d(X, Y) = \mathrm{rank}(X - Y)$; these codes are also useful in space-time coding [26] and distributed storage.

Relinde Jurrius investigated the rank weight enumerator of a rank metric code and some of its generalizations, namely the r-th generalized rank weight enumerator and the extended rank weight enumerators. Analogously to results for ordinary linear codes, these objects determine each other. Moreover, Jurrius used counting polynomials to extend her results from the case of codes over $\mathbb{F}_q$ to codes over a finite field extension.

Arguably the most important subclass of rank metric codes is given by the linear maximum rank distance codes (MRD codes) which were constructed by Gabidulin; these are analogues of the classical Reed-Solomon codes. Anna-Lena Trautmann addressed the practical problem of list decoding the Gabidulin codes, using minimal bases of linearized polynomial modules. Her decoding algorithm computes a list of all closest codewords to a given received word. Although the complexity of the algorithm becomes exponential as soon as the closest codewords are beyond the unique decoding radius, it still beats the complexity of exhaustive search.

John Sheekey considered MRD codes that are not necessarily linear; the first non-trivial example of a non-linear MRD code was recently given by Cossidente, Marino and Pavese for the case where $n = 3$ and the minimum distance d is 2. Sheekey studied the case $d = n$, which corresponds to a finite semifield (namely a non-associative division algebra). He gave an overview on semifields (which have been studied intensively in recent years for other reasons) and introduced a new family of linear MRD-codes for each parameter; using some of the theory of semifields, he proved that these are inequivalent to the Gabidulin codes.

Kai-Uwe Schmidt's talk focussed on subgroups of the set of $n \times n$ symmetric matrices over $\mathbb{F}_q$ for odd q, for which the rank of the difference of any pair of distinct matrices in the subgroup is at least d. (Such sets can be considered as rank metric codes that are subject to the additional constraints that the matrices of the code must be symmetric and the set must form a subgroup.) Schmidt derived an upper bound on the size of such a subgroup in terms of n, q and d, and showed how to construct subgroups for which the upper bound is attained. A key insight is a new understanding of the association scheme of symmetric bilinear forms. His results can be equivalently formulated in terms of the weight enumerators of certain cyclic codes.

Planar Functions and their Generalizations

A perfect non-linear (PN) function is a map $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ with the property that $x \mapsto F(x+a) - F(x)$ is a permutation for all $a \neq 0$. Such functions are also called planar functions, because they define projective planes. Most known PN functions are associated with semifields [30]. In the binary case $p = 2$, PN functions unfortunately cannot exist; this motivates the study of almost perfect nonlinear (APN) functions, where now $x \mapsto F(x+a)+F(x)$ is required to have 0 or 2 solutions for all $a \neq 0$. APN functions are of great interest in cryptography, as they provide optimal resistance of a block cipher to differential attacks. In his expository lecture, Alexander Pott introduced these notions and gave a comprehensive overview of the known constructions of PN and APN functions. He also discussed both the similarities and the differences between the PN and the APN case, and highlighted several important open problems.

Petr Lisoněk considered the existence of APN functions which are also permutations of $\mathbb{F}_{2^n}$; this additional property is desirable in the design of block ciphers. While many APN permutations are known when n is odd, their existence in even dimensions $n > 6$ is an open problem. An example for $n = 6$ was given by Browning, Dillon, McQuistan and Wolfe in 2009 [4]. Lisoněk related some parts of their construction to consideration of the number of rational points on a certain family of hyperelliptic curves of genus 2 over $\mathbb{F}_{2^6}$, and discussed the possibility of obtaining similar constructions in higher even dimensions.

Yin Tan studied the related notion of zero-difference δ-balanced functions, where one requires that the equation $F(x+a)-F(x) = 0$ has exactly δ solutions for all $a \neq 0$. All known quadratic planar functions are zero-difference 1-balanced, and some quadratic APN functions are zero-difference 2-balanced. After considering the relationship between this notion and differential uniformity, Tan gave new families of zero-difference p^t-balanced functions and used these to construct new partial difference sets and hence new strongly regular graphs.

Yue Zhou considered monomial negabent functions. Like the related but better-known bent functions (which arise as component functions of PN functions), negabent functions play an important role in both cryptography and coding theory. Here the defining property is that the map $x \mapsto F(x+a) + F(x) + \mathrm{Tr}(ax)$ (where Tr denotes the trace function) should be balanced for every $a \neq 0$. Zhou presented families and examples of quadratic and cubic negabent polynomials in the special case $F(x) = \mathrm{Tr}(\gamma x^d)$.

Sequences and Quantum Information Theory In his expository lecture, Bill Martin described two notoriously challenging problems of quantum information theory that he considered the workshop participants were "born to solve". The first problem is the construction of large sets of equiangular lines in $\mathbb{C}^d$ or in $\mathbb{R}^d$, namely sets of unit vectors for which distinct vectors have inner product of constant magnitude. The second problem is the construction of large sets of mutually unbiased bases in $\mathbb{C}^d$ or in $\mathbb{R}^d$, namely orthonormal bases for which unit vectors from distinct bases have inner product of constant magnitude. Much of what is currently known about these two problems is related to bent functions, PN functions and codes that are linear over the ring $\mathbb{Z}_4$ [5], [14]. Martin carefully and entertainingly explained how these problems arise in quantum information theory, and why he believes they should be regarded as fundamentally combinatorial problems.

Golay complementary sequences and arrays have the property that the sum of their aperiodic autocorrelations is zero at all non-zero shifts [16]. They have been applied to a wide range of digital communications technologies, including infrared spectrometry [15], optical time domain reflectometry [28], and especially multicarrier wireless communications [9]. Matthew Parker introduced the novel idea of constructing Golay sequences and arrays using mutually unbiased bases. This allows the construction of larger sets of Golay sequences/arrays than those described by Davis and Jedwab [9], and therefore a higher code rate when used for transmission; this advantage occurs at the cost of an increase in the size of the sequence/array alphabet. The new constructions lead to interesting enumeration problems.

Codes and Groups It has long been recognized that codes with strong error-correction capabilities are often related to finite simple groups, extremal graphs, and extremal finite geometries. These connections are still being fruitfully exploited.

Dimitri Leemans described a new method of studying primitive coset geometries, using the permutation representations of groups. This method enables the construction of new binary codes, from the row span over $\mathbb{F}_2$ of the incidence matrices of some strongly regular graphs associated with large groups. Leemans presented an algorithm for handling the calculations for these groups, that is at least 1000 times faster than the best previously known. It permits the classification of rank two primitive coset geometries for the five Mathieu groups, the first three Janko groups, the Higman-Sims group, and the McLaughlin group.

The Hoffman-Singleton graph and the Higman-Sims graph are associated with the finite simple group $PSU_3(5)$ and the Higman-Sims group, respectively. Bernardo Rodrigues examined the codes of these graphs, producing examples of codes having optimal or best-known minimum distance for their length and dimension, and examples meeting the classical Gilbert-Varshamov bound [13], [36]. He also constructed new 2-designs that are invariant under the Higman-Sims group.

Dean Crnković described a method for constructing self-orthogonal and self-dual codes using orbit matrices of symmetric 2-designs with prescribed automorphism group. The method employs Lander's results on linear codes spanned by incidence matrices of symmetric designs [24], and extends previous constructions due to Harada and Tonchev [19].

Emerging Applications in the Mathematics of Communications

Researchers are able to draw on an enormous body of coding theory knowledge, accumulated over many decades, in order to solve entirely new practical problems soon after they present themselves. This was powerfully illustrated by five of the workshop talks, whose topics were channel estimation, efficient spectrum allocation, chip design, tamper-resistant cryptography, and random number generator hardware.

Digital information can be transmitted over a noisy channel by modulating a carrier signal with a sequence of values drawn from a finite alphabet. The channel estimation problem is to find the parameters that determine how the channel transforms the transmitted sequence into the received sequence. In his expository lecture, Alexander Fish described the classical pseudo-random method for solving the channel estimation problem for a delay-Doppler channel. This method has complexity $O(N^2 \log N)$, where N is the length of the transmission sequence. Fish then introduced alternative solutions to this problem, developed with Gurevich and others, whose complexity is only $O(N \log N + r^2)$ for a channel of sparsity r.

Conventional coding theory is used to recover information in the presence of errors introduced by transmission over a noisy channel. Anant Sahai introduced the novel concept of an identity code, for determining the identity of the transmitter without necessarily being able to decode the actual message that was transmitted. This has potential application to the problem of allocating available electromagnetic spectrum more efficiently than under the current regulatory constraints.

On-chip data buses frequently experience problems of crosstalk, in which a signal travelling along one path experiences interference from signals on adjacent parallel paths. These problems are growing in severity as circuits are becoming progressively more miniaturized. Charlie Colbourn showed how balanced sampling plans from statistical experimental design theory can be modified to produce packing sampling plans, leading to coding schemes that eliminate various types of crosstalk while simultaneously achieving low power and error correction.

Designers of cryptographic systems always attempt to protect against attacks based on the theoretical properties of their cryptosystems. In addition, they must also guard against side-channel attacks exploiting information, such as timing or power consumption, that is leaked when the cryptosystem is physically implemented. Jon-Lark Kim discussed complementary information codes that reduce the cost of countermeasures against side-channel attacks. He showed how to construct such codes from strongly regular graphs and doubly regular tournaments.

The generation of truly random numbers by physical means is important for producing cryptographic keys and for resisting cryptographic attacks such as side-channel attacks and fault injection. In his talk, Florian Caullery assumed that a true random number generator is embedded in an electronic device. He then examined how one can test at run time whether the generator is operating correctly, using limited memory and processing. His tests are based on the computation of the nonlinearity and absolute indicator of Boolean functions.

Panel Discussion

The final formal event of the workshop was a panel discussion on future research directions in the mathematics of communications, moderated by Jonathan Jedwab. The panellists were Claude Carlet, Charlie Colbourn, Bill Martin, and Anant Sahai. The discussion began with each of the four panellists explaining their view of the important trends, emerging areas, major open problems, and new connections. This was followed by a lively and wide-ranging discussion among the workshop participants, which extended well beyond the allotted 90 minutes.

Many specific future research directions were identified during the discussion, including:

- decoding random linear codes
- using coding theory to manage distributed data storage
- developing new types of stream cipher
- applying the considerable body of existing knowledge about APN functions to the design of better cryptographic S-boxes
- attacking longstanding open conjectures in coding theory, such as the MDS conjecture [33] or Delsarte's constant-weight conjecture [10].
- developing codes suited for low power consumption, particularly as the "Internet of Things" (interconnecting computing devices embedded within existing infrastructure) emerges
- using coding theory to enable efficient version control for distributed file storage
- solving problems arising in the construction of practical quantum computers.

Special mention was made of Peter Keevash's spectacular and unexpected 2014 solution [22] of one of the most important open problems in design theory: the existence conjecture for Steiner t-designs. One of the panellists declared that this put design theory "at a crossroads", and challenged participants to try to find applications of this new theory to practical problems, rather than solely seeking to develop the theory further.

There was general agreement among panellists and participants that theory and application are both important, that neither one should be neglected in favour of the other, and that the study of the mathematics of communications is renewed each time a new connection is made in either direction. There was considerable discussion of specific strategies by which theoreticians can identify and explore possible applications, for example:

- teaching a course geared to students in application-oriented disciplines such as biology, engineering, or anthropology
- participating in an industrial problem-solving event such as the Graduate Industrial Mathematical Modelling Camp (Canada) or Mathematical Problems in Industry (USA)
- organizing cross-disciplinary seminars for graduate students
- maintaining contact with former graduate students who are now employed in industry
- browsing various IEEE journals in search of familiar combinatorial structures, and then trying to understand the underlying reason for their appearance.

One of the workshop participants remarked after the panel discussion that he had never seen such frank self-examination take place in public at a conference, and that he found it extremely interesting and helpful.

Interactions

The workshop schedule was designed with copious time for unstructured private discussions, and the participants eagerly took advantage of the opportunities. The following (decidedly not exhaustive) examples of participant interaction are intended to give a sense of the activity and excitement that occurred outside the formal sessions of the workshop, and to indicate that many discussions took place between researchers who had not previously collaborated.

Bill Martin hosted an open session, attended by over a dozen researchers, attempting to catalogue as many documented examples as possible of specific error-correcting codes used in practical applications.

The seven participants based at Canadian institutions had a group discussion about long-term plans for collaborating more closely with each other.

Several participants spoke to Emina Soljanin about her experience of working in an industrial research lab.

Brett Stevens and Daniel Katz began a collaboration, investigating a construction of covering arrays using multiplicative characters over finite fields.

Jim Davis and Anant Sahai had several conversations about legal and engineering questions arising from their respective presentations, as well as academic and public policy issues.

Bill Kantor had discussions with Claude Carlet about constructing new Kerdock codes, with Jim Davis about mutually unbiased bases and bent function and difference sets, and with Brett Stevens about PN and APN functions.

Participant Feedback This report concludes with some samples of participant feedback.

“This was one of the best workshops I have attended in years. The talks were all very interesting and the idea of including Jim Davis’ talk was just perfect. Jim had a unique experience and sharing it with us was so enlightening. I had no idea how the judicial system works in a scientific dispute, before his talk. Having someone from the industry was an excellent idea. Not jamming too many talks each day was very helpful in staying alert. Bill Martin’s session and talk were both very interesting.
The fact that we met and planned for a joint venture, if fruitful, would be a great highlight of the workshop.
Thank you very much for a great workshop.”

“Thank you for organizing such an amazing meeting. I really had a good time. And this may really cement [named graduate student] into this research area. He’s totally pumped to do research now.”

“Thanks for the excellent meeting.”

“Thank you, thank you, thank you for not scheduling too many talks!”

“Once again thank you for the opportunity I was given to attend a well run workshop. I think that the workshop was extremely useful to me in particular, since I had three collaborators attending the meeting and this was a good opportunity for us to have a look at outstanding projects and discuss ideas of how best to address them. Two papers which were in advanced stage of preparation are about to be submitted thanks to the fact that we met. We spoke and got new ideas about finishing some outstanding papers. In addition we were able to start new projects and discussed ideas regarding directions for joint work. I was approached by two colleagues on the possibility of joint work in the near future, and possible collaborative visit to our universities. The panel discussion was an essential component of the discussion to me and it enlightened me on the various problems that one can address. The idea of a common and yet beautiful remote research place is a plus for the meeting.”

“Thank you for a great workshop!”

“It was indeed an enjoyable, informative, and productive week!”

“I thought the quality of all the talks, both expository and contributed, was higher than the average conference in regards to both delivery and content. Quite a few of even the contributed talks included “big problems” that should be or were in the process of being tackled. There was a variety of topics discussed, yet the conference was very cohesive overall. The schedule made it possible to attend all the talks without feeling burnt out and while still having time for small collaboration sessions. I did not leave at the end being glad it was over, but rather looking forward to the next one!"

"I would add my voice to say that this workshop was one of the more useful gatherings I have had in the past decade. The talks were interesting, and those talks sparked conversations. There was plenty of free time that enabled participants to do the work we love to do. Well done!"

Participants

Carlet, Claude (University Paris 8)
Caullery, Florian (Institut de Mathmatiques de Marseille)
Colbourn, Charles (Arizona State University)
Crnkovi?, Dean (University of Rijeka)
Davis, James (University of Richmond)
Dynerman, David (University of Wisconsin-Madison)
Etzion, Tuvi (Technion IIT)
Fish, Alexander (University of Sydney)
Fujiwara, Yuichiro (California Institute of Technology)
Gurevich, Shamgar (University of Wisconsin-Madison)
Helleseth, Tor (University of Bergen)
Jedwab, Jonathan (Simon Fraser University)
Jurrius, Relinde (University of Neuchtel)
Kantor, William (Brookline, MA)
Katz, Daniel (California State University, Northridge)
Kharaghani, Hadi (University of Lethbridge)
Kim, Jon-Lark (Sogang University)
Kodalen, Brian (Worcester Polytechnic Institute)
Leemans, Dimitri (Universit Libre de Bruxelles)
Leung, Ka Hin (National University of Singapore)
Lison?k, Petr (Simon Fraser University)
Martin, William (Worcester Polytechnic Institute)
Panario, Daniel (Carelton University)
Parker, Matthew (University of Bergen)
Poskin, Jeff (University of Wisconsin-Madison)
Pott, Alexander (Otto-von-Guericke-University Magdeburg)
Rodrigues, Bernardo (Universuty of Kwazulu-Natal)
Sahai, Anant (University of California Berkeley)
Schmidt, Kai-Uwe (Otto-von-Guericke University)
Schmidt, Bernhard (Nanyang Technological University)
Sheekey, John (University of Ghent)
Soljanin, Emina (Bell Labs Research)
Stevens, Brett (Carleton University)
Tan, Yin (University of Waterloo)
Thomson, David (Carleton University)
Tonchev, Vladimir (Michigan Technological University)
Trautmann, Anna-Lena (University of Melbourne)
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Chapter 4

Discrete Geometry and Symmetry (15w5019)

February 8 - 13, 2015

Organizer(s): Karoly Bezdek (University of Calgary), Asia Ivic Weiss (York University), Egon Schulte (Northeastern University)

Symmetry is among the most frequently recurring themes in the natural sciences and has inspired much of modern geometry.

Highly symmetric figures that are discrete naturally fall into the realm of discrete geometry. In a broad sense, discrete geometry investigates discrete structures in geometry and combinatorics such as polytopes, polyhedra and maps, tessellations (tilings), complexes and graphs, efficient sphere arrangements, packing and covering arrangements, and lattices. The Workshop focused on aspects of symmetry in the analysis and classification of such structures, and exploited symmetry as a unifying theme. The Workshop brought together established experts and emerging researchers in discrete geometry and related areas, to share recent developments in the study of highly-symmetric discrete geometric structures, discuss emerging directions, encourage new collaborative ventures, and achieve further progress on fundamental questions in the field.

The last few decades have seen a revival of interest in discrete geometry and symmetry, and have produced groundbreaking new discoveries. The Workshop presented new directions of research, explored new approaches to old problems, and helped formulate conjectures pointing to a fascinating world of highly-symmetric structures still to be discovered. The themes of the Workshop reflected the rich mathematical traditions of the groundbreaking works of H.S.M. Coxeter, László Fejes Toth, Branko Grünbaum and Peter McMullen to whom we owe a great deal of our present understanding of discrete geometry, and the more recent progress in the field that they inspired. The 50th anniversary in 2014 of the publication of Fejes Tóth's classic on "Regular Figures" by Pergamon Press in 1964, as well as the upcoming Fejes Toth Centennial celebrations in Budapest in 2015, made the Workshop a particularly timely event.

In the area of polytope-like structures and symmetry, much of the recent progress has centered around the modern theory of abstract polytopes and combinatorial symmetry [12]. Abstract polytopes are combinatorial structures with distinctive geometric, algebraic, or topological properties, in many ways more fascinating than traditional polyhedra, polytopes and tessellations. While much of the fundamental work of Coxeter and Grünbaum in this area focused on highly regular structures, recent research also dealt with somewhat less restricted aspects of symmetry, thus broadening the classical approach while leading to many new and unexplored problems. There has been much recent research on the chirality in polyhedra, maps or polytopes. Chirality is a fascinating phenomenon which does not occur in the traditional theory of polyhedra and polytopes.

The rapid development of abstract polytope theory has resulted in a rich theory featuring an attractive interplay of methods and tools from discrete geometry (classical polytope theory), group theory and geometry (Coxeter groups and their quotients, as well as reflection and crystallographic groups), combinatorial group theory (generators and relations), and hyperbolic geometry and topology (tessellations and their groups). Still, even after an active period of research, many deep problems have remained open and await solution.

Efficient packings of solids have been investigated since the times of Kepler. He was the first to formulate the discrete geometric problem of finding the densest packings of spheres (balls) in ordinary space, and his conjecture about the most efficient arrangement became known as the Kepler Conjecture (finally confirmed by Hales). The systematic research on general packing and covering problems in space began in the late 1940's with the pioneering work of Fejes Tóth. The Hungarian geometry school of Fejes Toth greatly contributed to the growing field of discrete geometry and has attracted the interest of numerous other mathematicians, including prominent researchers such as Coxeter, Rogers, Penrose, and Conway.

Two particularly active subject areas, already highlighted in Hilbert's 18th Problem, stand out since the early days of discrete geometry and are naturally intertwined, namely dense sphere packings, and tiling theory. The interest in sphere packings generated a great deal of research on the geometry of Voronoi tilings with a staggering number of real world applications. Over the past few years, it has become increasingly evident that further progress on most "hard problems" about efficient arrangements of solids would also require new optimization techniques [1, 8]. Many central and by now classical problems in discrete geometry have an established record of strong connections with geometric analysis, coding theory, group theory (symmetry groups), number theory, and differential and integral geometry. The connections with combinatorics and optimization are of particular importance and have not yet been fully exploited. Advanced optimization techniques have significantly helped achieve recent breakthrough results on kissing numbers and densest lattice sphere packings [4, 13].

Discrete Structures and Symmetry

Polyhedra and Polytopes

Nikolay Abrosimov, Sobolev Institute of Mathematics, Novosibirsk, presented joint work with Alexander Mednykh and Ekaterina Kudina about the volume of hyperbolic octahedra with $\overline{3}$-symmetry, to appear in Proceedings of Steklov Institute of Mathematics. An octahedron is said to have $\overline{3}$-symmetry if its symmetries include a rotation of order 3 as well as he antipodal involution.

Abrosimov first discussed the Euclidean case exploiting intuition, and in particular derived both an existence criterion for an octahedron with $\overline{3}$-symmetry as well as a volume formula. Embedding the Euclidean octahedron in the projective Cayley-Klein model of hyperbolic 3-space then permitted to compute the hyperbolic edge lengths and dihedral angles in terms of the coordinates of the vertices; these parameters are non-invariant and depend on the choice of coordinate chart. Then the coordinates were eliminated and relationships between the edge lengths and the dihedral angles were obtained. Using this relationships, a Schläfli equation was solved and an explicit volume formula for a hyperbolic octahedron with $\overline{3}$-symmetry was presented. In addition, an existence criterion for a hyperbolic octahedron with $\overline{3}$-symmetry was described.

Javier Bracho, UNAM at Mexico City, discussed highly symmetric realizations of abstract polytopes as geometric polytopes in Euclidean spaces. A geometric polytope is geometrically regular if it has a flag-transitive geometric symmetry group; the polytope then has maximum possible symmetry by reflection. A geometric polytope is geometrically chiral if it has a geometric symmetry group with two orbit on the flags such that adjacent flags are in distinct orbits; the polytope then has maximum possible symmetry by rotation.

Bracho presented an unexpected example of a finite regular abstract polytope of rank 4 with a chiral embedding in $\mathbb{E}^4$, that is, a geometrically chiral 4-polytopes in $\mathbb{E}^4$. This disproved an earlier claim in the literature that no such geometric polytopes exist.

Frieder Ladisch, University of Rostock, gave a talk about affine symmetries of orbit polytopes. An orbit polytope is the convex hull of a point orbit under a finite subgroup G of $\mathrm{GL}(\mathrm{d}, \mathbb{R})$. Ladisch studied the possible affine symmetry groups of orbit polytopes. For every group, there is an open and dense set of "generic points" such that the orbit polytopes of generic points have conjugate affine symmetry groups and are minimal in a certain sense. The symmetry group of a generic orbit polytope coincides with G if G is itself the affine symmetry group of some orbit polytope, or if G is absolutely irreducible. On the other hand, there are some general cases where the affine symmetry group grows, for example representation polytopes (the convex hull of a finite matrix group). Their affine symmetries can be computed effectively from a certain character. The results presented were joint work with Erik Friese.

Nicholas Matteo, Northeastern University, discussed the classification of convex polytopes with few flag orbits under the geometric symmetry group action. The convex polytopes with a single flag orbit are precisely the regular convex polytopes. In earlier work, Matteo had classified the two-orbit convex polytopes (as well as the convex polytopes which have two flag orbits under the combinatorial automorphism group). In his talk, Matteo described a full classification of the convex polytopes with three flag orbits under the symmetry group. These polytopes exist only in eight dimensions or fewer. Tilings of Euclidean spaces with few flag orbits under the geometric symmetry group were also described. The results presented have appeared in Matteo's 2015 PhD thesis on "Convex Polytopes and Tilings with Few Flag Orbits" at Northeastern University.

Abigail Williams, Northeastern University, gave a lecture about uniform skeletal polyhedra in ordinary 3-space. In skeletal polyhedra, each face is considered to be a set of edges which is not spanned by a membrane as in traditional convex polyhedra. The faces, and indeed the polyhedra themselves, are hollow. The uniformity condition signifies that the polyhedra have regular faces and are vertex transitive under the geometric symmetry group. Williams described a construction which can be used to generate uniform skeletal polyhedra from the symmetry groups of the regular skeletal polyhedra. Also discussed was an extension of this construction which can be used to generate more uniform skeletal polyhedra. The results presented have appeared in Williams' 2015 PhD thesis on "Wythoffian Skeletal Polyhedra" at Northeastern University.

Surfaces, maps, and graphs

Marston Conder, University of Auckland, kicked off the workshop, with a talk on arc-types of vertex-transitive graphs. Let X be vertex-transitive graph of valency d, and let A be its full automorphism group. Then the arc-type of X is defined in terms of the lengths of the orbits of the action of the stabiliser A_v of a given vertex v on the set of arcs incident with v. Specifically, the arc-type is the partition of d as the sum $n_1 + n_2 + \cdots + n_t + (m_1 + m_1) + (m_2 + m_2) + \cdots + (m_s + m_s)$, where $n_1, n_2, \ldots, n_t$ are the lengths of the self-paired orbits, and $m_1, m_1, m_2, m_2, \ldots, m_s, m_s$ are the lengths of the non-self-paired orbits, in ascending order. For example, if X is arc-transitive then its arc-type is d, while if X is half-arc-transitive then its arc-type is $d/2 + d/2$. In his talk Conder explained how it can be shown that there are vertex-transitive graphs with every possible arc-type, except $1 + 1$ and $(1 + 1)$.

Undine Leopold, Technical University of Chemnitz, presented Part I of a joint talk with Tom Tucker on euclidean symmetry of closed surfaces immersed in 3-space. Given a finite group G of orientation-preserving euclidean isometries and a closed surface S, an immersion $f : S \to E^3$ is in G-general position if $f(S)$ is invariant under G, points of S have disk neighborhoods whose images are in general position, and no singular points of $f(S)$ lie on an axis of rotation of G. For such an immersion, there is an induced action of G on S whose Riemann-Hurwitz equation satisfies certain natural restrictions.

In the first part of this talk, Leopold introduced these restrictions and presented how models arise from the quotient surface S/G in the orbifold E^3/G. It may be particularly surprising that an orientable symmetric surface can lead to a nonorientable quotient. Leopold also pointed out that the problem of classifying which of the restricted Riemann-Hurwitz equations are realizable becomes intractable outside of G-general immersions.

Thomas W. Tucker, Colgate University, gave Part II of a joint talk with Undine Leopold on euclidean symmetry of closed surfaces immersed in 3-space. In the second part of this talk, Tucker focussed on additional group theoretic conditions that must be satisfied by G and the fundamental groups of the surface S and its quotient surface, before completing the classification of which restricted Riemann-Hurwitz equations are realized by a G-general position immersion of S. Exceptions arise, in particular, for low genus and little branching. One is then able to decide which genera of a surface allow a G-general immersion in 3-space.

Abstract Polytopes and Groups

Eric Ens, York University, discussed consistent colourings of polytopes. A colouring of the facets of a polytope is called consistent if the colouring is respected (though not necessarily preserved) by the automorphism group. Any polytope can be coloured trivially be assigned a different colour to each facet or by assigning the same colour to each facet. Interesting examples of consistent colourings were discussed, and then colourings of the regular and chiral toroidal polytopes of type $\{4, 4\}$ were examined in more depth.

Isabel Hubard, UNAM at Mexico City, lectured about products of abstract polytopes. Given two convex polytopes the operations of taking their join, their cartesian product, and their direct sum are well understood. In her talk, Hubard described how these three kinds of products can be extended to abstract polytopes. She also introduced a new product, called the topological product, which also arises in a natural way from geometry.

One is particularly interested in understanding the automorphism group of a product of $\mathcal{P}$ and $\mathcal{Q}$ in terms of the automorphism groups of $\mathcal{P}$ and $\mathcal{Q}$. To this end, Hubard introduced the concept of a prime polytope, for a given product. We shall see that highly symmetric non-prime polytopes are sparse; in fact, for the join product the only regular non-prime polytopes are the simplices, for the cartesian product the only regular non-prime polytopes are the hypercubes, for the direct sum the only regular non-prime polytopes are the cross polytopes and for the topological product the only regular non-prime polytopes are toroidal polytopes.

Kyle Meyer, Northeastern University, discussed face enumeration for the colorful associahedra and related structures. The classical associahedra can be formulated in terms of flipping the diagonals of triangulations of convex polygons. Similarly the colorful associahedra, an abstract polytope, introduced by Araujo-Pardo, Hubard, Oliveros, and Schulte, is formulated in terms of flipping diagonals of triangulations whose diagonals are colored (colored triangulations). In this talk, Meyer gave a modified formulation of the colorful associahedra in terms of partial colored triangulations, and using this formulation counted the number of faces of the colorful associahedra by dimension.

Egon Schulte, Northeastern University, gave a survey talk about colorful polytopes, associahedra and cyclohedra. Every n-edge colored n-regular graph G naturally gives rise to a simple abstract n-polytope $P(G)$, called the colorful polytope of G, whose 1-skeleton is isomorphic to G. Schulte described colorful polytope versions of the associahedron and cyclohedron. Like their classical counterparts, the colorful associahedron and cyclohedron encode triangulations and flips, but now with the added feature that the diagonals of the triangulations are colored and adjacency of triangulations requires color preserving flips. The colorful associahedron and cyclohedron are derived as colorful polytopes from the edge colored graph whose vertices represent these triangulations and whose colors on edges represent the colors of flipped diagonals. This was joint work with G.Araujo-Pardo, I.Hubard and D.Oliveros.

Micael Toledo, UNAM at Mexico City, spoke about the automorphism groups and the symmetry type graphs of maniplexes. A manicomplex is a generalization of an abstract polytope, in much the same way in which a map on a surface is a generalization of an abstract polyhedron. For a given maniplex M let O denote its set of flag orbits under the action of the automorphism group. Then an edge-coloured graph with vertex set O can be constructed by joining two vertices o_1 and o_2 by an i-coloured edge whenever there are flags f in o_1 and g in o_2 which are i-adjacent. This graph is called the symmetry type graph of M. In this talk, Toledo discussed symmetry type graphs of maniplexes. In particular, given a symmetry type graph, generators for the automorphism group of a maniplex with this symmetry type graph were presented

Polytopes and Incidence Geometries

Maria Elisa Carrancho Fernandes, University of Aveiro, talked about regular and chiral hypertopes. In 1983, Aschbacher proved that string C-groups are thin, residually connected, regular geometries. The talk concerned C-groups with nonlinear Coxeter diagrams. It was shown that thin, residually connected regular geometries are C-groups, but that the converse is not true. Nevertheless flag-transitivity is a sufficient condition to establish the converse: flag-transitive C-groups are thin, residually connected regular geometries (the Tits algorithm is used to get an incidence geometry from a C-group).

Abstract regular polytopes are string C-groups, as described by McMullen and Schulte in their book (2002). For this reason, the term (regular) hypertope is used to designate a thin, residually connected (regular) geometry. Abstract regular polytopes are regular hypertopes with linear Coxeter diagram. Guided by the ideas of chirality in the abstract polytope theory, we extend the concept to a more general setting of incidence geometries. Indeed, when the geometry is thin, it is possible to define chirality, as in the case of polytopes. We give characterisations of automorphism groups of thin residually connected chiral geometries and we show how to construct such chiral objects group-theoretically. One of our focus is the classification of hypertopes of a certain type. Here we consider spherical, locally spherical and locally toroidal hypertopes (hypertopes having all parabolic subgroups either spherical or toroidal).

Dimitri Leemans, University of Auckland, discussed the classification of abstract polytopes whose automorphism group is an almost simple group of PSL(2,q) type. The talk explained the classification of the regular polytopes for the groups PSL(2,q) and PGL(2,q) obtained in joint work with Egon Schulte, and then elaborated on more general results established jointly with Thomas Connor and Julie De Saedeleer. Leemans also described the current state of the classification of the chiral polytopes related to these groups; this is ongoing work with Eugenia OReilly-Regueiro and Jeremie Moerenhout.

Eugenia O'Reilly-Regueiro, UNAM at Mexico City, continued the theme of Leemans' talk and spoke about "Abstract polytopes and projective lines, the chiral case". The classification of abstract polytopes with almost simple automorphism group of PSL(2,q) type has been addressed separately for the regular and the chiral cases. The regular case was presented by Dimitri Leemans; it was completed jointly with Thomas Connor and Julie De Saedeleer following previous work with Egon Schulte. This talk presented some results on the chiral case, from ongoing joint work with Dimitri Leemans and Jeremie Moerenhout.

Discrete Convex Geometry

Convex Geometry

Ryan Trelford, University of Calgary and York University, spoke about X-raying of 3-dimensional convex bodies with mirror symmetry. Let K be a d-dimensional convex body. A point p on the boundary of K is said to be X-rayed along a line with direction vector $\mathbf{v}$ if the line through p with direction $\mathbf{v}$ intersects the interior of K. A collection of lines is said to X-ray K if every boundary point of K is X-rayed along one of the lines. The minimum number of lines required to X-ray K is called the X-ray number of K, and is denoted by $X(K)$. In 1994, K.Bezdek and T.Zamfirescu conjectured that $X(K) \leq 3 \cdot 2^{d-2}$ for any d-dimensional convex body K.

The talk explained how the X-ray Conjecture is related to the famous Gohberg-Markus-Hadwiger Covering Conjecture. Trelford briefly verified the X-ray conjecture for planar convex bodies, showing that three lines are needed if, and only if, the convex body is a triangle. Then it was proved that any 3-dimensional convex body exhibiting mirror symmetry also satisfies the X-ray Conjecture.

Vlad Yaskin, University of Alberta, discussed stability results for sections of convex bodies. Let K be a convex body in $\mathbb{R}^n$. The *parallel section function* of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}), \quad t \in \mathbb{R}.$$

If K is origin-symmetric (i.e. $K = -K$), then Brunn's theorem implies

$$A_{K,\xi}(0) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$$

for all $\xi \in S^{n-1}$.

The converse statement was proved by Makai, Martini and Ódor. Namely, if $A_{K,\xi}(0) = \max_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is origin-symmetric.

Yaskin, in joint work with Matthew Stephen, provided a stability version of this result. If $A_{K,\xi}(0)$ is close to $\max_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is close to $-K$.

Packing and Covering

Karoly Bezdek, University of Calgary, gave a survey about contact numbers, summarizing old and new results. Contact numbers are natural extensions of kissing numbers. The talk focussed on estimating the contact numbers in a packing of n unit balls in Euclidean d-space.

Muhammad Khan, University of Calgary, spoke about joint work with Karoly Bezdek on the covering index of convex bodies. Covering a convex body by its homothets is a classical notion in discrete geometry that has resulted in a number of interesting and long standing problems. Swanepoel introduced the covering parameter of a convex body as a means of quantifying its covering properties. Khan introduced a relative of the covering parameter called covering index, which turns out to have a number of nice properties. Intuitively, the covering index measures how well a convex body can be covered by a relatively small number of homothets having a relatively small homothety ratio. It was shown that the covering index provides a useful upper bound for well-studied quantities like the illumination number, the illumination parameter, the vertex index and the covering parameter of a convex body. Khan obtained upper bounds on the covering index and investigated its optimizers. Furthermore, it was shown that the covering index satisfies a nice compatibility with the operations of direct vector sum and vector sum that helps in determining the covering index of several convex bodies.

Marton Naszodi, Ecole Polytechnique Federale de Lausanne, and Eötvös Loránd University, gave a lecture about coverings in Euclidean space and on the sphere. The talk presented a method to obtain upper bounds on covering numbers. As applications of this method, Naszodi reproved and generalized results of Rogers on economically covering Euclidean n-space (resp. the sphere) with translates resp. rotated copies of a (spherically) convex body, or more generally, any measurable set. Using the same method, Naszodi sharpened an estimate by Artstein–Avidan and Slomka on covering a bounded set by translates of another.

The main novelty of the method described is that it was not probabilistic. The key idea, which made the proofs rather simple, is an algorithmic result of Lovász.

Convex Polytopes

Wendy Finbow-Singh, St. Mary's University, presented a talk on low dimensional neighbourly simplicial polytopes. Amongst the d-polytopes with v vertices, the neighbourly polytopes have the greatest number of facets. This maximum property has prompted researchers to compose lists of them. Finbow-Singh discussed an algorithm for generating the list of simplicial neighbourly d-polytopes with v vertices, for a given dimension d and number of vertices, v.

Alexander Litvak, University of Alberta, lectured about joint work with D. Alonso-Gutierrez and Nicole Tomczak-Jaegermann on the isotropic constant of random polytopes. Let $X_1, \dots, X_N$ be independent random vectors uniformly distributed on an isotropic convex body $K \subset \mathbb{R}^n$, and let K_N be the symmetric convex hull of X_i's. Litvak showed that with high probability $L_{K_N} \leq C\sqrt{\log(2N/n)}$, where C is an absolute constant. This result closed the gap in known estimates in the range $Cn \leq N \leq n^{1+\delta}$. Furthermore, Litvak extended the estimates to the symmetric convex hulls of vectors $y_1X_1, \dots, y_NX_N$, where $y = (y_1, \dots, y_N)$ is a vector in $\mathbb{R}^N$. Also discussed was the case of a random vector y.

Graph Drawings

Janos Pach, Ecole Polytechnique Federale de Lausanne, and Renyi Institute, gave a lecture on the number of crossings between curves.

David Richter, Western Michigan University, talked about algebraic universality of parallel drawings. Let Σ be a set of d fixed-point-free involutions on a given set $S = \{1, 2, 3, ..., 2n\}$. Graph-theoretically, this is the same as specifying a d-regular multigraph with vertex set S and an edge coloring by d colors. A *parallel drawing* of Σ is a drawing of the underlying graph in which every edge is represented by a segment and the segments sharing a common color are mutually parallel. The purpose of this talk was to explain "algebraic universality" for parallel drawings in the plane in the case when $|\Sigma| = 4$.

Helly's Theorem and Relatives

Deborah Oliveros, UNAM at Queretaro, spoke about joint work with J.A. De Loera, R.N. La Haye and E. Roldán-Pensado about Helly's Theorem over subgroups and other additive subsets of $\mathbb{R}^d$. In the usual Helly-type theorems, the convex sets are required to intersect in a proper subset S of $\mathbb{R}^d$. For instance, in the classical Helly's theorem this subset is $S = \mathbb{R}^d$ and the Helly number is $d+1$; and for Doignon's theorem, S is the set of integer points $\mathbb{Z}^d$ and the Helly number is 2^d. Oliveros presented some extensions of these results to the case when S is an arbitrary additive subgroup of $\mathbb{R}^d$, as well as some other interesting related results in dimension 2.

Convex and Combinatorial Geometry Fest

The 5-day Discrete Geometry and Symmetry Workshop was directly followed by the 2-day Convex and Combinatorial Geometry Fest at BIRS (15w2177), February 13-15, 2015. Organizer of this event were Abhinav Kumar (MIT), Daniel Pellicer (Universidad Nacional Autonoma de Mexico), Konrad Swanepoel (London School of Economics and Political Science), and Asia Ivić Weiss (York University). The programs of the two workshops were coordinated.

The 2-day Workshop explored the ways in which the areas of abstract polytopes and discrete convex geometry has been influenced by the work of Karoly Bezdek and Egon Schulte. The last three decades have witnessed the revival of interest in these subjects and great progress has been made on many fundamental problems. The Workshop was held to honor the occasion of Bezdek's and Schulte's 60-th birthday.

Participants

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Chapter 5

Advances in Numerical Optimal Transportation (15w5067)

Feb 15 - 20, 2015

Organizer(s): Jean-David Benamou (INRIA), Yann Brenier (Ecole Polytechnique), Adam Oberman (McGill U.)

Introduction

Optimal Transportation (OT) is a mathematical research topic which began two centuries ago the French mathematician Monge's work on "des remblais et déblais" in 1781. This engineering problem consists in minimizing the transport cost between two given mass densities. In the 40's, Kantorovitch solved the dual problem and interpreted it as an economic equilibrium. The Monge-Kantorovitch problem became a specific research topic in optimization and Kantorovitch obtained the 1975 Nobel prize in economics for his contributions to resource allocation problems. Following the seminal discoveries of Brenier in the 90's, Optimal Transportation has received renewed attention from mathematical analysts, resulting in the Fields Medal awarded in 2010 to C. Villani, who gave important contributions to Optimal Transportation, arrived at a culminating moment for this theory. Optimal Transportation is today a mature area of mathematical analysis. Numerical methods and applications are comparatively underdeveloped. The workshop was dedicated to recent results and methods pertaining to modelization and numerical simulations using Optimal Transportation tools and concepts.

Numerical approaches to Optimal Transportation

The Augmented Lagrangian method applied to the CFD Optimal Transportation [31] and its variants has been for a long time the main tool in numerical optimal transportation. It was used in particular for warping and registration in image processing. Three other numerical approaches have recently been explored :

Solving the Monge-Kantorovich linear program : Cuturi [8, 9, 10, 11, 4] presented the idea of regularizing optimal transport problems with an entropic penalty to enforce desirable properties for optimal couplings, such as balanced flows between equally expensive routes and overall smoothness. Only recently was it shown that such a regularization can provide an extremely efficient computational framework to approximate optimal transport using the toolbox of (strict) convex optimization. This presentation was followed by Peyré's with various applications of this family of numerical method to JKO gradient flows [20], and Wasserstein barycenters [4] in particular. For applications in neuroscience see [15] [14].

Ruan Yuanlong also presented a **multiscale linear programming solver** for optimal transportation (without Entropic regularisation).

Laguerre Cells for Semi-Discrete Optimal Transportation : Levy [17] presented a 3D implementation of quadratic OT between a density and a sum of Dirac masses. The modern and implementation relies of the efficient computation of Laguerre Cells and the Newton or Quasi Newton minimisation of a convex objective function. Merigot proposed a hierarchical algorithm that improves the speed of convergence, together with an implementation in 2D [32]. The numerical algorithm has been tested on several datasets, with up to hundred thousands tetrahedra and one million Dirac masses.

Finite difference Monge Ampère solvers and discretization of the cone of convex function : Mirebeau reviewed the different approaches to discretize optimisation of functionals under convexity constraint [35]. This was illustrated on the Principal Agent problem in 3D. He then explained how it can be use to solve the Monge-Ampère equation [34] [33]. This is the only monotone FD scheme for Optimal Transportation after [40, 41].

Extensions

Optimal Transportaion on graphs : Qinglan Xia [42] also presented his numerical simulations approach of the ramified optimal transportation.

Multi-Marginal Optimal Transport : Multi-Marginal optimal transportation is a new concept extending Optimal Transportation when data consists in more than 2 densities. Pass [19] gave a general overview of the multi-marginal optimal transport problem and outlined several applications.

JKO gradient flows and fluids, new modelisation and numerical simulations

Minimizing the Wasserstein distance to between a source measure and a target measure can be regarded as an infinite dimensional two point boundary problem. The mapping between the two densities also generates, by interpolation, a geodesic in the Wasserstein distance between the two measures (regarded as points).

A different extension is to consider a gradient flow in the Wasserstein metric, in other words, to evolve the density in order to minimize some functional. Computing the steepest descent direction with respect to the Wasserstein distance defines a semi discrete Wasserstein gradient flow, also known as a JKO gradient flow [30]. It is now well-known, that in the limit, several diffusion and aggregation equations of second and fourth order can be interpreted as gradient flows of an entropy in the L^2-Wasserstein metric. The theory is well developed [3] and continues to expand, see for instance new convergence results in [22]. An significant part of the workshop was devoted to new theoretical and numerical results in this field.

JKO schemes Theory : Santambrogio presented some new L^∞ estimates obtained in collaboration with J.-A. Carrillo. for the Keller-Segel model of chemiotaxis, based on a JKO-like scheme, and on a fine analysis of the optimality conditions at every time step, combined with the use of the Monge-Ampère equation.

Carlier and Agueh [1] presented a new JKO approach to the analysis of kinetic models of granular materials.

JKO schemes Numerical methods : The JKO scheme is semi-discrete in time. These works (try to) address the space discretisation and the convergence of the resulting shemes.

Wolfram [6] presented a finite element methods for two optimal transportation problems, in particular a class of nonlinear convection-aggregation equations and the Monge-Ampere equation which discretize the associated JKO scheme. OT provides a Lagrangian change of variable into the non-linear PDE to be solved.

Osberger and Mathes [36] use the Lagrangian change of variable into the JKO scheme and perform the gradient flow descent on a galerkin approximation of the transport map. The mass densities can be viewed as weighted moving particles. They check convergence.

Pataccini also uses the Lagrangian approach but chooses to represent the densities as balls of various size along particles. Gamma convergence results are available. This is joint work with J. A. Carrillo, Y. Huang, P. Sternberg and G. Wolansky.

See also Peyré contribution in section 2.

Applications

Fluids Dynamics : Maury presented an optimal transportation framework to pressureless Euler equation with a maximal density constraint. This is a second order extension of the the macroscopic congested crowd motion models proposed in [37].

Cullen [29, 7] presented and discussed computations of singular solutions of the semi-geostrophic eady problem using solutions of the Euler problem. The Semi-Geostrophic problem is the asymptotic limit as the Rossby number tends to zero.

Mérigot [18] presented a numerical method to solve Brenier Generalized Euler solutions in 2D which is an instance of Multi-Marginal optimal transportation.

Seismology : In seismic exploration a wave field is generated at the surface and reflections from the earths interior are recorded. The purpose is to find properties such as wave velocity and location of reflecting sub layers. This is done in an inverse process where the measurements are compared to a computed wave field with unknown coefficients in the wave equation. Engquist and Froese [13] propose to use the Wasserstein metric for this comparison. We will also remark on applications to registration in seismology.

Non-Imaging optics : Free-Forming or the automatic design of smooth refractors or reflectors can be modelled with Monge-Ampère type PDEs and optimal transportation.

Gutierrez [16] gave a complete mathematical overview of the refractor problem : design of interface surfaces between the two homogeneous materials, so the resulting lense refracts radiation in a prescribed manner.

The far-field reflector problem is also a well-known inverse problem arising in geometric optics. It consists in creating a mirror that reflects a given point light source to a prescribed target light at infinity. Thibert [12] presented how the combinatorics of this intersection can be efficiently computed using tools from computational geometry, thus providing an efficient algorithm for the far field reflector problem. This work is in common with Pedro Machado et Quentin Mérigot. [12]

Thije Boonkkamp [21] also addressed the reflector problem but using finite difference based discretisation of the reflector and in a far field regime.

Image Processing : Solomon [27, 28, 26, 25, 24, 23] reviewed several transportation techniques for Geometric Data Processing. Mainly based on accelerated LP solvers and entropic regularization.

Hongkai Zhao [39] uses a simplification of Wasserstein distance called Sliced-Wasserstein Distance to perform Multi-scale Non-Rigid Point Cloud Registration. He applies the Sliced-Wasserstein Distance density buit with the Laplace-Beltrami Eigenmap Operator.

Saumier [38] presented a method based on Optimal Optimal Transport for Particle Image Velocimetry.

Mesh Adaptation : Budd [5] borrows idea from optimal transport to construct a moving mesh adapted to a transient solution. He uses a parabolic Monge-Ampère equation to compute an approximation of the optimal map at each time step. This is applied to the Semi-Geostrophic problem mentioned above.

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Chapter 6

Hypercontractivity and Log Sobolev inequalities in Quantum Information Theory (15w5098)

February 23 - 27, 2015

Organizer(s): Patrick Hayden (Stanford University), Christopher King (Northeastern University), Ashley Montanaro (University of Bristol), Mary Beth Ruskai (University of Waterloo)

Overview of the Field

Quantum Information Theory (QIT) is a highly interdisciplinary field, and many different areas of mathematics have played key roles in its development. Recently the topics of hypercontractivity (HC) and logarithmic Sobolev (LS) inequalities have found applications in a variety of problems within QIT, and this has led to a growth of interest among researchers in the field. The purpose of this workshop was to bring together researchers from both the 'traditional' areas of application and also the new areas of interest in QIT. Some talks were expository, some covered current areas of research, and some presented ideas for new directions of research.

Hypercontractivity, which concerns the contractive properties of semi-groups e^{-tH} from L^p to L^q, was introduced in Nelson's seminal 1965 work on semiboundedness of certain Hamiltonians arising in two-dimensional quantum field theory. In that context H was an operator on the (infinite dimensional) bosonic Fock space. The essential mathematical content was later extracted and reformulated in finite dimensional settings. There are two basic 'classical' versions, the continuous and discrete. The statement of hypercontractivity is the same in both cases: for $1 \leq p \leq q$, and for a suitable operator A,

$$\|e^{-tA}\|_{p\to q} = \sup_{\|f\|_p \leq 1} \|e^{-tA} f\|_q \leq 1 \quad \text{if and only if} \quad t \geq \frac{1}{2} \log\left(\frac{q-1}{p-1}\right)$$

The related concept of the logarithmic Sobolev inequality is an infinitesimal version of hypercontractivity, obtained by setting $p = 2$, $q(t) = 1 + e^{2t}$ and taking the derivative at $t = 0$. These methods and concepts have been generalized to the quantum setting (essentially by considering matrix spaces in place of function spaces), and then applied to a variety of problems in QIT. Some recent examples include: Kastoryano and Temme's extension of classical Markov chain results to estimate convergence rates for semigroups which arise in quantum Markov processes [2]; Gavinsky et al's use of hypercontractivity to give the first proof of exponential separation between one-way quantum and classical communication complexity of partial boolean functions [1]; and Klartag and Regev's use of hypercontractivity on the n-sphere to resolve the long-standing conjecture that one-way quantum communication is exponentially stronger than even two-way classical communication [3].

Logistics and organization The schedule of the workshop was arranged with four lectures per day, in order to allow time for discussions and follow-up. The success of this workshop was due, in part, to a decision to focus on a particular area of mathematics of potential importance to quantum information and to bring together people with very different backgrounds in mathematics and in the physical sciences. Several participants commented that they found the focused nature of the workshop particularly inspiring and productive. The intensive nature of a BIRS workshop naturally encouraged interaction and new collaborations. An important feature of the workshop was a session devoted to the presentation and discussion of open problems. This led to discussions and collaborations, and several preprints as described below.

Major topics of discussion **Overviews and historical surveys** William Beckner provided a survey talk on LS and HC, describing the context of the original work of Gross and others, and connected the different formulations of LS for Gauss measure on $\mathbf{R}^n$ and uniform measure on the sphere. Chris King described the origins of HC in Nelson's work on semi-boundedness of the Hamiltonian for the interacting ϕ^4 field theory in two space-time dimensions. Eric Carlen reviewed the work of Gross, and Ball, Carlen, Lieb on HC for fermions, and described the important links between this work and the subsequent proofs of uniform smoothness and uniform convexity for non-commutative L^p spaces. Ronald de Wolf described the formulation of HC for functions on the Boolean cube, and reviewed inportant applications of this notion in computer science, including the famous KKL theorem. Boguslaw Zegarlinski described the extensions of HC and LS to noncommutative spaces. These concepts, dating back to the late 90s, have recently come to prominence in QIT, being used for example by Kastoryano and Temme to prove bounds on the mixing time of quantum Markov processes. Ashley Montanaro provided an overview of recent applications of HC and LS in quantum information theory, including bounds for the bias of lcoal games, separation of quantum and classical communication complexity, and limitations on quantum random access codes.

Mixing and Markov chains One of the principal applications of LS inequalities is to bound the mixing times of classical Markov chains. Using the methodology of Olkiewicz and Zegarlinski, Kastoryano and Temme extended the classical results to one-parameter semigroups of quantum channels, and obtained bounds for mixing times in terms of LS constants. The extension to the most general case would require the property of strong L^p regularity, which is an open problem in the general case. For classical Markov chains there is an easy 'additivity' property for LS constants which implies that the mixing time of independent copies of a chain is the same as for a single copy. The corresponding additivity result has not been demonstrated for quantum channel semigroups, and is suspected to fail in general. This failure would prevent an easy derivation of the mixing time bound for multiple copies of a channel semigroup.

Fernando Brandao discussed the quantum version of a property of thermal models in statistical mechanics, namely the equivalence of mixing in time (size-independent spectral gap) and mixing in space (bounded correlation length). For commuting Hamiltonians the equivalence goes through, however in general the relation is not so clear. David Perez-Garcia discussed the relation between how the mixing time scales with the system size, and the area law. The area law is a conjectured property of quantum systems, namely that the entanglement of the ground state grows with the size of the boundary rather than the bulk. Oleg Szehr reported on a new approach to estimating mixing times for quantum channel semigroups. This approach yields spectral bounds on norms of functions of transition maps of (classical or quantum) Markov processes and implies new estimates for the convergence of such processes to stationarity. The main technical ingredient is the use of a Wiener algebra functional calculus in the context of bounded semigroups.

Nonlocal games and complexity A productive way to explore the power of quantum correlations has been via the concept of nonlocal games. These are games where two separated parties, who are not allowed to communicate, aim to each output some value which depends on their joint inputs. An especially interesting class of nonlocal games is XOR games, where whether the players succeed depends only on the sum modulo 2 of their outputs. Quantum correlations are known to yield advantages over classical correlations in XOR games, one example being the famous CHSH game. As well as their practical importance as experimental tests of quantum mechanics, these games have a number of links to different areas of mathematics, and in particular are closely connected to Grothendieck inequalities. Harry Buhrman presented an overview of the evolution of XOR games and their applications. These include a recent resolution of a 35-year-old problem of Varopoulos in operator algebras, showing that the space of compact operators on a Hilbert space is a Q-algebra under Schur product.

Thomas Vidick described recent advances in generalisations of XOR games to the setting where the questions to the players are quantum, including some intriguing connections to noncommutative Grothendieck inequalities.

Other presentations described work concerned with different notions of quantum complexity. In the computational setting, SDP hierarchies are a rather canonical approach to find relatively efficient algorithms for optimization problems in quantum information. Aram Harrow reviewed this notion of SDP hierarchy and described work showing that SDP hierarchies can sometimes be replaced with nets. This can be a significant simplication (conceptually and algorithmically), and also provides insight into why the class of 1-LOCC measurements is "easier" to optimise over than general measurements. In information theory, hypercontractivity has been used classically to prove limitations on transformation of nonlocal correlations, via the concept of hypercontractivity ribbon due to Ahlswede and Gács. Salman Beigi described his work with Delgosha on a generalisation of this concept to the noncommutative setting in order to put bounds on the asymptotic transformation of quantum correlations.

Links to other research areas Several speakers described new directions of research with connections to other areas of mathematics and physics. Reinhard Werner discussed how to combine classical and quantum harmonic analysis via a direct sum construction. Mathilde Perrin described new results on hypercontractivity for group von Neumann algebras. Michael Kastoryano presented results on quantum Gibbs samplers. A Gibbs sampler is a Markov process which drives a system to thermal equilibrium (its Gibbs state). Recent work has built on the theory of mixing times of quantum channels to show that, for a number of physical systems of interest, Gibbs samplers provide an efficient way of preparing the associated Gibbs states on a quantum computer.

Research resulting from the workshop Discussions at the workshop led to several research collaborations and results. In the open problem session Ashley Montanaro posed the question of proving a reverse hypercontractivity result in QIT. Subsequently Toby Cubitt, Michael Kastoryano, Ashley Montanaro and Kristan Temme solved this problem, leading to the preprint 'Quantum reverse hypercontractivity' (arXiv:1504.06143) which also contains several applications of their results. Another collaboration between Aram Harrow and Ashley Montanaro led to the preprint 'Extremal eigenvalues of local Hamiltonians' (arXiv:1507.00739). Also Aram Harrow made important progress on his paper 'Approximate orthogonality of permutation operators' at the workshop. Salman Beigi and Chris King started a collaboration on the problem of determining the form of HC and LS inequalities for the completely bounded norms.

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Chapter 7

Computability in Geometry and Analysis (15w5005)

March 8 - 13, 2015

Organizer(s): Mark Braverman (Princeton), Michael Yampolsky (Toronto)

The workshop brought together experts and junior mathematicians in various areas of Analysis, Dynamics, Geometry, and Topology where computability and complexity ideas play a key role. Theory of computation of analytic objects is an emerging field of mathematics which has seen much exciting progress recently. The talks by the workshop participats outlined the main current directions of study, which we describe below.

Computability problems in topology of knots

Two beautiful talks on the subject were given by M. Lackenby (Oxford) and N. Dunfield (UIUC). We briefly survey the field below.

Computational complexity of knot genus

A fundamental property of a null-homologous knot K in a closed 3-manifold Y is the minimal genus $g(K)$ of an embedded surface in Y with boundary K. In the 1960s, Haken used normal surface theory to give an algorithm which computes $g(K)$, opening the door to a whole subfield and the discovery of algorithms for determining a wide range of topological properties. However, algorithms based on normal surface theory are usually exponential-time both in theory and practice [**?**]. Moreover, in some cases the underlying problems have been shown to be in complexity classes which are thought to be fundamentally difficult. For example, consider the following decision problem:

KNOT GENUS. *Given an integer g_0 and a null-homologous knot K embedded in the 1-skeleton of a triangulation $\mathcal{T}$ of a closed 3-manifold, is $g(K) \leq g_0$?*

Agol, Hass, and W. Thurston showed that KNOT GENUS is NP–complete [**?**], and so has the same computational complexity as e.g. the Traveling Salesman Problem. The conjecture that P $\neq$ NP thus implies that there is no algorithm for KNOT GENUS which runs in time polynomial in the size of $\mathcal{T}$.

However, when $b_1(Y) = 0$, for instance $Y = S^3$, then KNOT GENUS should be considerably easier than NP–complete, perhaps even solvable in polynomial time. Here are three pieces of evidence for this. First, as discussed below in Section 7, Hirani and Dunfield showed that a closely related but more geometric problem can be solved in polynomial time when $b_1(Y) = 0$ despite being NP–complete in general. Second, Lackenby, using an approach of Agol, has recently proved

Theorem. [(Lackenby 2015)] *When $b_1(Y) = 0$, then KNOT GENUS is in coNP.*

The Agol-Lackenby approach uses Gabai's sutured manifold hierarchies to bound the genus from below.

If P is the class of decision problems for which there are polynomial-time algorithms, then the standard conjectures are that the known containments P $\subset$ (NP $\cap$ coNP) $\subset$ NP are all proper. Thus, as KNOT GENUS is in

NP $\cap$ coNP when $b_1(Y) = 0$, there is the potential for an algorithm in this special case which is substantially faster than for an NP-complete problem. Third, using a very different approach relying on results in gauge theory and arithmetic geometry, Kuperberg had previously proved that for $Y = S^3$ the $g_0 = 0$ case of KNOT GENUS is in coNP [**?**].

Minimal spanning area

Returning to the general setup, let K be a null-homologous knot in a closed 3-manifold Y with no restriction on $b_1(Y)$. Let M be a simplicial complex triangulating the exterior $Y \setminus \operatorname{int}\big(N(K)\big)$ of K, and assign to each 2-simplex in M a positive real number which we refer to as its area. Consider the set $\mathcal{F}$ of simplicial maps $f \mapsto (S, \partial S) \to (M, \partial M)$ where S is an orientable surface with boundary and $f_*\big([\partial S]\big)$ generates the kernel of $H_1(\partial M) \to H_1(M)$. Thus, when we collapse ∂M back to K, the image $f(S)$ is a (possibly non-embedded) surface homologically bounding K. The areas of 2-simplices of M naturally define the area of each $f \in \mathcal{F}$. Consider:

LEAST SPANNING AREA. *Given $A_0 \in \mathbb{N}$ and the exterior M of a null-homologous knot $K \subset Y$ is there an $f \in \mathcal{F}$ with $\operatorname{Area}(f) \leq A_0$?*

The work of [**?**] shows this problem is NP-complete. Despite this, Hirani and Dunfield proved the following, which made Dunfield conjecture in his talk KNOT GENUS is in P for knots in S^3.

Theorem. [?] *For manifolds with $b_1(Y) = 0$, the LEAST SPANNING AREA problem can be solved in polynomial time.*

Random knots

Dunfield also described his experimental work with a graduate student Malik Obeiden on random prime knots with 100 to 1,000 crossings, both to probe algorithmic complexity in practice and to better understand the properties of random knots in the spirt of [**?**, **?**]. Their initial findings give an evidence of linear growth (with little spread) with respect to crossing number of the following invariants: hyperbolic volume (slope ≈ 2), knot genus (slope ≈ 0.25), and bridge number (slope ≈ 0.15). One pattern that demands explanation: these knots have triangulations where most of the tetrahedra are "fat" in the hyperbolic structure, that is, have volumes near that of the regular ideal one.

Algorithmic randomness and computable Ergodic theory

D. Hirschfeldt (Chicago) gave a beatiful introduction to Algorithmic Randomness. Building on his talk, C. Rojas (Andres Bello) and J. Avigad (Carnegie Mellon) described applications of Computability to Ergodic Theory, which is an area of research relating measurable dynamics to theoretical computer science. Its general goal is to understand in a precise mathematical way the theoretical simulation and computation of the long term behavior of dynamical systems. Among the objects describing this limiting behavior, of particular interest are invariant measures and generic points which provide a complete statistical description of the system. Their computability and complexity properties – in the sense of theoretical computer science – and its relationship to the dynamical, geometrical and analytical properties of the system, are the main questions of the subject. The tools and techniques vary according to the nature of the different systems involved, but they have all one thing in common: one has to deal with rigorous notions of computability for infinite objects. This is done in the field known as "Computable Analysis" ([**?**, 9, **?**]). The main idea here being that an object x is computable if there exists an algorithm (a Turing Machine) which, upon input $\varepsilon \in \mathbb{Q}$, will produce a "finite" object describing x at accuracy ε with respect to some suitable notion of distance. In what follows we recall the most relevant known results on the subject, and state some directions of active research.

Invariant measures

Computability of invariant measures in the rigorous sense of computable analysis is a very recent topic of research. One way to prove computability of an invariant measure is to use some known strong statistical properties of the system, like for example decay of correlations as done in [**?**]. The algorithms obtained in this way, however, usually depend on some finite information that might be not known, and therefore they are non-uniform. The most general technique to obtain uniform algorithms consists, very roughly speaking, on finding a list of suitable semi-decidable conditions that together characterize a particular invariant measure of interest. The computability of this measure then follows from general computability considerations involving the effectiveness of certain compact sets. These results have the advantage of being simple and quite general with all the needed assumptions made explicit. They have a wide range of application including several different classes of systems like hyperbolic systems (see [?]) or rational functions (see [?]). On the other hand, they are not well suited for a complexity (in time or space) analysis, so it is not clear if they can be implemented and used in practice. The rigorous framework in which they are proved, however, allows to see them as a study about the theoretical limits of (Turing-)computation of invariant measures and, in fact, also negative results can be obtained. For example, there exists computable systems for which every invariant measure is non-computable (see [?]), and systems having measures of maximal entropy, all of which are non computable ([?]). There are also examples of computable invariant measures which are a computable combination of finitely many ergodic measures all of which are non computable (see [?]). An important direction of research here is the study of the robustness properties of these kind of non computable phenomena. In [?] it is shown for example that the addition of a small amount of noise at each step of the evolution of the system is sufficient to destroy the non computability of the ergodic measures – the noise turns them from non computable into computable. However, a more fine analysis of how persistent can the non computable phenomenon be, is still lacking. Another important open research direction is the analysis of the computational resources required to achieve the computation of an invariant measure and the way this complexity relates to the dynamics. In particular, how sensitive is the computational complexity with respect to small changes to the dynamics. Some progress was made in [?] where the authors show that for noisy systems, if the noise itself is not a source of additional complexity, then the invariant measures can be computed efficiently, namely in time polynomial in the number of bits required to specify the precision of the computation.

Effective ergodic theorems, randomness and pseudorandomness

The main question here is to understand the effectiveness of the rate of convergence in the ergodic theorems. It has long been known that this rate can be arbitrarily slow [?, ?]. However, the point here is not directly about the speed of convergence, but rather about the information required to algorithmically extract a bound on this rate [?, ?]. These effective results are important because they allow to obtain more concrete information about how the finite structures underlying the ergodic behavior are constructed. For example, they are useful in developing algorithms to compute points exhibiting good statistical properties ([?, ?]). They have played a role in recent results in number theory and combinatorics ([?, ?]). They are also important in the theory of algorithmic randomness ([?]) as a tool to calibrate the degree of randomness required by points in order to satisfy the ergodic theorem (or other almost everywhere convergence results) with respect to specific observables. For example, the understanding achieved on the computable content in the ergodic theorem for ergodic measures versus non ergodic ones, gave rise to a series of results ([?, ?, ?, ?, ?, ?, ?, ?]) culminating in a sharp characterization of generic points in terms of algorithmic randomness.

Computability questions in Complex Analysis

Computability of the Riemann mapping Theoretical aspects of computability of the Riemann mapping were discussed by I. Binder (Toronto). We briefly summarize the discussion below.

An open set U in the plane is called *lower-computable* if there exists a computable sequence of rational balls whose union exhausts the set U. Similarly, a closed set K is lower-computable if rational balls which intersect K can be computably enumerated.

Let Ω be a simply-connected proper subdomain of $\mathbb{C}$ and let $w_0 \in \Omega$. The celebrated Riemann Mapping Theorem states that there exists a unique conformal homeomorphism

$$g : \Omega \to \mathbb{D} \text{ such that } g(w_0) = 0 \text{ and } g'(w_0) > 0. \tag{7.0.1}$$

We will denote $f \equiv g^{-1} : \mathbb{D} \to \Omega$. Hertling showed in [?] that g and f are computable if and only if: Ω is a lower-computable open set, $\partial\Omega$ is a lower-computable closed set, and $w_0 \in \Omega$ is a computable point.

The computational complexity of the Riemann mapping were addressed by Binder, Braverman, and Yampolsky in [?]. They showed, in particular:

Theorem. [?] *Suppose there is an algorithm A that given a simply-connected domain Ω with a linear-time computable boundary, a point $w_0 \in \Omega$ with $\mathrm{dist}(w_0, \partial\Omega) > \frac{1}{2}$ and a number n, computes $20n$ digits of the conformal radius $f'(0)$), then we can use one call to A to solve any instance of a $\#SAT(n)$ with a linear time overhead.*

In other words, $\#P$ is poly-time reducible to computing the conformal radius of a set.

Note, that any algorithm computing values of the uniformization map f will also compute the conformal radius with the same precision, by Koebe Distortion Theorem. Conversely,

Theorem. [?] *There is an algorithm A that computes the uniformizing map in the following sense:*

Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain Ω, $\partial\Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to A by an oracle. Then A computes $g(w)$ with precision n with complexity PSPACE(n).

Rettinger later observed that the proof of [**?**] actually gives a better complexity bound, $\#P$.

Binder also described the work [**?**], in which he, Rojas, and Yampolsky developed the computable version of the Carathéodory Theory of prime ends, proving a computable version of the Carathéodory Theorem.

A special case of this theorem states that f continuously extends to the unit circle if and only if $\partial\Omega$ is locally connected. The computable version of this proved in [**?**] relies on the definition of the *Carathéodory modulus*. Namely, a non-decreasing function $\eta(\delta)$ is called the Carathéodory modulus of Ω if $\eta(\delta) \to 0$ as $\delta \to 0$ and if for every crosscut γ with $\operatorname{diam}(\gamma) < \delta$ we have $\operatorname{diam} N_\gamma < \eta(\delta)$. Here N_γ is the component of $\Omega \setminus \gamma$ not containing w_0.

It was shown in [**?**], that the Carathéodory extension of $f : \mathbb{D} \to \Omega$ is computable iff f is computable and there exists a computable Carathéodory modulus of Ω. Furthermore, it was shown that there exists a domain Ω with computable Carathéodory modulus but no computable modulus of local connectivity.

Computational aspects of the conformal mapping: Zipper algorithm

S. Rohde's (Washington) was based in part on his joint work with Don Marshall. The best known algorithm for computing the conformal mapping is Zipper algorithm proposed by Marshall [**?**]. Its convergence was proven by Rohde and Marshall in [**?**]. Rohde described the algorithm, together with the related Loewner equation and conformal welding. He explained how the zipper algorithm can be viewed as a discretization of the Loewner differential equation, and how this discretization can be implemented to produce an approximation to a given conformal map as a composition of a large number of conformal maps onto half-planes slit by a hyperbolic geodesic (or straight line). He sketched the proof of the convergence of the algorithm, a key idea being the use of Jorgensen's theorem about the hyperbolic convexity of Euclidean discs in planar domains.

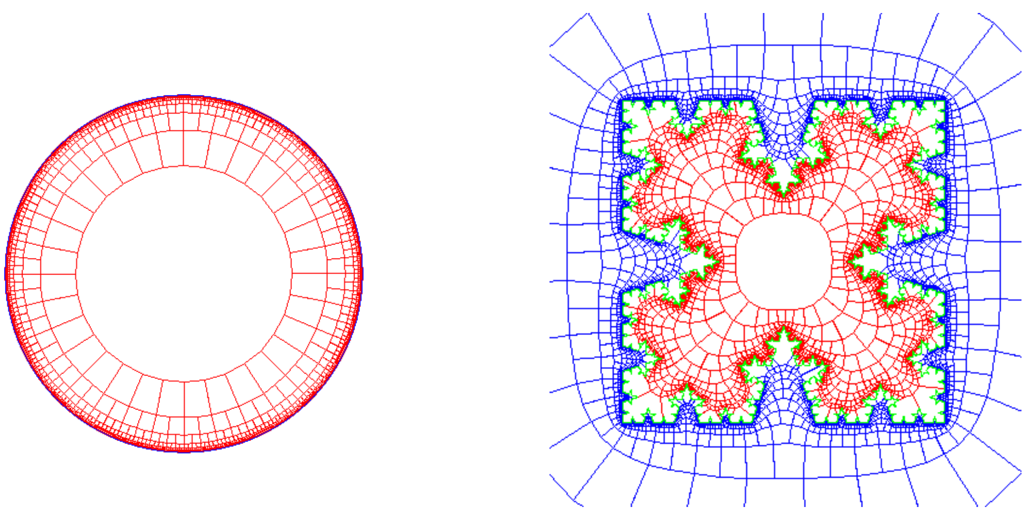

Figure 7.1: Zipper algorithm at work: Carleson grid in the unit disk (left) and its image under the Riemann map inside a "snowflake" domain.

The celebrated Schramm-Loewner Evolution SLE is obtained by using Brownian motion as the driving function for the Loewner equation. Rohde gave a brief overview of some highlights regarding SLE, some of their path properties, and a comparison to the corresponding deterministic results. He then proceeded to the seemingly unrelated topic of Grothendieck dessins d'enfants and the associated Belyi functions: Every connected graph drawn on the sphere can be realized (up to homeomorphism) as the preimage $f^{-1}(L)$ under a rational map with at most three critical values (Belyi function), where L is a line segment joining two of the critical points. In the special case when the graph is a tree (no cycles), one of the critical points can be normalized to be infinity, and the Belyi function is a polynomial (Shabat polynomial). The computation of Belyi functions has spurred a lot of research and led to many publications in computational algebra, but the algebraic methods fail as soon as the degree of the polynomial exceeds about 10. Rohde explained how the zipper algorithm can be modified to numerically approximate Shabat polynomials and their trees when the degree is very large (even in the thousands!). The method also allows to re-construct a dendrite from its lamination, and Rohde illustrated this in the setting of quadratic Julia sets. He then proceeded to a discussion of random trees, an object of very high current interest: He explained how a bijection between trees and their Dyck paths leads to the Aldous Continuum Random Tree in the scaling limit, and how this CRT arises as a building block of the scaling limit of large random maps, the so-called Brownian map. He concluded by discussing partial results to the existence question of the distributional limit of the Shabat trees when the degree tends to infinity.

Decidability of equivalence problems in topological dynamics

Decidability of the Thurston equivalence problem N. Selinger (Stony Brook) spoke about the Thurston equivalence problem for branched covering maps. A *Thurston mapping* $f : S^2 \to S^2$ is a branched covering of a finite topological degree $d > 1$, and such that the orbits of the branched points are finite. f is *Thurston equivalent* to g if there exist homeomorphisms ψ_1, $\psi_2 : S^2 \to S^2$ such that $(\psi_2)^{-1} \circ f \circ \psi_1 = g$, and there is an isotopy between ψ_1 and ψ_2 which does not move the orbits of the branched points of the coverings. A celebrated theorem of Thurston [**?**] answers the question when a Thurston mapping is equivalent to a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The criterion of Thurston is formulated in terms of non-existence of certain finite collections of homotopy classes of loops on the sphere, known as *Thurston obstructions*.

A question, which had remained open for some time, was whether there exists an algorithm which given a finite description of the branched covering f can answer whether a rational map R exists or not. We resolved this long-standing problem in a recent work of M. Braverman, S. Bonnot, and M. Yampolsky [**?**]. A more general problem is:

Thurston Equivalence Problem. *Is there an algorithm which decides whether two Thurston mappings f and g are equivalent or not?*

N. Selinger and M. Yampolsky [**?**] we have made significant progress towards resolving this questions by proving that any Thurston mapping can be constructively canonically geometrized. This allowed them to partially resolve the problem of comparing two maps, yet complete solution requires further work.

Conjugacy problem for expanding maps

V. Nekrashevych (Texas A& M) reported on his current work on the conjugacy problem for expanding maps. Let (X, d) be a compact metric space. A map $f : X \to X$ is said to be <u>expanding</u> if there exist $\varepsilon > 0$ and $L > 1$ such that $d(f(x), f(y)) \geq Ld(x, y)$ for all $x, y \in X$ such that $d(x, y) < \varepsilon$.

Expanding covering maps $f : X \to X$ can be given (modulo topological conjugacy) by a finite amount of information. For example, one can use the fact that they are <u>finitely presented</u>: such a map is conjugate to a quotient of a shift of finite type by an equivalence relation which is also a shift of finite type.

If X is locally connected and connected, then there is also a natural group associated with $f : X \to X$ which is a complete invariant. For $t \in X$ consider the tree of preimages T_t with the set of vertices equal to the (formal) disjoint union of the sets$f^{-n}(t)$ for $n \geq 0$. Here a vertex $v \in f^{-n}(t)$ is connected to $f(v) \in f^{-(n-1)}(t)$. If γ is a path from t_1 to t_2, then for every $v \in f^{-n}(t_1)$ there exists a unique lift γ_v of γ by f^n starting at v. Denote the end of γ_v by $S_\gamma(v)$. Then $S_\gamma : T_{t_1} \to T_{t_2}$ is an isomorphism. It also induces a homeomorphism of the boundaries $S_\gamma : \partial T_{t_1} \to \partial T_{t_2}$.

Fix $t \in X$, and consider two finite sets $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_n\}$ of vertices of T_t such that ∂T_t is equal to the disjoint union $\bigsqcup_{i=1}^n \partial T_{v_i}$ and to the disjoint union $\bigsqcup_{i=1}^n \partial T_{u_i}$. Choose paths γ_i from v_i to u_i. Then union of the maps S_{γ_i} is a homeomorphism of ∂T_t with itself. Denote by $\mathcal{V}_f$ the set of all such homeomorphisms.

Nekrashevych reported the following result:

Theorem. *The set $\mathcal{V}_f$ is a group. It is finitely presented, its derived subgroup is simple. It is a complete invariant of topological conjugacy: two dynamical systems are topologically conjugate if and only if the corresponding groups are isomorphic as abstract groups.*

He then posed a natural question, whether the problem of topological conjugacy of expanding self-coverings is algorithmically solvable. While the question is open, there is hope that it can be resolved using group-theoretic methods.

Words in linear groups, random walks, automata, and P-recursiveness I. Pak (UCLA) described his current work with S. Garrabrant. An integer sequence $\{a_n\}$ is called polynomially recursive, or P-recursive, if it satisfies a nontrivial linear recurrence relation of the form

$$(*) \qquad q_0(n)a_n + q_1(n)a_{n-1} + \ldots + q_k(n)a_{n-k} = 0,$$

for some $q_i(x) \in \mathbb{Z}[x]$, $0 \le i \le k$. The study of P-recursive sequences plays a major role in modern Enumerative and Asymptotic Combinatorics. They have D-finite (also called holonomic) generating series

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} a_n t^n,$$

and various asymptotic properties.

Let G be a group and $\mathbb{Z}[G]$ denote its group ring. For every $g \in G$ and $u \in \mathbb{Z}[G]$, denote by $[g]u$ the value of u on g. Let $a_n = [1]u^n$, which denotes the value of u^n at the identity element. When $G = \mathbb{Z}^k$ or $G = F_k$, the sequence $\{a_n\}$ is known to be P-recursive for all $u \in \mathbb{Z}[G]$. Maxim Kontsevich asked in 2014 whether $\{a_n\}$ is always P-recursive when $G \subseteq \mathrm{G}L(k, \mathbb{Z})$. Pak and Garrabrant gave a negative answer to this question:

Theorem. *There exists an element $u \in \mathbb{Z}[SL(4, \mathbb{Z})]$, such that the sequence $\{[1]u^n\}$ is not P-recursive.*

Pak described two proofs of the theorem. The first proof is completely self-contained and based on ideas from computability. Roughly, one gives an explicit construction of a finite state automaton with two stacks and a non-P-recursive sequence of accepting path lengths. One then converts this automaton into a generating set $S \subset \mathrm{S}L(4, \mathbb{Z})$. The key part of the proof is a new combinatorial lemma giving an obstruction to P-recursiveness.

The second proof is analytic in nature, and is the opposite of being self-contained. The problem is interpreted in a probabilistic language, a number of advanced and technical results in Analysis, Number Theory, Probability, and Group Theory are used to derive the theorem.

Other complexity questions in groups with applications M. Sapir (Vanderbilt) gave a beautiful talk on various computing devices (Turing machines, S-machines, Minsky machines) used in dealing with algorithmic problems in group theory.

A. Nabutovsky (Toronto) described his joint work with B. Lishak. In his earlier works [**?**, **?**, **?**], Nabutovsky described the geometric complexity of the space of Riemannian metrics on a manifold of dimension $d \ge 5$ which arises from undecidability of word problem in the fundamental group, and related geometric complexity phenomena resulting from non-computability in groups. The work with Lishak extends these results to dimension $d = 4$. Namely, Nabutovsky and Lishak proved that: 1) There exist infinitely many non-trivial codimension one "thick" knots in $\mathbb{R}^5$; 2) For each closed four-dimensional smooth manifold M and for each sufficiently small positive ε the set of isometry classes of Riemannian metrics with volume equal to 1 and injectivity radius greather than ε is discon- nected; 4) For each closed four-dimensional PL-manifold M there exist arbitrarily large values of N such that some two triangulations of M with $< N$ simplices cannot be connected by any sequence of $< M(N)$ bistellar transformations, where $M = 2^{2^{2^{\cdots^{2}}}}$ ($[\log_2 N]$ − const times).

Information and communication complexity A.Garg (Princeton) gave a talk on communication and information complexity. Two-party communication complexity is the study of how much communication two parties need to exchange to compute a function of their private inputs. It is one of the few models of computation in which strong unconditional lower bounds are known and has applications throughout complexity theory, for example to lower bounds on circuits, streaming algorithms, data structures etc. In the past 5-10 years, the study of information complexity has greatly advanced the state-of-the-art knowledge about communication complexity. Information complexity is the study of how much information two parties need to exchange to compute a function of their private inputs. It has helped in tackling hard questions in communication complexity, for example direct sum and direct product theorems. The best known direct sum and direct product theorems for randomized communication complexity are proven via information complexity. Roughly, the current best direct sum theorem [**?**] says that the amount of communication required to compute n independent copies of a function is at least $\Omega(\sqrt{n})$ times the amount of communication required for one copy. Furthermore, it has also been proven [**?**] that the success probability decays exponentially in n. Such statements are known as direct product theorems. Improving the known direct sum and direct products theorems is equivalent to the compression question: whether uninformative conversations can be compressed down to their information content. It is known that the compression question does not have a positive answer in its full generality [**?**], but it still largely remains an open problem. Information complexity also has other applications within communication complexity and complexity theory in general. For example, it can be used to pin down the exact communication complexity of disjointness up to low order terms [**?**]. Also the techniques used for proving direct product theorems in communication complexity have helped in getting new proofs (and improvements) of the parallel repetition theorem [**?**, **?**].

A complexity theory of constructible functions and sheaves

S. Basu (Purdue) gave a talk on constructible functions and sheaves. Constructible functions and more generally constructible sheaves play a very important role in algebraic geometry with many applications, including in theory of D-mocules, algebraic theory of partial differential equations, and even in more applied areas such as computational geometry and signal processing. In his talk Basu described an approach towards developing a complexity theory for these objects, which generalizes the Blum-Shub-Smale model over $\mathbb{R}$. More precisely, he introduced a class of sequences simple constructible sheaves, that could be seen as the sheaf-theoretic analog of the Blum-Shub-Smale class $\mathrm{P}_{\mathbb{R}}$. He also defined a hierarchy of complexity classes of sheaves mirroring the polynomial hierarchy, $\mathrm{PH}_{\mathbb{R}}$ in the B-S-S theory. He proved a singly exponential upper bound on the topological complexity of the sheaves in this hierarchy mirroring a similar result in the BSS setting. He obtained as a result an algorithm with singly exponential complexity for a sheaf-theoretic variant of the real quantifier elimination problem. Finally, he posed the natural sheaf-theoretic analogs of the classical P vs NP question, and also discussed a connection with Toda's theorem from discrete complexity theory in the context of constructible sheaves.

Participants

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Chapter 8

Laplacians and Heat Kernels: Theory and Applications (15w5110)

March 22 - 27, 2015

Organizer(s): Denis Grebenkov (Ecole Polytechnique), Peter Jones (Yale University), Naoki Saito (University of California - Davis)

Overview of the Field

The investigation of eigenvalues and eigenfunctions of the Laplace operator in bounded domains, manifolds, or graphs is a subject with a history of more than two hundred years [1]. This is still a central area in mathematics, physics, engineering, and computer science and activity has increased dramatically in the past twenty years. Laplacian eigenfunctions appear as vibration modes in acoustics, as electron wave functions in quantum waveguides, or as a natural basis for constructing heat kernels in the theory of diffusion. For instance, vibration modes of a thin membrane (a drum) with a fixed boundary are given by Dirichlet Laplacian eigenfunctions, with the drum frequencies proportional to the square root of the eigenvalues. The Laplacian eigenvalue problem is archetypical in the theory of elliptic operators, while the properties of the underlying eigenfunctions have been thoroughly investigated in various mathematical and physical disciplines, including spectral theory, probability and stochastic processes, dynamical systems and quantum billiards, condensed matter physics and quantum mechanics, the theory of acoustical, optical, and quantum waveguides, and the computer sciences. Various aspects of Laplacian eigenfunctions and eigenvalues have been studied: high-frequency asymptotic behavior, isoperimetric inequalities and the related shape optimization problems, inverse spectral problems, geometric properties of nodal lines/surfaces or nodal domains. More recently, the intricate relation between the shape of the domain and the spatial distribution of eigenfunctions has been analyzed, in both high-frequency and low-frequency regimes. The intriguing properties of eigenfunctions find physical and engineering applications. For instance, the localization of some eigenfunctions in small regions of the domain was used to build noise protective walls. More generally, eigenmethods have found numerous applications in data analysis, e.g., low-dimensional parameterization of point clouds or page ranking in search algorithms (Google). Despite of a long history of investigations of Laplacian eigenfunctions, many questions remain open or have been recently formulated.

Recent Developments and Open Problems

A considerable progress has been recently made in understanding the geometry and topology of Laplacian eigenfunctions. For example, F. Nazarov and M. Sodin have made major advances in studying the nodal lines and domains of a random linear combination of Laplacian eigenfunctions on the sphere (i.e., spherical harmonics) belonging to the same eigenspace [2]. The interest of studying such random spherical harmonics comes from the fact that they can serve as a good model for the typical behavior of high frequency Laplacian eigenfunctions on a compact surface endowed with a smooth Riemannian metric. A. Hassell made a breakthrough in the field of quantum billiards by showing that generic stadiums are not quantum uniquely ergodic [3]. Yet another significant discovery on the structure of eigenfunctions has been made by M. Filoche and S. Mayboroda who developed a unifying mathematical approach to localization phenomena [4]. These and many other important results have been recently reviewed by one of the organizers [1]. Despite such progress, much remains open; for instance it is not understood how scaling limits of these random spherical harmonics (i.e., a random linear combination of plane waves) behave.

Beside classical spectral expansions in mathematics, Laplacian eigenfunctions turn out to be a natural tool for a broad range of areas, e.g., in data analysis to reduce dimensionality of datasets by using diffusion maps, or to study brain regions functionality. The eigenfunctions also appear as intrinsic local coordinate system with low distortion mappings.

In physics, Laplacian eigenfunctions were successfully applied for a better interpretation of nuclear magnetic resonance measurements of diffusive transport. For instance, the asymptotic properties of the heat kernel allowed experimental determination of the surface to volume ratio in porous media like sandstones or sedimentary rocks. These results are also related to Weyl asymptotic for the eigenvalues and its Weyl-Berry conjectural extension, with opening connections to number theory (e.g., spectral zeta functions).

The use of Laplacian eigenvalues as natural fingerprints to identify geometrical shapes was suggested for different applications, e.g., copyright protection, database retrieval, and quality assessment of digital data representing surfaces and solids. The related inverse spectral problems have attracted much attention in the last decades.

In this light, this workshop gathering experts in different fields, from mathematics to physics, engineering, and computer science, has provided an exciting opportunity to discuss various aspects of these long-standing problems.

Presentation Highlights

The presentations at this workshop can be roughly categorized into 7 topics although some of them are not localized in a single category: 1) Localization and geometry of eigenfunctions (Grebenkov, Maltsev, Beliaev, Filoche, David); 2) Heat/wave kernels (van den Berg, Harrell, Jakobson, Jones, Jerison); 3) Brownian motions, harmonic functions (Burdzy, Mayboroda); 4) Physical applications (Berkolaiko, Amitai, Song); 5) High dimensional data analysis, data manifolds, shape analysis (Maggioni, Bronstein, Lai, Cloninger, Saito, Beg, Meyer); 6) Shape optimization, numerical methods (Kao, Antunes, Shivakumar); and 7) Eigenvalue inequalities (Laugesen, Siudeja, Henrot, Benguria, Hermi, Mahadevan)

Disclaimer: To keep the brevity of this report, we will not include the names of the collaborators of the speakers below, which can be obtained from the abstracts of the speakers as well as their presentation slides.

Localization and geometry of eigenfunctions

Denis Grebenkov (Ecole Polytechnique) kicked off the workshop by the overview of the field of Laplacian eigenfunctions closely based on his SIAM Review paper [1]. He discussed the localization and concentration phenomena of the low-frequency Laplacian eigenfunctions due to the geometry of the domains. In particular, he focused on the exponential decay of the low-frequency eigenfunctions on domains with elongated branches, and explained a sufficient condition on the minimal length of such branches leading to localization of the low-frequency eigenfunctions.

Anna Maltsev (Bristol) discussed on localization/exponential decay of eigenfunctions of Schrödinger operator on quantum graphs. She used the so-called "Agmon metric" $\rho_E(x,y) := \int_x^y \sqrt{(V(x)-E)_+}\mathrm{d}x$ where V is the potential and E is the energy level (or eigenvalue) of the system. If $\exp(\rho_E(0,x))\psi \in L^2$ is shown, then ψ clearly decays exponentially. She started from the review of the known results for path graphs and discussed her approach to generalize them for rooted trees, regular trees, and finally general trees. She concluded her talk with an example of "harmonic millipede" graphs and some open problems, e.g., how to tackle more general graphs containing cycles.

Dmitry Beliaev (Oxford) talked about the nodal geometry of random plane waves, which behaves in a similar manner as that of high-energy eigenfunctions in stadium-shape domains that often appear in the quantum chaos literature. In particular, he reviewed the above Nazarov-Sodin result on the estimate of the number of nodal domains generated by random linear combinations of spherical harmonics of degree n (forming $2n+1$-dimensional eigenspaces). Moreover, he presented numerical results that suggest a way to amend the Bogomolny-Schmit conjecture on the percolation model by considering the random graphs generated by random plane waves on the square lattice.

Both Filoche and David talked about their program that led to another small-group workshop, Focused Research Group: Localization of Eigenfunctions of Elliptic Operators that was held immediately after our workshop. Marcel

Filoche (Ecole Polytechnique) discussed the localization of eigenfunctions of both the Laplace and biharmonic operators on complicated domains including some slits and punctures. His discussion on the "inverse" problems, i.e., where to put punctures on a thin plate in order to form a desired localized pattern, was interesting and important for real-life applications. In addition, he described his work on the Anderson localization (i.e., the Schrödinger eigenvalue problems with random potentials), and the conditions under which the delocalization occurs. The key tool he used for all these is the so-called "landscape" function, i.e., the solution of $\mathcal{L}u = 1$ with the appropriate boundary condition where $\mathcal{L}$ is an elliptic partial differential operator (e.g., Laplace, biharmonic, Schrödinger, etc.). His numerical simulations agreed well with the physical experiments conducted at the Langevin Institute.

Guy David (Paris-Sud) discussed a related free boundary problem: how to decompose a given domain into N subdomains via minimization of a certain functional that includes a usual energy term and a term penalizing subdomains of unbalanced sizes. It turns out that the Laplacian/Schrödinger eigenfunctions tend to localize on such subdomains, which were investigated further in the above-mentioned FRG.

Heat and wave kernels

Michiel van den Berg (Bristol) discussed his work on heat flow on Riemannian manifolds. He started off with his earlier results on heat content asymptotics on a simpler case of a compact Riemannian manifold with C^∞ boundary, then discussed the case of a compact Riemannian manifold without boundary before reaching his latest results on a complete, non-compact Riemannian manifold with non-negative Ricci curvature. In particular, he discussed heat flow from a subdomain Ω of such a manifold M into $M \setminus \Omega$ if the initial temperature distribution is the characteristic function of Ω. For $|\Omega| = \infty$, he obtained a necessary and sufficient condition to have finite heat content for all $t > 0$ and also obtained upper- and lower bounds for the heat content. Two-sided bounds are obtained for the heat loss of Ω in M if $|\Omega| < \infty$.

Evans Harrell (Georgia Tech) presented two rather distinct results whose common theme is a certain optimality about heat traces. The first one is an optimal placement problem for an obstacle in a domain so as to maximize or minimize the heat trace. More specifically, given a region Ω, excluding a subset B of a fixed shape (in practice round) at an unspecified position, how the extremal values of an eigenvalue or other spectral function can be achieved by moving B around? In the case of the Dirichlet boundary condition, the maximum is achieved when B is in a distinguished subset while the minimum is attained when B touches the boundary $\partial\Omega$ for some class of regions Ω (e.g., annular regions). These results are mainly due to Harrell-Kröger-Kurata (2001), but he also traced some of more recent works done by the others. The second topic of his talk was a set of sharp semiclassical inequalities for sums of eigenvalues, which are closely related to traces of heat kernels via Karamata's theorem and its variants. These are based on a new variational principle that incorporates averages of the lowest k eigenvalues, which implies sharp estimates for Laplacians of various kinds, including those on graphs and on quantum graphs.

Dmitry Jakobson (McGill) presented his construction of Gaussian measures on the manifold of Riemannian metrics with the fixed volume form. He showed that diameter, Laplacian eigenvalue and volume entropy functionals are all integrable with respect to these measures. He also presented his computation of the characteristic function for the L^2 (also called Ebin) distance from a random metric to the reference metric.

Peter Jones (Yale) discussed the importance of randomness that often gives rise to certain orders and structures. In particular, he discussed the notion of Gaussian free fields (mainly in 2D) and contrasted their differences from the Brownian motion, and explained that the cosmic microwave background may be viewed as a realization of such Gaussian free field.

David Jerison (MIT) discussed his work on wave trace: a spectral invariant function $W(t) := \sum_j \cos(\lambda_j t)$, in which the sum is over all eigenvalues λ_j^2 of $-\Delta$ on a compact Riemannian manifold. It is called the wave trace because it is the trace of the solution operator for the wave equation. It is well known that $W(t)$ is singular at times t that are equal to the length of geodesics. The singularities are well understood if the length is an isolated point in the time line. He presented his theorem in the case of the unit disk in $\mathbb{R}^2$ with the Dirichlet boundary condition: $W \in C^\infty[2\pi, 8)$.

Brownian motions, harmonic functions

Chris Burdzy (Univ. Washington) discussed his recent results on obliquely-reflected Brownian motion in the unit disk in $\mathbb{R}^2$ and some fractal domains. Dealing with obliquely-reflected Brownian motion is tough since classical Dirichlet form approach to Markov processes is limited to symmetric processes and obliquely-reflected Brownian motion is not symmetric. Also, such a Brownian motion could jump along the boundary in the tangential direction if it hits the boundary with certain angles. He showed his theorem stating that for such a Brownian motion X_t, $(1/t)\arg X_t - \mu$ converges to the Cauchy distribution as $t \to \infty$ where μ is a rate of rotation of this Brownian motion. For fractal domains, he presented two possible approaches: the one based on smooth approximation of a domain and the other based on conformal mapping.

Svitlana Mayboroda (Univ. Minnesota) discussed the relationship between uniform rectifiability and harmonic functions. In relatively friendly geometric settings, e.g., on Lipschitz domains, harmonic measure is absolutely continuous w.r.t. the Lebesgue measure. Moreover, quantitative absolute continuity of elliptic measure is equivalent to solvability of the Dirichlet problem in L^p, square function estimates, Carleson measure estimates, and a certain approximation property, so called ε-approximability of solutions. In particular, she addressed the following question: what are the key geometrical properties of the boundary responsible for such a behavior of harmonic functions and whether these can be generalized for uniformly rectifiable domains without assuming the connectivity, whose answer is 'Yes.'

Physical applications

Gregory Berkolaiko (Texas A & M Univ.) talked about the so-called Dirac points in the spectra of graphene (honeycomb) structure. Many exciting physical properties of graphene can be traced to the presence of conical singularities ("Dirac points") in its dispersion relation. He first traced a history of the proof of the presence of such Dirac points in the context of Laplacian on domains in $\mathbb{R}^2$, discrete Laplacian, and quantum graph Laplacian. He then presented his general yet very simple proof that works in all the above models. His proof of the presence of the Dirac points uses quotients of the operator by the (co)representations of the symmetry group, and a proof of stability of Dirac points uses the so-called Berry phase: a phase gained by an eigenfunction after the parameters specifying that eigenfunction rotate around a contour in the parameter space. He also showed some animations to illustrate these ideas.

Assaf Amitai (MIT) presented his results on the mean first encounter time (MFET) between two polymer sites, which is an interesting application of a Brownian motion in high dimensional manifolds. The encounter between two sites on chromosomes in the cell nucleus can trigger gene regulation, exchange of genetic material, and repair of DNA breaks. By forming a DNA loop, a protein located on the DNA can trigger the expression of a gene located far along the chain. He computed asymptotically the MFET between the head and tail of the polymer as a function of the distance ε between them, using the classical Rouse polymer model, in which the polymer is described as a collection of bead monomers connected by harmonic springs. This novel asymptotic relies on the expansion of the spectrum of the Fokker-Planck operator as a function of ε, and the explicit computation of the Riemannian volume for Chavel-Feldman formula, which gives the shift in the spectrum of the Laplace operator when Dirichlet boundary conditions are imposed on the boundary of tubular neighborhood of a constraint manifold (removal of a manifold with a small volume).

Yiqiao Song (Schlumberger-Doll Research & Mass. Gen. Hospital, Cambridge, MA), who was the only participant from industry, talked about diffusion dynamics from the viewpoint of multi-point correlation functions, and its applications to study the microstructure of a wide range of porous materials, including geological formations and biological tissues. In order to characterize the non-Gaussian diffusive behavior in heterogeneous media, the fourth order cumulant (kurtosis) can be expressed through the 4-point correlation function which allows one to distinguish contributions from pore size distribution. The related NMR experiments on asparagus and avocados were discussed. In the second part of the talk, he presented the concepts of Diffusion Eigenmode Spectroscopy. This experimental tool relies on the properties of Laplacian eigenfunctions with the Robin boundary condition. By manipulating the nuclei magnetization with magnetic fields, one can probe the geometric properties of a porous medium through the measurable relaxation curves.

High-dimensional data analysis, manifold learning, shape analysis

Mauro Maggioni (Duke) discussed a geometry-based statistical learning framework for performing model reduction and modeling of stochastic high-dimensional dynamical systems. He considered two complementary settings. In the first one, given long trajectories of a dynamical system (e.g., molecular dynamics governed by a Fokker-Planck equation), he presented new techniques for estimating in a robust fashion: (i) an effective number of degrees of freedom of the system; and (ii) a local scale where the dynamics is well-approximated by a reduced dynamics with a small number of degrees of freedom. He then used these ideas to produce an approximation to the generator of the system and obtain, via eigenfunctions, reaction coordinates for the system that captures the large time behavior of the dynamics. In his second setting, the only available data are short-time/local simulations (of various different stages) of a high-dimensional stochastic system due to the often expensive nature of the numerical simulators. Then, he introduced a statistical learning framework for determining automatically a family of local approximations to the system. He then presented applications of this method including homogenization of rough diffusions and deterministic chaotic systems in high-dimensions.

Michael Bronstein (Univ. Lugano, Switzerland) discussed his long-time and large-scale project on manifold correspondence, a fundamental and notoriously hard problem with a wide range of applications in geometric processing, graphics, computer vision, and machine learning. He started off with the classical method, i.e., decomposing each given object into a linear combination of the Laplace-Beltrami eigenfunctions, and considering the correspondence between two different eigen-coordinate systems. Unfortunately, those eigenbasis methods suffer from several issues such as sign ambiguities of eigenfunctions and the differences in eigenfunctions for non-isometric manifolds, which are more pronounced for the high-frequency eigenfunctions. In order to solve these problems, he discussed the use of coupled (or joint) diagonalization of two sets of eigenbases for drastic improvement for the correspondence problems of two non-isometric objects (e.g., an elephant vs a horse). Finally, he discussed a method to estimate the matrix describing the deformation from one manifold to the other based on geometric matrix completion under the sparsity and smoothness constraints. This method does not require any computation of eigenbases. He demonstrated all of these methods using many 3D geometric objects.

Rongjie Lai (RPI) discussed his work on processing and analysis of point clouds using Laplace-Beltrami operator. Point clouds are the simplest and most basic forms for data representation in 3D modeling, imaging science, the Internet and many others. Although raw data appear as an unstructured and unorganized set of points, they are usually with certain coherent structures which allow one to model them as points sampled from lower dimensional Riemannian manifolds in a high dimensional space. Analyzing and inferring the underlying structure from the point clouds are crucial in many applications. Lai discussed his work on solving PDEs on point clouds, which requires only local information such as k nearest neighbors of each point. In particular, the Laplace-Beltrami eigenfunctions can be used to embed the high-dimensional point clouds onto a low-dimensional Euclidean space where one can often see the global organization of these point clouds. Again, similar to what Bronstein listed, the difficulties using such Laplace-Beltrami eigenfunctions are: their sign ambiguities, ambiguities due to the multiplicities of the corresponding eigenvalues (if any), and natural ordering of the eigenfunctions. In order to resolve those difficulties, he incorporated the idea of optimal transportation after computing the two sets of the Laplace-Beltrami eigensystems. He also discussed a few possible ideas to speed up the computation of such an optimization problem utilizing the intrinsic multiscale nature of the eigenfunctions.

Alexander Cloninger (Yale) discussed ways to augment diffusion kernels on datasets using knowledge of function values on a subset of the data. An external function on the dataset, whose random and finite realizations (e.g., noisy versions of the true function values) are available to the user, is used to discover features in the data that are locally invariant as well as features that are locally insignificant, and to build a diffusion kernel on the data that diffuses quickly along these local irrelevant features. The new metric and diffusion kernel, generated via iterations of stacked neural nets, create an embedding which captures the geometry of the data while yielding a low Lipschitz constant with respect to the functions of interest. Along with this, the new kernel does not depend explicitly on the function values, which makes it trivially extendable to new points. He demonstrated the performance of their algorithms using synthetic datasets as well as the real medical dataset. The latter is quite intriguing: the dataset consists of about 80 different measures of more than 1,000 hospitals in US, e.g., the frequency of prescribing aspirin hospitalization for heart attack; time spent in the emergency department prior to discharge home; the patients' experience and comments on their doctors; patients' mortality at 30 days after hospitalization for heart failure, etc. From this dataset, he successfully characterized the overall hospital performance as well as the ranking and grouping of these hospitals.

Naoki Saito (UC Davis) discussed his work on Laplacian eigenfunctions that 'do not feel the boundary.' These are in fact the eigenfunctions of the integral operators commuting with the Laplacian on domains in $\mathbb{R}^d$ and their discretized versions. These eigenfunctions satisfy the Helmholtz equation inside the domain, and can be extended smoothly and harmonically outside of the domain. The kernel of these integral operators are the so-called free space Green's functions or the fundamental solution of the Laplacians. He discussed the similarities and differences of these eigenfunctions with the so-called Krein-von Neumann Laplacian eigenfunctions. Also, he described the intimate relationship between the discretized version of these integral operators on the lattice graphs, their distance matrices, and their graph Laplacians, and in particular, showed that the eigenvectors of the graph Laplacian of a lattice graph coincide with those of the distance matrix when it is sandwiched by the projector onto the orthogonal complement of the DC component. He concluded his talk by raising a natural question: what is the corresponding integral operator (or the kernel matrix) that commutes with the graph Laplacian of a general graph.

Faisal Beg (Simon Fraser Univ.) and his postdoc Karteek Popuri talked about their work on medical image analysis and computational anatomy using Laplacian eigenfunctions and heat kernels. They reviewed the use of the ratios of the Laplacian eigenvalues as a feature vector for classifying hippocampus shapes (obtained from MRI images) into those of Alzheimer's Disease patients and those of control patients. Moreover, the nodal lines of the eigenfunctions are also used for partitioning various anatomical manifolds. As for the use of heat kernels, they discussed cortical thickness smoothing, which is similar in spirit to the popular nonlocal means denoising algorithm. They concluded their talk by posing two open problems: 1) how to capture cortical structure beyond thickness; 2) how to capture and characterize vascular structures.

François Meyer (Univ. Colorado, Boulder) talked about his work on prediction of evolution of epilepsy from electrophysiological measurements around hippocampuses of 23 rats. His hypothesis is that the response to an auditory stimulus changes during epileptogenesis. Hence, he needed to quantify changes in the auditory response that are correlated to changes in the hippocampus associated with epileptogenesis. The challenge is to decode the progression of the disease from the auditory response. To do so, he first computed the stationary wavelet transform of the successive segments of electrophysiological responses of rats to auditory stimuli. Then, he chose a subset of those wavelet coefficients that best separate the four conditions of rats under the forced epileptogenesis experiments: baseline, silent, latent, and chronic. Then, he tracked the evolution of these coefficients in the phase space by embedding these high-dimensional coefficients in $\mathbb{R}^d$ where d is much smaller than the number of those coefficients followed by training the hidden Markov model, which in the end estimates the probability of being in one of the four states listed above. His result was quite impressive: his algorithm reliably predicted eventual spontaneous recurring seizures. In rats that did not develop spontaneous recurring seizures, his algorithm predicted a return to baseline.

Shape optimization, numerical methods

Chiu-Yen Kao (Claremont McKenna College) talked about her work on shape optimization for eigenvalue problem involving biharmonic operators. In particular, she focused on the vibration of a plate with the nonhomogeneous thickness or density under the clamped ($u = 0 = \partial_\nu u$) or the simply-supported ($u = 0 = (\partial_\nu^2 + \partial_\tau^2)u$) boundary conditions. Her optimization problem is as follows: for a given domain shape under one of the above two boundary conditions, find the thickness or density distribution of the plate material that minimizes or maximizes the lowest eigenvalue. Her numerical scheme iterates between the two stages: 1) computation of the optimal eigenpair under the fixed density and 2) computation of the optimal density under the fixed eigenfunction. This process converges to the optimal density ρ^* that gives the extremal λ_1. The same iterative scheme also works for material thickness. She demonstrated the efficiency and robustness of her numerical schemes using a variety of examples including square, disk, annulus in 2D. It is interesting to note that the optimal thickness distribution on the 2D annulus under the simply-supported boundary condition turns out to be symmetric eigenfunction u_1 if the inner radius is small, but become asymmetric if the inner radius becomes large (i.e., the annulus becomes thin).

Pedro Antunes (Univ. Lisbon, Portugal) presented his numerical scheme and his results for the shape optimization problem associated with localized Laplacian eigenfunctions. Motivated by the Shnirelman theorem on the quantum ergodicity, his aim was to provide a numerical scheme for the following shape optimization problem: given a planar domain V with $|V| < 1$, find a convex shape Ω that minimizes (or maximizes) the energy concentration $\|u_j\|^2_{L^2(V)}$ on V subject to $V \subset \Omega$ and $|\Omega| = 1$ where u_j is the jth Dirichlet-Laplacian eigenfunction on Ω, $j \in \mathbb{N}$. His method uses a clever parameterization of the boundary curve $\partial\Omega$, the so-called Hadamard shape derivatives, and the Method of Fundamental Solutions (MFS). His numerical results nicely confirmed the theory of Harrell, i.e., the minimization of the energy concentration $\|u_1\|^2_{L^2(V)}$ is achieved when V touches Ω while the maximum is achieved when V is centered in Ω and away from $\partial\Omega$. He also obtained those Ω's for different eigenfunctions u_j with $j = 23, 160, 200$, etc. for various fixed V.

Pappur N. Shivakumar (Univ. Manitoba) discussed infinite linear algebraic systems with diagonally dominant matrices and its important application, e.g., computing the Dirichlet-Laplacian eigenvalues of various planar domains. His approach first represents a domain with biaxial symmetry by an analytic curve in the complex plain as a power series of $z\overline{z}$ or $z + \overline{z}$. Such a representation converts the Dirichlet-Laplacian eigenvalue problem to an infinite system of linear equations whose coefficients are polynomials of the eigenvalues depending on the domain shape. The zeros of the determinant of this linear system determine the eigenvalues. To compute the eigenvalues numerically, he truncated the infinite system matrix and demonstrated that this strategy generated extremely good approximation to the eigenvalues for various domains. He finally illuminated the classical question "Can you hear the shape of a drum," by his constructive analytic approach: a pre-knowledge of eigenvalues yields the information about the boundary for many domains with analytic boundary curves (e.g., circle, ellipsis, annulus, etc.) while in the case of a square, this approach fails. This may indicate why the non-analytic (i.e., polygonal) boundary cases could be the counter-examples of Kac's question.

Eigenvalue inequalities

Rick Laugesen (Univ. Illinois, Urbana-Champaign) gave a talk on Steklov spectral inequalities through quasi-conformal mapping. The Steklov eigenvalue problem is an eigenvalue problem with the spectral parameter in the boundary conditions while the governing equation itself is the Laplace equation. Vibration of a free membrane whose mass is concentrated at the boundary is a typical example. Recently, there has been a growing interest in the Steklov problem from the viewpoint of spectral geometry as well as its applications including liquid sloshing. Its spectrum coincides with that of the Dirichlet-to-Neumann operator. He reviewed the basics of the Steklov eigenvalue problems including a question "can you hear the shape of a Steklov membrane?" The answer might be 'yes' and so far, no counterexamples are known. Things that can be heard include boundary length (by Weyl) and the number of boundary components (by Girouard et al. 2014). He then showed that the disk maximizes various functionals of the Steklov eigenvalues, under normalization of the perimeter and a kind of boundary moment. The results cover the first eigenvalue, spectral zeta function and trace of the heat kernel.

Bartłomiej Siudeja (Univ. Oregon) discussed his work on nearly radial Neumann modes on highly symmetric domains. Schiffer's conjecture states that if a Neumann eigenfunction is constant on the boundary of a domain, then either the eigenfunction is constant on the whole domain, or the domain is a disk. The disk is special, due to the presence of radial modes. He discussed the existence of Neumann modes on regular polygons and boxes which are nearly radial (do not change sign on the boundary). Siudeja got a very interesting result: if the domain is a regular polygon with more than 4 sides, then there exists an eigenfunction that is strictly positive on the boundary but not constant inside the domain. He then also discussed his recent work on the 3D Steklov eigenvalue problem with the Neumann boundary condition at the sides of the container. In particular, the so-called "high-spots" conjecture (the highest spot of the second eigenfunction is on the boundary of the container) was discussed. He showed two results: (i) it is possible to slosh a fluid so that the liquid surface moves in unison on the boundary of a cylindrical cup whose base is a regular polygon with more than 4 sides; (ii) in the case of the triangular base, the fluid moves up in one place on the boundary and moves down in another on the boundary.

Antoine Henrot (Univ. Lorraine) discussed how to maximize the first eigenvalue of the Laplacian with Dirichlet boundary conditions by placing some obstacle K inside a fixed domain Ω. He first reviewed some known results for that problem, mainly when the shape of the obstacle K is already given (then the question is to find its location), which was also discussed by Harrell. Here the shape of K is free but connected and with a fixed perimeter. He proved the existence of a maximizer. He also discussed more specific case where Ω is a ball, for which he proved that the obstacle should be a concentric ball.

Rafael Benguria (Pontificia Universidad Católica de Chile) discussed the Brezis–Nirenberg problem on $\mathbb{S}^n$ in spaces of fractional dimension, in particular, $2 < n < 4$. This is a nonlinear eigenvalue problem described as:

$$-\Delta_{\mathbb{S}^n} u = \lambda u + |u|^{4/(n-2)} u,$$

with $u \in H_0^1(\Omega)$, where Ω is a geodesic ball in $\mathbb{S}^n$ contained in a hemisphere. This problem has its origin in the Lane-Emden equation, a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. In 2002, Bandle and Benguria identified the range of the eigenvalues, which depends on the geodesic radius of the ball, where this problem has a unique positive solution in the case of $n = 3$. Then, he discussed his recent result for the case $2 < n < 4$ where he also identified such eigenvalue range that depends on the indices of the associated Legendre functions.

Lotfi Hermi (Univ. Arizona) discussed the isoperimetric upper bound for the fundamental tone of the membrane problem for a class of wedge-like domains. Such upper bound can be used to estimate the fundamental tone of a right-angle triangular membrane or more generally regular α-polygon with n sides. Moreover, he proposed a new lower bound for its "relative torsional rigidity" of such domains, i.e., the torque required for unit angle of twist per unit length when the shear modulus is 1 relative to the weight function (e.g., density function of samples over manifolds, etc.).

Rajesh Mahadevan (Universidad de Concepcion, Chile) discussed eigenvalue minimization for the clamped plate under compression, $(\Delta^2 + \tau\Delta)u = \lambda u$ in Ω with $u = |\nabla u| = 0$ on $\partial\Omega$. In particular, he addressed the isoperimetric problem: among domains of equal volume, for which domain the first eigenvalue of the clamped plate with compression is a minimum? In the case of no compression (i.e., $\tau = 0$), Rayleigh, in 1877, conjectured that the ball is the minimizer of this problem. The Rayleigh conjecture for the clamped plate with $\tau = 0$ has been proved by Nadirashvili for $d = 2$ (1995) and Ashbaugh and Benguria for $d = 3$ (1995). He demonstrated that for the small positive values of τ, the first eigenvalue is a minimum when the domain is a ball.

Scientific Progress Made

In joint work carried out during the workshop, Laugesen and Siudeja answered a quested of J. Cima about the location of the maximum points of ground state solutions of semilinear Poisson equations. Specifically, they found examples in which the maximum points of two different equations are in two different locations albeit remarkably close together (differing only on the scale of 10^{-4}). These researchers also developed a road map for proving a Weyl Law for the Steklov spectrum on piecewise smooth domains (work in progress, initiated during the workshop).

Beliaev, Grebenkov, and Jones have completed their joint work on multiscale representations of Gaussian processes and fields. Antunes and Grebenkov have started a new project on localization of Laplacian eigenfunctions in three-dimensional domains, which theoretical exponential estimates will be combined with the efficient numerical method of fundamental solutions.

David, Filoche, Jerison, Mayboroda, together with Doug Arnold, organized an extended focused research group workshop on localization of Laplacian eigenfunctions after our workshop.

Outcome of the Meeting

We feel this workshop was a success. We also believe that more collaborations have been established, particularly, between scientists from pure and applied fields. For instance, the workshop participants working on numerics of the eigenfunctions such as Pedro Antunes seem to be on high demand for revealing and checking mathematical hypotheses. Through the presentations and interactions during this workshop, the participants also realized that they encountered similar problems in their own research and shared their experiences and discussed potential solutions. Examples include ordering and sign ambiguities of eigenfunctions by people working in data manifolds; shape optimization associated with obstacle residing within a given domain. eigenfunction localizations and how to control them; and the other extensions to the traditional Laplacian eigenvalue problems, e.g., biharmonic eigenvalue problems, Steklov eigenvalue problems, nonlinear eigenvalue problems.

We also discussed a possibility to propose a semester-based long program at ICERM (Brown Univ.) or IPAM (UCLA), as a continuation of our program. Peter Jones, knowing both places firsthand, explained the operational mechanisms of these institutes, and discussed how to write proposals for short and long programs.

We also got many positive comments from the participants, some of which are (for all the comments, see the BIRS testimonial website):

> "Always a pleasure to be at Banff. I heard for the first time the talks by Mayboroda, Filoche, David and Jerison on eigenfunction concentration, that was quite inspiring, I will probably give a survey lecture on this a few times in the next few months. I also started a new paper with Pedro Antunes from Lisboa at the conference, not sure how long it will take." — D. Jakobsen

> "The Meeting was a wonderfully stimulating event, bringing together researchers who use a wide variety of techniques to approach problems in the same field." — R. Laugesen

> "During the workshop I have had the occasion to work with Michiel van den Berg on some questions related to the first eigenvalue of the Dirichlet-Laplacian and the torsion (in particular we are interested in getting sharp lower bounds for the product of these two natural quantities). I also discussed with Bartek Siudeja and Rick Laugesen in the one hand and Pedro Antunes on the other hand about a project of a collective book on recent results of spectral theory." — A. Henrot

> "I found the conference very stimulating. I had a chance to talk to some people who I know well, for example, Michiel van den Berg, Peter Jones, Marcel Filoche and Bartek Siudeja. I have also

renewed some friendships, for example, with Antoine Henrot, and make new friends, for example, with Dima Jakobson. It was great to see what is done in fields that are somewhat different from mine. I learned about the relationship between eigenfunctions and billiards, and about results and conjectures on the Gaussian character of the value function for high frequency eigenfunctions. Overall, the idea of bringing people in different fields to one conference was excellent." — C. Burdzy

"I found last week's workshop at BIRS very stimulating: the range of topics was quite broad, and the talks were on the whole of good quality. It also allowed me to work with some colleagues notably Antoine Henrot. However, I also had some interesting discussions with several other participants. The facilities at BIRS are first class, and the surrounding scenery spectacular. This was my second time at BIRS, the first visit was in 2004. I very much hope it won't be my last!" — M. van den Berg

"This was one of the most interesting conferences I have attended in years, and I believe it might bring some profound insights to my research. I was also able to make contacts with great people and seed new collaborations." — M. Bronstein

Participants

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Beliaev, Dmitri (University of Oxford)
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Chapter 9

Perspectives on Parabolic Points in Holomorphic Dynamics (15w5082)

March 29 - April 3, 2015

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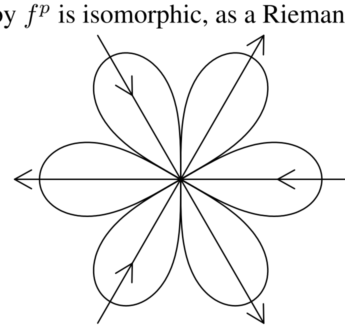
History and recent developments

Parabolic points, classification

One dimensional holomorphic dynamics is concerned with the iteration of holomorphic functions on $\mathbb{C}$ or more generally on a collection of one or more Riemann surfaces.

If a holomorphic map f has a periodic point z, i.e. $f^p(z) = z$, this point is called parabolic if the multiplier, i.e. $(f^p)'(z)$, is a root of unity. The classification of parabolic points, up to local conjugacy (by a continuous or a holomorphic map, or by formal power series) has been carried out mainly by Leau, Fatou, Camacho, Écalle, Voronin, Ramis and Martinet.

The Leau-Fatou theorem, [28, 24] describes the dynamics near a parabolic point. There is some $q > 1$ such that a neighborhood can be covered by q attracting petals and q repelling petals, alternating, on which the dynamics is conjugate to "the translation by 1 on the right half-plane $\Re z > 0$" for attracting petals, with right replaced by left for repelling petals. Such conjugacies are called Fatou coordinates. The number $q+1$ is equal to the order of tangency between identity and a sufficiently high iterates of f^p. Since the conjugacy can be taken holomorphic, this implies that the quotient of a petal by f^p is isomorphic, as a Riemann surface, to $\mathbb{C}/\mathbb{Z}$.



The classification up to homeomorphism is thus determined by the number q. The formal classification, depends on the number of petals and on a complex number called the formal invariant: $A \in \mathbb{C}$. Each germ can be conjugated by a formal power series to a unique $\rho \times z \times (1 + z^q + Az^{2q})$ with $\rho = (f^p)'(0)$. The classification up to a holomorphic change of variable involves a more complicated invariant, which has been expressed in analytic terms by Écalle (coefficients appearing in alien derivations associated to a resurgence phenomenon of the Borel resummation of asymptotic expansion of Fatou coordinates) and in terms of conformal geometry by Voronin: the change of variables between repelling and attracting Fatou coordinates defines a holomorphic function, the horn map on the upper/lower half-plane that commutes with $z+1$ (they thus in fact live in the cylinders), and provides conjugacy invariants, the so called Voronin invariants. The link between the two points of view is that Écalle's coefficients are nothing, but the Fourier coefficients of Voronins horn maps.

The approach by Écalle is rich but extremely complicated, and 30 years after it was proposed, its elucidation is still under way.

Practical ways to obtain estimates on the Fatou coordinates, the horn maps or the Écalle invariants, have been studied by many researchers, including Lanford-Yampolsky in their study of parabolic renormalisation, L.-Y. and Chritat in the production of computer pictures, and Sylvain Bouillot for Écalle's invariants.

Let us also cite extension toward higher dimension: dynamics near fixed points tangent to identity and its classification has been studied by calle, Hakim, Abate, Vivas and Rong, among others. See [2] for references and an exhaustive survey or [3], pages 38–49 for a shorter one.

Families, parabolic implosion

Consider the following heuristic statement, concerning families of one-dimensional holomorphic dynamics: "Parabolic points are the source of instability." In this introduction, we will formulate a few theorems that motivate it.

Consider a family of rational maps $R_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ depending analytically on a parameter λ that belongs to a complex manifold Λ. Let the stability set Ω be the set of λ for which R_λ does not undergo a bifurcation, i.e. is conjugate on its Julia set to all nearby maps in the family. Define the bifurcation set B as the complement of Ω. The most famous example is $B = \partial M$, the boundary of the Mandelbrot set, for the family $R_\lambda(z) = z^2 + \lambda$.

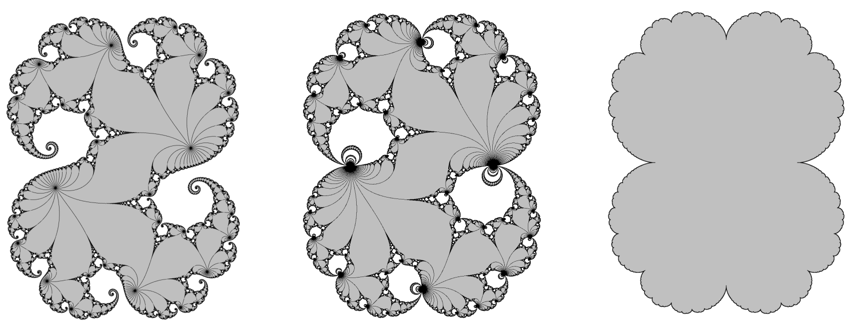
Theorem : (Ma, Sad, Sullivan [32])

1. *Ω is open and dense,*
2. *B is the closure of the set of parameters λ for which the family has a non persistent parabolic point.*

A parabolic point is called persistent if nearby maps in the family still have a nearby periodic point with the same multiplier. We refer to [32] for more precise statements. In this report, we call parabolic parameters those for which R_λ has a non-persistent parabolic point λ.

If $B \neq \emptyset$, then the set B also contains many non-parabolic parameters. However, the parabolic parameters are those for which the discontinuity of the dynamical system is the worst. First, consider the following theorem, concerning the filled-in Julia sets $K(P_\lambda)$ for holomorphic families of polynomials P_λ:

Theorem : (Douady [17]) *The map $\lambda \mapsto K(P_\lambda)$ is upper semi-continuous for the Hausdorff distance on the set of compact subsets of $\widehat{\mathbb{C}}$, discontinuous at all non-persistent parabolic parameters, and continuous at all other parameters.*



The set of pictures above illustrates the discontinuity case of the theorem: let $P_\theta(z) = e^{2\pi i\theta}z + z^2$. On each image, black denotes the Julia set and gray denotes $K - J$. On the right we have $K(P_0)$, on the left, $K(P_{1/22})$ and in the middle, $\lim K(P_{1/n})$ together with $\lim J(P_{1/n})$. Note also the upper semi-continuity, the filled Julia set $K(P_0)$ of the limit contains $\lim J(P_{1/n})$.

For rational maps, there is no filled-in Julia set. A similar statement can be formulated and the discontinuity will occur iff there is a parabolic point, Siegel disk or Herman ring, (in each case we mean non-persistent). However, the discontinuity generated by parabolic points is much richer.

The first precise study of parabolic implosion was undertaken by Douady and Lavaurs in the case of quadratic and cubic polynomials. As an illustration of these results follows below a precise statement for the family of quadratic polynomials P_λ as above. Let $\lambda_0 = 1$ (similar statements follow for λ_0 a different root of unity, but these are slightly more involved). Let $\phi : B(0) \to \mathbb{C}$ denote an attracting Fatou-coordinate for P_{λ_0} where $B(0)$ denotes the parabolic basin of 0. Similarly let $\psi : \mathbb{C} \to \mathbb{C}$ denote a repelling Fatou parameter for P_{λ_0}. Finally for $A \in \mathbb{C}$ denote by

$$g_A := \psi(A + \phi(z)) : B(0) \to \mathbb{C}$$

the family of Lavaurs maps for P_{λ_0}.

Theorem : (Lavaurs [27]) *Given any $A \in \mathbb{C}$ there exists a sequence of parameters $\lambda_n \longrightarrow \lambda_0$ and there exists a sequence of corresponding integers $N_n \to +\infty$ such that the N_n-th iterate of P_{λ_n} tends, on the parabolic basin of P_{λ_0}, to the Lavaurs map g_A.*

Note that Lavaurs gave in fact an explicit sufficient condition on the sequences λ_n and N_n, depending on A, for the above convergence to occur: write $\lambda_n = e^{2\pi i \alpha_n}$ with α_n a complex number sequence tending to 0; then the condition is that

$$\frac{\pm 1}{\alpha_n} + N_n \longrightarrow A + a$$

where $a \in \mathbb{C}$ is another constant that depends in an explicit way on the choice of Fatou coordinates.

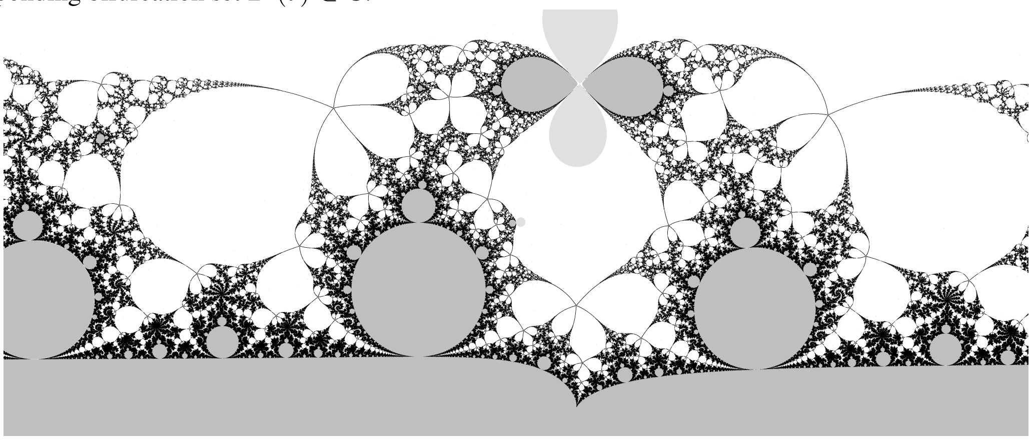
The convergence above of higher and higher iterates of nearby polynomials to Lavaurs's maps of the limiting polynomial is very similar to the phenomenon called geometric limits for Kleinian groups, see [43, 26]. For this reason Lavaurs maps are also called geometric limits. Presumably Lavaurs was aware that his results had generalizations to generic perturbations in families with a non persistent parabolic point, though he only gave a description of these limits in particular cases.

Here is a small list of striking consequences:

- (Douady [17], Lavaurs [27], late 1980's) The limit of the Julia set is strictly bigger than the Julia set of the limit. It corresponds to the Julia set of an enriched dynamical system, containing the limit map and the Lavaurs map. (This phenomenon is called parabolic implosion)
- (Lavaurs [27], late 1980's) Non-local connectivity of the cubic connectedness locus (analog of the Mandelbrot set for cubic polynomials)
- (Lavaurs [27]) Limit shape of some specific zooms on the Mandelbrot set.
- (Shishikura [39, 40], early 1990's) The Hausdorff-dimension of the boundary of the Mandelbrot set is equal to 2.
- (Douady, Hubbard, [20] mid 1980's) Discontinuity of renormalisation for polynomial-like maps.
- (Douady, Sentenac, [19] 1980's) Existence of external rays tending to the root of hyperbolic components of the Mandelbrot set.
- (Buff-Chritat, [10], late 2000's) Fine estimates on the size of Siegel disks.
- (Buff-Chritat, [9, 11], mid 2000's) Existence of Julia sets with positive Lebesgue measure.

Later work has focused on understanding more general bifurcations of parabolic points than the particular cases studied by Lavaurs and on developing the theory for generic families of holomorphic maps with a non-persistent parabolic orbit. This includes, but is not limited to work by Douady Estrada and Sentenac [18], pursued by Branner and Dias [7] (studying polynomial vector fields, a model and tool for a parabolic point), Oudkerk [34] (gate structures associated to a given perturbation), Christopher and Rousseau [38, 16] (classifying the full unfolding of the underlying bifurcation in general). Bedford-Smillie-Ueda [6] looked at parabolic implosion in a special case, semi-parabolic points, where the theory has a lot in common with the one-dimensional case, but proofs are harder: here one works in $\mathbb{C}^2$ and assumes that the eigenvalues of the fixed point are $\{1, \lambda\}$ with $|\lambda| < 1$. Few people have looked at the perturbation of maps tangent to the identity (all eigenvalues are 1).

Parabolic implosion also occurs in parameter space: one can consider the bifurcation set $B(\theta)$ for the family $P_\lambda(z) = e^{2\pi i\theta}z + \lambda z^2 + z^3$ ($\lambda \in \mathbb{C}$). These families are one dimensional slices in the 2-parameter family of all cubic polynomials. Note that if B denotes the bifurcation set for the latter, then $B(\theta)$ is contained in the θ-slice of B, but is not necessarily equal. Nonetheless, the difference can be precisely described. Similarly, one can look at quadratic rational maps with a fixed point of multiplier $e^{2\pi i\theta}$. It can also be parameterised by $\Lambda = \mathbb{C}$, and there is a corresponding bifurcation set $B'(\theta) \subset \mathbb{C}$.



In the picture above, black represents part of $B'(2/5)$. Gray indicates copies of the Mandelbrot set and light gray denotes parameters for which the parabolic point is parabolic-attracting (see [33]).

Describing $B(\theta)$ has attracted a lot of attention in recent work (Petersen-Roesch [37] for $B'(0)$, unpublished works by Roesh and Roesh-Nakane, Zakeri [49] for $B(\theta)$ where θ is a bounded type irrational). Eva Uhre [44] has studied similar slices, for quadratic rational maps. See also the work of Buff-Écalle-Epstein [12] and Petersen (double implosion, work in progress) concerning the parameters for which the number of petals doubles in $B'(1/n)$.

A striking feature is the following: If θ tends to p/q, the set $B(\theta)$ has a richer limit than just $B(p/q)$, and this limit is also described by parabolic implosion. This is the subject of study of I. Zidane (thesis, work in progress). Similar phenomena occur for $B'(\theta)$.

Parabolic renormalisation

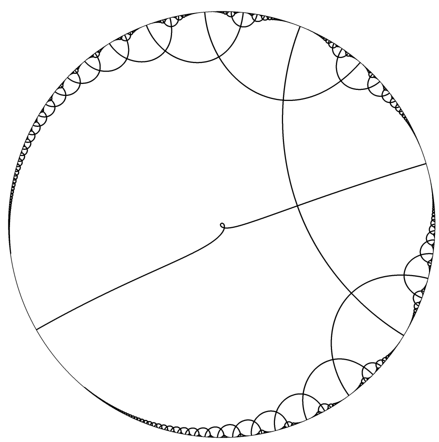
There may be further geometric limits occurring, depending on the fine behaviour of the perturbation. If this happens then the corresponding horn map has a Siegel or parabolic point. These second order geometric limits are defined on the basin of the parabolic point of the first order geometric limit, thus on a strict subset of the basin of the original parabolic point. The second order geometric limits (of parabolic type) played a crucial role in Shishikura's proof that the boundary of the Mandelbrot set has Hausdorff dimension 2, [39, 40]. There may also be third order geometric limits, and there exist even arbitrarily large possibly infinite order geometric limits, so-called towers. This led Adam Epstein [21] to develop in his thesis a formal framework and theory for formulating and proving statements not only about towers, but for general families of finite type maps and finitely generated dynamical systems from such maps. Following Epstein a finite type map is a co-compact holomorphic map with finitely many singular values (values which do not possess an evenly covered neighbourhood). This class of maps generalizes the class of rational maps. Finite type maps have rigidity properties that allow to extend many classical results. Epstein's theory has proven to be very fruitful: not only does it provide a precise frame work for discussing parabolic implosion, it also has produced strong results of its own such as the Fatou-Shishikura inequality for general finitely generated finite type dynamical systems, the proof of which specializes to a new proof of the Fatou-Shishikura inequality for rational maps, and generalizes the tools developed by Thurston in order to characterize post critically finite branched coverings. Also Epstein's theory has led to general transversality theorems.

A horn map of a finite type map has a natural maximal extension, which is again a finite type map. The extended horn maps have a dynamics that mimics (by a semi-conjugacy) the dynamics of the geometric limits. And there is a clear advantage in looking at the dynamics of the extended horn maps.

The set of all possible limits is an interesting object in itself. Epstein was developing his theory from the point of view of geometric limits, expressed in the language of sheaves. For an other perspective, Hubbard now considers the Julia sets as points in the space of compact sets and looks at the closure of the set of all Julia sets of $z^2 + c$, exploring its topological features. Bachy's thesis [5] explored one branch in this project: the Siegel disks (the other branch are the parabolic points).

Consider the operator that, to a parabolic point, associates say its (upper) horn map. We normalize Fatou coordinates so that the upper end of the fundamental cylinder is parabolic with multiplier 1 (the case of lower end is similar). Then the operator is called the (upper horn map) parabolic renormalisation operator. It can also be seen as the limit of a procedure of cylinder renormalisation (see the work of Yampolsky on circle maps [15, 16]; it is also a central heuristics in the work of Yoccoz on indifferent germs and circle maps [47], and the work of Perez-Marco on Hedgehogs, partially unpublished).

Shishikura [41] was the first to produce an invariant class SH_1 of maps for the parabolic renormalization operator for horn maps. Following this idea Petersen observed that there is a similar invariant class of "belt maps" for parabolic renormalization of critical circle maps with a parabolic fixed point. This observation was the starting point for Yampolsky's study of (parabolic) renormalization of critical circle maps [15]. Later Shishikura has defined a renormalization invariant subclass $SH_2 \subset SH1$, for which he jointly with H. Inou has developed a theory of near-parabolic renormalization. This theory in turn provided a missing link in the Douady-Cheritat strategy to prove existence of quadratic Julia sets with positive area [11]. Recently Cheritat has generalized the Inou-Shishikura theory of near-parabolic renormalization to families of maps with multiple critical point [15]. Yampolsky defined an invariant class for the (upper) horn map parabolic renormalization operator, which is slightly smaller than SH_1 and which allowed him to give an independent proof of the existence of a renormalization fixed point the parabolic renormalization operator.



The Parabolic chessboard, illustrated above, helps understanding the covering properties of maps in the invariant class. This proved useful in later works.

This striking feature hinted at contraction properties of the parabolic renormalisation, but actually showing this proved difficult. Inou and Shishikura [25] managed to do this by considering particular ramified coverings that contain only part of the structure. Their proof was computer assisted in 2002, they later gave a proof that can be checked by hand. Lanford III and Yampolsky developed a numerical scheme that enabled them to draw the domain of definition of the parabolic map f^*, the attracting fixed point of the renormalisation operator. Unfortunately Lanford-III passed away before the computer assisted proof was written up.

As a consequence of the contraction of parabolic renormalisation, and the techniques employed, it was possible to build a near-parabolic renormalisation invariant class for cylinder renormalisation [25], which proved particularly fertile in solving conjectures, especially for indifferent fixed points in the family of quadratic polynomials: control on the post-critical set (boundaries of Siegel disks, shape of hedgehogs, upper semi-continuity of closure of critical orbit is crucial in the proof of the positive measure theorem of Buff and Chritat), size of Siegel disk and the Marmi-Moussa-Yoccoz conjecture, computability questions... People busy with this harvest include Buff, Cheraghi [14], Chritat, Shishikura, Yampolsky, Zidane, with many collaborations between them.

It is to be noted that in [22], Yampolsky and Gaydashev gave a computer-assisted proof of the existence of a hyperbolic fixed point of cylinder renormalisation, with rotation number $\frac{\sqrt{5}+1}{2} = [1, 1, 1, \ldots]$ at one end of the cylinder, applicable to the family of quadratic polynomials. This particular example is outside the present reach of the techniques of Inou and Shishikura. Indeed the latter can only cover rotation numbers whose continued fraction entries are all bigger than an integer N whose value has not been determined (it should lie between 20 and 1000).

The Douady-Hubbard polynomial-like maps give rise to another notion of renormalization. Many polynomials P have iterates having polynomial-like restrictions f of degree equal or lower than the degree of P. Their Julia sets are attached to that of P at the pre-fixed points (points whose iterates are eventually fixed by f), which are either repelling or parabolic, and in the latter case all attracting petals are in the filled-in Julia set of f. Given a particular P, to prove the existence of f is usually obtained by cutting along external rays, equipotentials (see [19] for the definition of these notions) and modifying the construction near repelling fixed points. However, it has long been known that a similar construction should be possible when we do not want to keep all petals in the restriction. The problem is that f cannot be polynomial-like in this case. To handle this, Lomonaco developed in her thesis the notion of parabolic-like map and extended part of the Douady-Hubbard theory to this case, see [29, 30].

Current work and open problems

- **What are all limits?** As mentioned above, the set of all possible limits is under exploration by Epstein and Hubbard, with different approaches. To simplify, let us consider only the case of $P_c = z^2 + c$. Epstein looks at all possible limits of the set of iterates, i.e. all limits of sequences $P_{c_n}^{N_n}$, as $c_n \longrightarrow c$ and N_n is any sequence of integers. He develops a sheaf-theoretic flavored approach, that bears many significant subtleties. Hubbard wants to describe the closure, in the space of all compact subsets of the complex plane (or the Riemann sphere), endowed with the Hausdorff topology, of the set of all Julia sets $J(P_c)$, $c \in \mathbb{C}$.

  Let us mention that rescaling limits is the keyword for an even wider project: understanding all possible limits of (finite or infinite iterates)/(Julia sets), where one is allowed to rescale, i.e.: $a_n \circ P_{c_n}^{N_n} \circ a_n^{-1}$ or $a_n(J(f_n))$ where a_n is a complex-affine map of $\mathbb{C}$.

- **The general parabolic implosion.** There remains open questions in the classification of unfolding: Rousseau obtained a set of invariants that is complete in the sense that two unfoldings are equivalent iff their invariant are the same. However, the set of all values that can be taken by the invariant is not known. In other words there is a map $\mathcal{U} \to \mathcal{I}$ that is injective but its image still has to be determined.

  One also should look at the consequences of the above theory on the global dynamics of perturbation of parabolic points. In particular, it has consequences on the description of bifurcation loci in parameter spaces, near parabolic points.

- **Near parabolic renormalization.** The consequences of the Inou-Shishikura invariant class (that applies in particular to degree 2 polynomials) are still under study. In particular: for high type[1] rotation numbers θ, prove that the closure of the critical orbit of $P_\theta : z \mapsto e^{2\pi i\theta}z + z^2$ is a Jordan curve when the rotation number satisfies Herman's condition (see [48] for the definition of Hermans's condition) or a Cantor Bouquet otherwise (see for instance [1] for the definition of a Cantor Bouquet), determine its Hausdorff dimension, find a topological model for the whole Julia set, understand the ergodic properties for various measures (for instance the Lebesgue measure for those which have a positive one).

  Another promising direction is progresses on the famous MLC conjecture, that states that "the Mandelbrot set is Locally Connected". We are still very far from it but at least, the I.S.-near parabolic renormalization seems to be the right tool to attack the infinitely satellite case, at least partially. In this direction, see the work of Cheraghi and Shishikura (in preparation).

  Also, for θ having bounded type, Petersen [35] and McMullen [31] proved many properties of the Julia set (Local connectivity, density) and of the boundary of the Siegel disk (asymptotic self similarity). These proof rely heavily on the fact that there is a Quasi-conformal model for the dynamics (Ghys (Unpublished),

[1] This means that, for some integer N given by the theory, the entries of the continued fraction expansion $\theta = a_0 + 1/(a_1 + 1/\ldots)$ are required to satisfy $a_n \geq N$ for all $n \geq 1$.

Swiatek [42], Herman, (Unpublished), see also [36]). Chritat and Yang Fei are investigating the possibility of reproving some of these results without the use of a quasiconformal model. The motivation is that for the family $e^z + c$, there cannot be such a model, but experimentally there seems to be an asymptotic self similarity.

In this spirit, Chritat has recently tried to give a proof of a version of the Inou-Shishikura result, that would apply to $z^d + c$ (preprint, [15]). This work has not been peer reviewed. Hopes are that similar ideas could give something for $e^z + c$.

This is only the beginning of a broader story, as there are at least two directions for extending this approach:

- Build a useful near-parabolic renormalization operator for maps with 2 (or any $k > 1$) singular values, so that it applies to polynomials or rational maps of higher degree. The repelling direction of this operator would have $\mathbb{C}$-dimension 2 (resp. k) instead of 1 in the classical case. Part of Zidane's ongoing thesis is concerned with part of this program. See also the section below about fixed-point slices.
- Prove that the near parabolic operator (for $k = 1$ or bigger) extends to a sufficiently big class of maps to cover all rotation numbers, an not only the high type ones. This has also application to infinitely satellite renormalizable polynomials and the MLC conjecture.

- **Fractal analysis.** Maja Reisman and her collaborators are studying the box dimension of orbits in parabolic petals for the standard and also more refined gauges. This leads functional equations similar to the Fatou-Abel equation and to subtle and surprising estimates. This problem can find applications in the interpretation of pixel counting experiments.

- **Understanding resummation.** As mentioned earlier in this report, Sauzin, Lopez, Bouillot, Menous and many others are exploring the consequence of calle's work on parabolic points (and other systems), and are trying to make it more accessible. One task is to write down complete proofs (in some cases, calle only gave hints). Another is to understand what it tells us on dynamical systems. The calle coefficient depend holomorphically but not algebraically on the coefficients. One of the objective in Bouillot's work is to understand this dependence, or, loosely stated: how is it structured? In the earlier work [12], some characteristic of this dependence was used to get information on the repartition of a specific sequence of dynamically defined points in the parameter space of degree 2 rational maps.

  Conversely, what can the dynamics tell us on the calle coefficients?

- **Fixed-point slices.** Eva Uhre has been studying the slices $\mathrm{Per}_1(e^{2\pi ip/q})$ of the parameter space of quadratic rational maps. They are defined as the set of such maps that have a fixed point of multiplier $e^{2\pi ip/q}$. They form a one complex dimensional family, parameterizable by $\mathbb{C}$. Ideally, one would like to prove that the parameter picture is the mating between the Mandelbrot set minus a limb and the filled-in Julia set K of $z \mapsto e^{2\pi ip/q}z + z^2$ modified by some surgery. The part corresponding to the interior of the modified K is completely proved, see [44] which also includes some points in the boundary.

  In the case of cubic polynomials, the description of $\mathrm{Per}_1(e^{2\pi ip/q})$ (also parameterizable by $\mathbb{C}$) is different. P. Roesch has given a description (manuscript in preparation). Roesch and Nakane are currently working on the elucidation of the possible limits and accumulation sets of the Branner-Hubbard stretching rays (see [8]).

  For a parameter in $\mathrm{Per}_1(e^{2\pi ip/q})$ (in the cubic polynomials family or in the quadratic rational maps family), the parabolic point has either $2q$ or q petals because of a theorem of Fatou that states that every cycle of petals must contain a critical point in its basin. The set $D_{p/q}$ of parameters for which this number is $2q$ is finite. This set is of particular interest and has been studied by several people. For θ a bounded type irrational, the analog of this set in $\mathrm{Per}_1(e^{2\pi i\theta})$ is the set Z_θ of parameters for which both critical points are on the boundary of the Siegel disk. Zakeri [49] proved that Z_θ is a Jordan curve. The next steps would be:

  - try to extend Zakeri's theorem to θ a Brjuno number (see [48] for the definition)
  - try to figure out what could be the analog when θ is a non-Brjuno irrational: then there will be no Siegel disk at all for most values of the parameter

– understand the interactions between the sets Z_θ and $D_{p/q}$: for instance if p_n/q_n denotes the convergents of θ, does D_{p_n/q_n} tend to Z_θ for the Hausdorff topology on compact subsets of $\mathbb{C}$?
– There are naturally associated sub-harmonic functions whose laplacian are supported by $D_{p/q}$, resp. Z_θ. They may help in studying the questions above.

Several of these questions are studied in Zidane's thesis.

- **Enrichment in slices.**

  Let θ_n tends to a rational p/q in such a way that the corresponding cylinder renormalization has a rotation number that is independent of n. In other words, $p/q = [a_0; a_1, \ldots, a_k]$ and $\theta_n = [a_0; a_1, \ldots, a_k, n + \theta]$ for some $\theta \in \mathbb{R}$. Then one should be able to describe the limits of the parameter slice $\mathrm{Per}_1(e^{2\pi i\theta_n})$. Of particular interest are the limits of Z_{θ_n} or (D_{θ_n} if $\theta \in \mathbb{Q}$). For $\theta =$ the golden mean, computer experiments show an interesting phenomenon occurring. Zidane is studying this phenomenon. Buff and calle and Epstein studied in [12] the case ($\theta = 0, p/q = 0$). Petersen is reinterpreting and extending their work by considering double parabolic implosion or equivalently the unfolding of simply degenerate parabolic points (work in progress).

- **Higher dimensions.** One of the major achievements of Sullivan in dynamics is the proof, that there are no wandering Fatou components for polynomials in one dimensional complex dynamics. The question was open in more dimensions until recently: Astorg, Buff, Dujardin, Peters and Raissy proved in [4] that wandering Fatou components occur for some polynomial endomorphism of $\mathbb{C}^2$. Their example is a fibered system (skew product) $(w, z) \mapsto (P(w), P_w(z))$ where P is polynomial, with a parabolic fixed point at the origin, and P_w is a perturbation of apolynomial P_0 that also has a parabolic fixed point at the origin. The question remains open in the case of polynomial automorphisms.

  Ueda, Smillie and Bedford are studying the extended Fatou coordinates and horn maps associated to a fixed point of a polynomial automorphism of $\mathbb{C}^2$ such that one eigenvalue has modulus < 1 and the other eigenvalue is equal to 1, see [6]. The study of the dynamic of nearby automorphisms (parabolic implosion) is also under way.

  The dynamics of fixed points whose linear part is the identity is much harder in dimension at least 2. We mentioned earlier the survey [2], and a few names. Good progress has been made, concerning the existence of petals (open or sub-manifolds) in specific direction but there are still a lot of dark points, and the full local dynamics still is unknown. The dynamics of perturbations (parabolic implosion) does not seem to have been studied much.

Outcome of the meeting

The meeting presented us with a selection of talks and mini-courses about hot topics and classical works, and allowed the present people to keep up to date with recent progress and established theory. In particular, it was nice to see that the field is still active, that a lot of open questions exists and that many of them seem addressable. The audience consisted in a mix of young and more experienced researchers, from all around the world. Almost all of the 19 participants gave a lecture: in alphabetic order we had Bedford, Bouillot, Chritat, Epstein, Inou, Lomonaco, Mukherjee, Peter, Petersen, Reisman, Roesch, Rousseau, Shishikura, Uhre and Yampolsky. The list of talks and their abstract can be found elsewhere in this book. In the following we now list a few highlights among the outcomes of this meeting:

Hubbard and Epstein gave an interesting view on the ongoing endeavour consisting in trying to describe of all geometric limits of degree 2 polynomials (or degree d polynomials or rational maps).

Understanding calle's invariants are not less of a long task, and Bouillot's talk gave us a interesting glimpse involving beautiful, yet dense, formulas with multizetas and other fascinating quantities.

Inou and Mukherjee both studied non-local connectivity features of the tricorn (bifurcation/connectedness locus of the family $z \mapsto \overline{z}^2 + c$, generalization of which leads to an interesting open question: if two polynomials have parabolic points whose horn maps are equal, then what can one say about the relationship of the polynomials (for instance: are they necessarily semi-conjugates of a common polynomial?)

Yampolsky made a point that for renormalization problems there are often many different invariant subspaces to start from. He introduced two new such invariant space for circle maps. He particularly advocated the use of a specific invariant space for renormalization of commuting pairs. This space consists of holomorphic pairs, which commute up to third order Taylor coefficients. He showed how starting from this space one can obtain new and old renormalization results with much less effort than starting from the other invariant spaces which have been used over time.

A collaboration has begun between A. Chritat and C. Rousseau about generic one-dimensional slices of unfoldings.

Participants

Arfeux, Matthieu (SUNY StonyBrook)
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Dudko, Artem (Stony Brook University, IMS)
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Inou, Hiroyuki (Kyoto University)
Lomonaco, Luna (University of Sao Paulo)
Morris, David (University of Warwick)
Mukherjee, Sabyasachi (Jacobs University Bremen)
Peters, Han (University of Amsterdam)
Petersen, Carsten Lunde (Roskilde University)
Resman, Maja (University of Zagreb, Faculty of electrical engineering and computing)
Roesch, Pascale (Institut of Mathematics of Marseille)
Rousseau, Christiane (Université de Montral)
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Chapter 10

Geometric Flows: Recent Developments and Applications (15w5148)

April 12 - 17, 2015

Organizer(s): Gerhard Huisken (Universität Tübingen and MFO), Jeff Streets (University of California, Irvine), Peter Topping (University of Warwick), Toby Wiseman (Imperial College London), Eric Woolgar (University of Alberta)

Overview of the Field

Geometric analysis emerged as a field of its own about 40 years ago, having formed at the nexus of PDEs, Riemannian geometry, and related fields such as Kähler geometry, general relativity, and applied mathematics. Few fields of mathematics can claim to have had such success over that period of time. Much of this success has arisen out of the study of geometric flow equations. The study of geometric flows has led to proof of the Poincaré conjecture [12, 13, 10, 11], the Thurston geometrization conjecture [12, 13, 10], the Riemannian Penrose conjecture [8, 9], the differentiable sphere theorem [1, 1], and the n-dimensional Rauch-Hamilton spherical space forms conjecture.

Geometric analysis has also benefited greatly from its interaction with physics. The positive energy conjecture of general relativity motivated the work of Schoen and Yau, who proved the conjecture [15, 16] as part of a much wider programme of using analytical techniques to study the geometry of manifolds. But this example shows that physics has also gained much from geometric analysis. Another example is provided by Yau's proof of the Calabi conjecture [18, 19], which made it possible to use Calabi-Yau manifolds to construct models of string theory phenomenology [3]. A third example is Geroch's fundamental observation that inverse mean curvature flow has a monotonic quantity [5], which could be used as the basis of a proof of a conjecture of Penrose related to cosmic censorship. The eventual proof was given by Huisken and Ilmanen 25 years later [8, 9], after much progress in mean curvature flow (which, in turn, is important in engineering and materials science).

It is therefore natural for physicists, engineers, and applied mathematicians to search for ways to exploit the mathematical progress made in geometric flow equations, and natural as well for mathematicians to use these fields, and physics in particular, as a source for new geometric flow problems. There have been a number of meetings in recent years, typically consisting mostly of geometric analysts together with a smaller number of physicists and applied mathematicians, who gather together to discuss progress in the field and possible new applications. The last such meeting was a workshop held at BIRS in 2011. There has been tremendous progress since then, and it had become time to meet again.

Recent Developments and Open Problems

In the last few years, new themes have emerged in the subject. Traditionally, Ricci flow has been applied to Riemannian metrics on closed manifolds. The non-compact case has been less well studied and the case of manifolds-with-boundaries has been studied least of all. Pulemotov has, however, advanced the latter subject [14] and recent work of Gianniotis has produced the current state of the art for Ricci flow with boundary [6, 7]. Meanwhile, physics has motivated the study of asymptotically hyperbolic Ricci flow, and also numerical Ricci flow. Wiseman, Figueras, and their collaborators have used a numerical Ricci flow with boundary to produce compelling numerical evidence for certain black holes (for an overview see [17]), including one which resolves an important question in the Randall-Sundrum "braneworld" scenario and another which may shed light on conformal field theory in Schwarschild spacetime.

There has been progress on questions arising from Hamilton's original programme to prove the Poincaré conjecture and Perelman's remarkable completion of it. Two such questions are whether there exists a (weak) Ricci flow that can be continued through singularities and whether scalar curvature always blows up when a singularity forms. Among the highlights of this workshop were the new results of Haslhofer and the remarkable work of Lott (joint with Kleiner) on the former question, and the new developments presented by Bamler on the latter question.

The Kähler-Ricci flow has also been a focus of recent developments. The objective here has been to obtain a classification of low-dimensional compact Kähler manifolds. Several speakers at the workshop presented recent developments. Most notably, Prof Gang Tian of Princeton University gave an excellent overview of progress within the last year.

Presentation Highlights

The organizing committee decided to emphasize early-career researchers. We decreased the length of most talks to 30 to 40 minutes, allowing more of the younger people to speak while still preserving a large portion of free time between lunch and afternoon coffee to encourage collaboration and informal discussions. The strategy was very successful, both in providing opportunities for younger researchers to present work and in encouraging collaboration. After lunch most days, the small break-out rooms in the basement of BIRS were full with small groups discussing work. And the quality of the work presented by so many relatively young researchers in the field was remarkable, as can be perceived from the contents of the next section.

Sigurd Angenent: Rigorous Asymptotics for Ancient Solutions of Curve Shortening and Mean Curvature Flow

Angenant discussed the construction and asymptotic description of ancient solutions in two settings: first ancient solutions with finite total curvature to curve shortening in the plane, then, a proof of the precise asymptotics of the White-Haslhofer-Kleiner ancient convex solution to mean curvature flow. He also commented on the possible existence of other compact ancient solutions to mean curvature flow.

Eric Bahuaud: The Ricci flow of asymptotically hyperbolic metrics

Bahuaud discussed two projects related to behavior of the normalized Ricci flow evolving from a conformally compact asymptotically hyperbolic metric. In the first part, he discussed his joint work with Mazzeo and Woolgar on the behavior of the renormalized volume along the flow of asymptotically Poincaré-Einstein metrics (APEs). In the second part, he discussed his joint work with Woolgar on the long-time existence of the flow for rotationally symmetric asymptotically hyperbolic initial data.

Richard Bamler: On the scalar curvature blow up conjecture in Ricci flow

It is a basic fact that the Riemannian curvature becomes unbounded at every finite-time singularity of the Ricci flow. Sesum showed that, more precisely, even the Ricci curvature becomes unbounded at every such singularity. Whether the same can be said about the scalar curvature has since remained a conjecture, which has resisted several attempts of resolution.

Bamler presented a new result that partially confirms this conjecture in dimension 4 and motivates some interesting questions in 4 dimensional Ricci flow. Its proof relies on a combination of multi-scale arguments and Perelman's Harnack inequality on the conjugate heat equation. As a byproduct, he obtains an unconventional backwards pseudolocality theorem, which holds in any dimension. This project is joint work with Qi Zhang.

Paul Bryan: A viscosity equation and applications of the maximum principle for the isoperimetric profile

The distance function on a Riemannian manifold holds much (all?) geometric information. As such, studying how it evolves under geometric flows can prove a very useful, and powerful technique. The chord-arc profile of a curve in the plane is defined to be the least planar distance between two points a given length apart along the curve. Such a geometric functional, as an extremum, is amenable to study via variational techniques, leading to a weak differential inequality. The profile is strongly related to curvature and so may be used to control the curvature of a family of curves of evolving by a geometric evolution equation. A comparison result yields an explicit curvature bound for curves evolving by curve shortening, which may be applied to deduce long time convergence results in a very direct manner.

Mauro Carfora: Heat kernel embedding, dilatonic sigma models and Ricci flow extensions

By generalizing the heat kernel injection of a Riemannian manifold into Wasserstein spaces of probability measures introduced by N Gigli and C Mantegazza, Carfora presented a non-perturbative (toy) model for the renormalization group flow for the dilatonic non-linear sigma model. The beta functions of this flow characterize a non-perturbative extension of the Hamilton-Perelman version of the Ricci flow.

Jingyi Chen: Compact branched shrinkers to Lagrangian MCF in the complex plane: Rigidity, compactness, F-stability

Chen discussed properties of the space of compact self-shrinking solutions to Lagrangian mean curvature flow in the complex plane. He is able to show that there is no compact branched Lagrangian shrinker of genus zero, and that the space of compact Lagrangian immersed shrinkers can be compactified by the branched ones, under the assumption that the areas and the conformal structures are bounded. He also explained how to prove that compact branched Lagrangian shrinkers are all F-unstable.

Alix Deruelle: Conical expanding gradient Ricci solitons

Deruelle discussed various questions about Ricci gradient expanders coming out of smooth metric cones, focusing on the positively curved case and the asymptotically Ricci flat case.

Pau Figueras: Numerical Ricci flows and black holes

Figueras described a novel use of Ricci flows that has attracted a lot of interest in recent years in the theoretical physics community. Black hole spacetimes are (Lorentzian) Einstein manifolds that play a central role in our understanding of general relativity, Einsteins theory of gravity. In his talk, Figueras explained how one can use Ricci flows to find, numerically, equilibrium black hole spacetimes. He provided some simple examples, emphasising their physical relevance. Note that the flows that often arise in black hole physics are Ricci flows on Lorentzian non-compact manifolds, with various asymptotic boundary conditions, including (but not restricted to) asymptotic flatness.

Panagiotis Gianniotis: The Ricci flow on manifolds with boundary

The behaviour of the Ricci flow on manifolds with boundary seems to be a hard problem and little is known. Gianniotis described progress towards developing a theory for the Ricci flow on such spaces. In particular, he addressed the issues of local existence and uniqueness, Shi-type a priori estimates, break down criteria, and compactness of flows.

Christine Guenther: Second order renormalization group flow

The Ricci flow arises in physics as the first order approximation of the renormalization group flow for the nonlinear sigma model of quantum field theory. The *second* order approximation is given by the system $\frac{\partial}{\partial t}g = -2\mathrm{Rc} - \frac{\alpha}{2}\mathrm{Rm}^2$, and can be considered as a natural nonlinear perturbation of the Ricci flow (here g is a Riemannian metric, Rc is Ricci curvature, $\alpha > 0$ is a small parameter and $\mathrm{Rm}^2_{ij} := R_{iklm}R_j^{klm}$). In this talk, Guenther surveyed what is known mathematically about this system, including a recent proof of short term existence in n-dimensions from joint work she did with James Isenberg and Karsten Gimre. She concluded with a list of open problems.

Robert Haslhofer: Weak solutions for the Ricci flow

Haslhofer characterized solutions of the Ricci flow in terms of various infinite dimensional estimates. Namely, given an evolving family of Riemannian manifolds, he considers the path space of its space-time. His first characterization says that the family evolves by Ricci flow if and only if a certain infinite dimensional gradient estimate holds for all L^2 functions on its path space. He proved further characterizations in terms of the regularity of martingales, a log-Sobolev inequality, and a spectral gap. Based on these characterizations he can define a notion of weak solutions for the Ricci flow. This was joint work with Aaron Naber.

James Isenberg: Asymptotic Behavior of Non-Round Neckpinches in Ricci Flow

Neckpinch singularities are a prevalent feature of Ricci flow, and recent work has given us a good picture of their asymptotic behavior, so long as the geometries are rotationally symmetric. Isenberg discussed this asymptotic behavior, both for degenerate and non-degenerate neckpinches. It has been conjectured that neckpinch singularities which develop in non-rotationally symmetric Ricci flows do asymptotically approach roundness, and consequently have very similar asymptotic behavior to those which are rotationally symmetric. He discussed recent work which supports this conjecture.

Brett Kotschwar: An energy approach to uniqueness for geometric flows

Kotschwar described a simple alternative method by which the uniqueness of solutions to a variety of curvature flows of all orders, including the Ricci flow, the cross-curvature flow, and the L^2-curvature flow, can be established without recourse to DeTurck's trick.

John Lott: Ricci flow through singularities

Perelman's Ricci flow-with-surgery involves a surgery parameter δ, which describes the scale at which surgery is performed. Lott is able to show that there is a subsequential limit as δ goes to zero, thereby partially answering a question of Perelman. The limiting object is called a singular Ricci flow. Such objects can considered to be flows through singularities, and studied in their own right. Lott proved some geometric and analytical properties of such singular Ricci flows. This was joint work with Bruce Kleiner.

Warner Miller: Discrete Hamiltons Ricci Flow in Higher Dimensions

Recently Miller and coworkers defined a discrete form of Hamiltons Ricci flow equations for an n-dimensional piecewise flat simplicial geometry S, where $n \geq 2$. These algebraic equations are derived using a discrete formulation of Einsteins theory of general relativity known as Regge calculus, or equivalently discrete exterior calculus. An algebraic Regge-Ricci flow equation is naturally associated with an edge l in S and is construction using the circumcentric dual lattice S^*. The inherent orthogonality between elements of S and their duals in S^* provide a simple geometric representation of Hamilton's Ricci flow equations. In this talk, Miller outlined the construction of these equations in 3-dimensions and discussed their solutions for a few illustrative examples including neck-pinch singularities.

Andrea Mondino: Properties of non-smooth spaces with Ricci curvature lower bounds

The idea of compactifying the space of Riemannian manifolds satisfying Ricci curvature lower bounds goes back to Gromov in the 1980s and was pushed by Cheeger and Colding in the 1990s who investigated the structure of the spaces arising as Gromov-Hausdorff limits of smooth Riemannian manifolds satisfying Ricci curvature lower bounds. A completely new approach via optimal transportation was proposed by Lott-Villani and Sturm almost ten years ago; with this approach one can a give a precise meaning of what it means for a non-smooth space to have Ricci curvature bounded from below by a constant. This approach has been refined in the last years by a number of authors (see the fundamental work of Ambrosio-Gigli-Savaré, among others) and a number of fundamental tools have now been established (for instance the Bochner inequality, the splitting theorem, etc), permitting further insight into the theory. In his seminar, Mondino gave an overview of the topic.

Reto Mueller: Harmonic Ricci Flow on Surfaces

The Harmonic Ricci Flow is a coupling of the Harmonic Map Flow and the Ricci Flow (on the domain manifold of the map). While it is known that for this flow the energy density cannot blow up without the curvature of the domain manifold also becoming unbounded, Mueller showed that if the domain manifold is of dimension two then also the converse result holds. From this, he can immediately obtain smooth long-time existence for the flow for large enough coupling constants. This was joint work of Mueller with Melanie Rupflin.

Eleonora di Nezza: Smoothing properties of the Kähler-Ricci flow

In connection with the "analytic analogue" of the Minimal Model Program, it is important to analyse the long-term behaviour of the Kaehler-Ricci flow. This motivated attempts to run the flow on a compact compact Kähler manifold X from degenerate initial data. Di Nezza showed that the Kähler-Ricci flow can be run from any arbitrary positive closed current, and that it is immediately smooth in a Zariski open subset of X. This was joint work with Chinh Lu, Chalmers University of Technology.

Huy Nguyen: Mean Curvature Flow of Surfaces of Codimension Two

Nguyen described joint work with Charles Baker, considering surfaces of co-dimension two in Euclidean space moving by the mean curvature flow. He showed that if the initial surface lies satisfies a nonlinear curvature condition depending on the normal curvature tensor then the mean curvature flow deforms the surface to a round point.

Artem Pulemotov: The prescribed Ricci curvature problem on homogeneous spaces.

Pulemotov discussed the problem of finding a Riemannian metric whose Ricci curvature coincides with a given invariant $(0, 2)$-tensor on a homogeneous space.

Frédéric Rochon: Uniform construction of the heat kernel under a neck pinching

Rochon presented a uniform construction of the heat kernel under a neck pinching leading to the formation of cusps, for instance when a geodesic is pinched on a hyperbolic surface. This allows one to study the limit of spectral invariants like the determinant of the Laplacian or analytic torsion under such a neck pinching. In particular, this leads to a Cheeger-Müller theorem on manifolds with cusps. This was joint work with Pierre Albin and David Sher.

Felix Schulze: A local regularity theorem for mean curvature flow with triple edges

Schulze considers the evolution by mean curvature flow of surface clusters, where along triple edges three surfaces are allowed to meet under an equal angle condition. He shows that any such smooth flow, which is weakly close to the static flow consisting of three half-planes meeting along the common boundary, is smoothly close with estimates. Furthermore, he shows how this can be used to prove a smooth short-time existence result. This was joint work with B White.

Benjamin Sharp: Compactness theorems for minimal surfaces with bounded index

Sharp presented a new compactness theorem for minimal hypersurfaces embedded in a closed Riemannian manifold N^{n+1} with $n < 7$. When $n = 2$ and N has positive Ricci curvature, Choi and Schoen proved that a sequence of minimal hypersurfaces with bounded genus converges smoothly and graphically to some minimal limit. A corollary of Sharp's main theorem recovers the result of Choi-Schoen and extends this appropriately for all $n < 7$.

Miles Simon: Some integral curvature estimates for four dimensional Ricci flows and consequences thereof

Simon presented some integral curvature estimates for four dimensional solutions to Ricci flow. These estimates hold for any compact, connected, smooth, four dimensional solution. In the special case that the existence time $T > 0$ is finite and the scalar curvature is uniformly bounded on $[0, T)$, he presented various consequences thereof.

Gang Tian: Some progress on Káhler-Ricci flow

Tian started with a brief tour onf the Analytic Minimal Model Programme through Ricci flow. Then he discussed several new results on Káhler-Ricci flow.

Bing Wang: Regularity scales and convergence of the Calabi flow

Wang defined regularity scales to study the behavior of the Calabi flow. Based on estimates of the regularity scales, we obtain convergence theorems of the Calabi flow on extremal Kähler surfaces, under the assumption of global existence of the Calabi flow solutions. His results partially confirm Donaldson's conjectural picture for the Calabi flow in complex dimension 2. Similar results hold in high dimension with an extra assumption that the scalar curvature is uniformly bounded. This was joint work with HZ Li and K Zheng.

Claude Warnick: Stability problems in Anti-de Sitter

The anti-de Sitter spacetime is the simply connected spacetime of constant sectional curvature -1. Einsteins equations in this spacetime have the character of an initial-boundary value problem, with boundary data specified on the timelike conformal infinity. Warnick discussed the possible boundary conditions, and presented some recent results with Holzegel, Luk, and Smulevici which indicate that for a particular, dissipative, choice of boundary condition the spacetime is stable against small perturbations to the initial data.

Burkhard Wilking: The $\hat{A}$ genus of almost non-negatively curved manifolds vanishes

Wilking investigated the sequences of spin manifolds with lower sectional curvature bound upper diameter bound and the property that the Dirac operator is not invertible. By comparing averaged curvature with averaged holonomy he is able to establish the topological conclusion indicated in the title.

Scientific Progress Made and Outcome of the Meeting

It is of course not possible to judge the outcome of such a meeting until long after it finishes. Nearly all of the work that was presented was very recent and it will take time for it to be disseminated and its effects to be felt. Amongst the recent scientific breakthroughs presented at the meeting, we have already mentioned the work of Haslhofer and of Kleiner and Lott on the flow through singularities, the work of Bamler on scalar curvature blow-up at singular times, and the recent progress in Káhler-Ricci flow and the minimal model programme described by Tian.

Many participants took away new ideas arising from interaction with others who brought their own perspectives. The Regge-calculus inspired discrete Ricci flow of Miller may help to catapult work on numerical geometric flows. The work of Schulze on mean curvature flow of networks of surfaces may have applications in cosmology, as observed by Wiseman. The new Ricci flow estimates of Simon may simplify proofs of long-time exitence and convergence of Ricci flow in certain cases.

A number of new collaborations appear to have arisen from the meeting, and the work of other colaborations was advanced. The physical structure of the small meeting rooms in the basement of the lecture facility helped quite a bit with this, as did the organizational decision to have a long lunch break during which people could meet in small groups. This will likely be an important part of the legacy of the meeting.

Participants

Angenent, Sigurd (University of Wisconsin
Bahuaud, Eric (Seattle University)
Bamler, Richard (UC Berkeley)
Bryan, Paul (U.C. San Diego)
Carfora, Mauro (University of Pavia)
Chau, Albert (University of British Columbia)
Chen, Jingyi (University of British Columbia)
Conboye, Rory (Florida Atlantic University)
Daskalopoulos, Panagiota (Columbia University)
Deruelle, Alix (University of Warwick)
Di Nezza, Eleonora (Imperial College)
Figueras, Pau (University of Cambridge)
Gianniotis, Panagiotis (University College London)
Guenther, Christine (Pacific University)
Guo, Siao-Hao (Rutgers University)
Haslhofer, Robert (Courant Institute)
Hershkovits, Or (Courant Institute)
Huisken, Gerhard (Universitaet Tuebingen)
Huxol, Tobias (University of Warwick)
Isenberg, Jim (University of Oregon)
Kotschwar, Brett (Arizona State University)
Lott, John (UC Berkeley)
Miller, Warner (Florida Atlantic University)
Mondino, Andrea (ETH)
Mueller, Reto (Queen Mary University of London)
Nguyen, Huy (University of Queensland)
Parkins, Scott (University of Wollongong)
Pulemotov, Artem (The University of Queensland)
Rochon, Frdric (Universit du Qubec Montral)
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Chapter 11

New Perspectives for Relational Learning (15w5080)

April 19 - 24, 2015

Organizer(s): Daniel Lowd (University of Oregon Eugene), Sriraam Natarajan (Indiana University Bloomington), David Poole (University of British Columbia), Oliver Schulte (Simon Fraser University)

Overview

The workshop sessions featured a number of technical talks. Day 1 talks described the state-of-the-art "what we can do", and Day 2 talks open problems "what we can't yet do". The technical talks in Day 4 concerned mainly relational inference problems. A popular session featured demos and success stories. In addition to technical content, a number of panels and small group discussions focused on strategic initiatives for increasing the impact of relational learning techniques. The next sections summarize the presentations and discussion. For further details, videos and abstracts are available from the BIRS workshop website http://www.birs.ca/events/2015/5-day-workshops/15w5080.

Presentation Highlights

There were many excellent presentations. In addition to the core topics of inference and learning, presentations addressed newer topics such as planning, robotics, outlier detection, and plan recognition. It was noted that modelling with continuous variables is essential in some of these domains, like planning and robotics. More research into continuous variable modelling, and support from relational learning systems would be valuable. An emerging topic of interest for several groups is latent variable modelling and community discovery. Participants noted that the high-level representation strengths of relational models has the potential to support an interface layer for latent variable modeling. Applications include clustering, dimensionality reduction, and imputation. While it was not addressed at the meeting, participants suggested relational causality as an important topic for future research.

We had popular special sessions on success stories and demos. Success stories were presented for problems from areas such as vision (e.g. collective activity recognition), sports, and health. The systems demonstrated include the following.

Problog Probabilistic Logic Programming https://dtai.cs.kuleuven.be/problog/.

PSL Probabilistic Soft Logic http://psl.umiacs.umd.edu/.

Primula Inference and Parameter Learning in Relational Bayesian networks http://people.cs.aau.dk/ jaeger/Primula/.

Event Registry Real-time event extraction from news items http://www.eventregistry.org/.

BayesBase Structure Learning http://www.cs.sfu.ca/ oschulte/BayesBase/BayesBase.html.

We discussed options for possible publication from the workshop. Peter Flach suggested organizing a special issue of the Machine Learning journal based on the workshop. It was noted that the StarAI workshop is planning a special issue with the Artificial Intelligence journal, and coordinating the special issues would be desirable.

Strategic Initiatives

Participants agreed that two new initiatives should have top priority.

Relational Learning Website A website that provides a single point of entry to the field was deemed a strategic necessity.

Competitions Relational learning competitions would bring a shared focus for different research groups within relational learning, and showcase the strenghts of relational learning to outside researchs.

Website

Guy van de Brook secured relationallearning.org as a domain name, and agreed to champion the development of the website together with Wannes Meert. The website could be modelled on deeplearning.net. It should contain pointers to the following.

- Tutorials.
- Demos.
- Software.
- Conferences.
- Datasets. With links to tools, papers, state-of-the-art results for each dataset.
- News.

A Wiki format would allow members of the community to help develop content rapidly.

Competitions Group discussions led to a number of proposals for competition topics. Topics with champions and much support from participants were the following.

Knowledge Graph Construction Champions: Stephen Bach, Jay Pujara, Achim Rettinger.

- Connects with Knowledge Graph Community.
- Could be part of the AKBC workshop (Automatic Knowledge Base Construction).

Relational Planning Champion: Scott Sanner.

- Learn the dynamics of planning domains, including hidden variables.
- Build on the International Probabilistic Planning Competion (IPPC).

Collective Activity Recognition in Vision Champion: Ben London.

Sports Analytics Champions: Jesse Davis, Oliver Schulte.

Other suggestions for competition areas included health, robotics, and recommendation with multi-relational social networks. Participants discussed general criteria for a relational-friendly competition and benchmark problems, including the following.

- Clear evaluation metric.
- Interest to outsiders, target challenges as service to other communities.
- Varying difficulty levels, dataset sizes.

- Multi-relational data, with a complex schema.
- Propositional i.i.d. baselines leave room for improvement.
- Showcase generality:
  - Variety of possible questions, different types of outputs.
  - Structure of outputs not specified in advance. Question answering over a range of queries.
  - Short time to develop solutions, "lightning Kaggle".

Other Initiatives

A number of other initiatives were considered worthwhile by the participants, including the following.

Building Bridges Connecting with other communities.

- Given tutorials for application areas.
- Invite outside speakers to relational learning meetings.
- Organize workshops at conferences. (Not everyone agreed this was worthwhile.)

Name Change There was some discussion of adopting a new name for the field, instead of statistical-relational learning or even relational learning. A popular candidate was high-level learning. However, participants generally felt that the field is not ready for a name change.

Nature and Scope of Relational Learning

A number of discussions concerned defining the scope of statistical-relational learning. Where does the field fit within the landscape of machine learning in general? What distinguishes the field from related communities that analyze relational data, such as network analysis, inductive logic programming, and multi-relational data mining? The viewpoints expressed include the following.

Problems Delineate the field by the type of problems for which statistical-relational learning methods provide the best solutions.

Data Type Relational learning is the part of machine learning that analyzes relational data. (Or more generally, structured data).

Representation and the Interface Layer Statistical-relational learning provides complex, expressive, structured models compared to traditional i.i.d. machine learning.

A number of strengths of relational representations were noted, including the following.

- The model syntax fits the data, rather than requiring preprocessing the data into a different format.
- Compatibility with knowledge representation formalism used in Artificial Intelligence and Logic Programming.
- Declarative Representation, which has the following advantages.
  1. An interface layer between problem representation and problem solving computations [2].
  2. Rapid Prototyping and Iteration.
  3. Fewer Errors.
  4. Model Sharing.
  5. Less Feature Engineering.

Motivated by these attractive features, statistical-relational learning has developed a plethora of model representation formalisms such as MLNs, PBNs, PRMs, BLPs, PSL, ProbLog, etc. Luc deRaedt has dubbed the collection of acronyms the "alphabet soup" of relational learning [1]. On the face of it, a unifying common model formalism would offer at least two advantages. (1) It facilitates the development of an advanced tool kit for learning and inference. (2) It would make relational learning more accessible for outsiders. Participants discussed a number of options for digesting the alphabet soup.

Find Common Core Instead of seeking a single consensus formalism, it may be possible to find common themes shared by different formalisms. Participants identified a number of commonalities.

- A combination of graphical models with logical syntax.
- A relational structured data model.
- A template model approach that provides an interface layer.
- A log-linear probabilistic model based on par-factors (template factors) [3].
- Inference as weighted model counting.

Algorithmic translations between different formalisms can help us understand their relationships theoretically [1]. A practical advantage of such translations is that allow transfer of techniques developed for one formalism to another.

Ontology of Formalisms To understand the difference between different representations, an ontology could be developed. One vision is a decision tree for classifying formalisms. Representations can be evaluated along the following important dimensions.

- Expressivity.
- Ease of Use.
- Tractability of Inference.
- Learnability.

There has been previous work on comparing relational formalisms; some results are available at http://people.cs.aau.dk/ jaeger/plsystems/.

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Chapter 12

Stochasticity and Organization of Tropical Convection (15w5023)

April 16 - May 1, 2015

Organizer(s): Boualem Khouider (University of Victoria), Andrew J. Majda (Courant Institute), Chidng Zhang (University of Miami)

Overview

The Madden-Julian Oscillation (MJO) is the most dominant subseasonal variability in the tropics. Understanding its physics and correctly represent it in numerical weather prediction and climate models have been a challenging problem since MJO was first documented in 1972. There are two general approaches to tackle this problem in numerical models: (a) continue improve and develop cumulus parameterization schemes; (b) by-pass cumulus parameterization for deep convection by using cloud-permitting models. Cloud-permitting models are promising but computational demanding. Without a revolution in the computer industry that would yield new computing capability not available today, cloud-permitting models will not be feasible for operations of weather forecast and climate projection, and cumulus parameterization will have to be used in weather and climate models.

Development and improvement of cumulus parameterization has been a diligent but painstaking process. Exciting and promising recent advancement in this field includes incorporating the stochastic nature of convection in cumulus parameterization and theoretical models. At the same time, it is know that tropical convection can be organized into repeatable structures with trackable dynamics in multiscale characteristics. Mesoscale convective organization has been a known key element for convective-largescale interaction and must be represented in cumulus parameterization schemes. Synoptic and intraseasonal convective organizations are main targets of tropical prediction.

Meanwhile, the study of the tropical atmosphere has progressed into a new era in which mutiscale interaction and nonlinear process can be explicitly investigated within newly developed theoretical and numerical models. Multiscale and nonlinear models have been furthered to include the stochastic behavior of tropical convection. Two outstanding examples are the stochastic multi-cloud model and stochastic skeleton MJO model.

This workshop was motivated by the need to better measure and understand stochasticity and organization of tropical convection and to better represent them in numerical and theoretical models.

Presentations and scientific progress highlights

Thirty-two presentations were given (see Appendix A for the agenda). They discussed stochastic and organized convection associated with the MJO from observational, theoretical, and modeling perspectives.

Organized Convection

Slantwise overturning circulation in MSCs and simultaneous eastward and westward propagation of tropical multiscale disturbances. Coherent structures are a challenging problem for climate models. Parametrization are to blame. CRM simulations are important tools to understand dynamics and momentum transport effect due to mesoscale systems and their influence on synoptic and planetary scale organized convection such the MJO that are at the intersection between weather and climate. New perspective for the parametrization of mesoscale systems are emerging. Multicloud models with two baroclinic modes capable to represent overturning circulation and downdrafts associated with stratiform clouds. (Moncrieff)

Stepwise evolution of the tops of convection and a moisture layer, troposphere-stratosphere interaction; diurnal air-sea interaction during MJO suppressed phases. Stratospheric gravity waves are believed to play a central in creating pockets of humid air in the upper troposphere that further destabilize the troposphere through radiation feedback and help trigger new convection. (Johnson)

Shallow to deep convective transition is more related to large-scale (200 km x 200 km) than small-scale (20 km x 20 km) moisture increase at 4 km, which is due mostly to increasing vertical moist advection and decreasing meridional dry advection. (Hagos)

Moistening by shallow cumulus is much larger than clear-air turbulent flux. According to various studies, the mesoscale moisture tendency associated with shallow convection can be interpreted as the resultant of the individual action of shallow clouds. Although on the same order of magnitude, large-scale advection can be stronger than moisture tendencies associated with shallow convection. On the other hand the later are less variable so that shallow convection appears as an important steady background moistening. (Bellenger)

Conventional definition of convective organization: size (< 100 km), lifetime (> 6 hrs), shape (linear), dynamics (updraft); organization mechanisms: warm-moist PBL, wind shear, tropospheric moisture, cold pools, gravity waves, synoptic-meso-scale boundaries, self-aggregation; upscale impact: Q1, Q2, Q3. (Schumacher).

Stochastic convection There is an anology between tropical cycle dynamics and the MJO. In both cases, stochastic processes on different scales lead to different predictability limit. This stochasticity imposes a natural barrier for grid resolution in cloud resolving modeling (CRM) ; cold pool recovery time depends on surface fluxes, which in turn depends on surface wind speed influenced by downward convective momentum transport. (S. Chen)

Among the stochastic aspects of tropical variability, the variance of higher-frequency perturbations within MJO convective envelopes is found to vary substantially from events to events. The MJO increases the overall convective activity at all scales within its envelope, but it does not change the overall characteristics of any one scale. For example the distribution of sizes, lifetime, propagation, and cloud top characteristics of mesoscale systems remain similar within and outside the MJO envelope. Also, while the frequency of equatorial waves coupled to convection increases within the MJO, the distribution (probability density function) of these waves is not appreciably altered, with the possible exception of more equatorial Rossby wave (ER) activity. On the other hand, mixed Rossby-gravity (MRG) and eastward inertio-gravity (EIG) wave variance is enhanced over the central and eastern Pacific ocean during the enhanced phase of the MJO over the warm pool. (Kiladis).

Single-column and Spatiotemporal stochastic models were used to study spectral power and statistical physics (precipitation phase transition, cloud size distribution, self-organized criticality). Close attention was payed to the background noise observed to characterize tropical rainfall and circulation variability. It is found that a simple linear stochastic Langevin-like equation, with forcing and damping, is able to reproduce bulk features of the tropical red noise spectrum. Space-time plots suggest a self-criticality behaviour of organized convective systems (Stechmann).

Multicloud Model MJO initiation by dry Kelvin waves, according to aquaplanet simulations using the HOMME GCM (at coarse resolution) coupled to the deterministic multicloud model; Northward vs eastward propagation depends on latitude of heating (Ravindran). Higher latitude heating leads to westward propagation.

Stochastic cloud transition: Based on a lattice multiple particle interacting system where lattice site are either cloudy, occupied by a certain cloud type, of are clear sky. Random transition between lattice site states occurs accoridng to intuitive propability rules depending on the background state, namely, convective instability and moisture.

Warm pool helps organizing convection into lower-frequency variability of the MJO, which depends on strong stratiform heating, when the SMCM is tuned to produce the right proportions of congestus versus deep convective clouds. Moreover, smaller startiform heating fraction lead to synoptic scale Kelvin wave-type organization. (Deng)

Including coupling to large-scale convergence in the stochastic generation and transition of shallow/congestus and deep convection leads to enhanced equatorial waves. (Brenowitz)

A Bayesian inference model was used to calibrate the stochastic cloud transition timescales in the multi-cloud model using in-situ and simulated data. Fast and slow transitions are distinguished. Transition may depend on grid size but invariant parameters are desirable. (De La Chevrotiere)

Conditional Markov Chains have been used to train the cloud transition parameters using observations of cloud types, large-scale conditions in terms of CAPE, vertical velocity, low-level humidity (Siebsma).

MJO Skeleton Model MJO events can be identified by projecting reanalysis data onto skeleton model solution with a zonally varying background and stochasticity. This leads to an MJO index based on skeleton Model. One important advantage is that the MJO skeleton solution, combines information based on shear, geopotential gradient (potential temperature) low-level moisture, and ORL, which provides robustness, unlike traditional indices that are based solely on ORL and/or the velocity potential. In addition, this approach does not need filtering or orthogonality assumption like the RMM index for example which is based on EOFs (Ogrosky).

The warm pool affects MJO statistics (length, location, primary vs. successive) in the stochastic skeleton model. Not all MJO events are mirror reproductions of previous events. They vary substantially one another in shape and strength rather caotically. The stochastic skeleton model reproduces this highly intermittent behavior quite well (Stachnik).

Adding a second baroclinic and a multi-cloud heating profile, with the natural transitions from congestus to deep and from deep to stratiform, to the MJO skeleton model produces vertical tilted structure of convection consistent with observations. It also allows the simulation of MJO wave trains as in nature (Thual).

The model of barotropic and baroclinic wave interaction for the skeleton model is proposed. A multiscale asymptotic analysis of resonant triads was performed. It leads to three-wave consistent of wither of an MJO, baroctropic Rossby and Kelvin triad or MJO, baroctropic Rossby and baroclinic Rossby triad. It allows a clear distinction and interpretation of tropical-extratropical interactions involving the MJO dynamics such as the influence of the MJO on midlatitude weather patterns (Rossby waves) and extra-tropical initiation of primary MJO's. (S. Chen)

Parametrization Two stochastic parameterization methods to improve simulations in different ways: Spectral Kinetic Energy Backscatter (SKEB) improves the mean, and Stochastically Perturbed Parametrisation Tendencies (SPPT) improves variability. (Christensen)

A unified parameterization is built upon a probabilistic plume model with a pdf in mixed-layer temperature. (Park)

Super parameterization in CFS improves simulations of the Indian summer monsoon in the amplitude of the annual cycle of rainfall, tropospheric temperature, intraseasonal spectral power. (Goswami)

Both cumulus and microphysics parameterization need to be improved for better simulations of intraseasonal variability. Putting back advective tendencies into large scale cloud condensate (liquid and ice) mixing ratios is found to improve large scale/stratiform precipitation to convective precipitation ratio (Mukhopadhyay).

Triggering functions and closures can be quantitatively evaluated using field observations and CRM simulations. Some are scale-aware, others are not. Eight trigger functions were considered and compared. They include dilute CAPE and non-dilute CAPE triggers. The dilute are found to out perform the non-dilute ones in many ways including the diurnal cycle and many stastical climate properties. CRM simulations suggest a 20 minute lead time of moisture convergence prior to deep convection. Both moisture convergence and dilute CAPE closure are found to be scale aware while pure CAPE closures are not (Zhang).

Stochastic parameterization needs to meet several criteria: resolution dependent, larger variability than initial uncertainties, improved forecast skill, unique interaction with large-scale environment not reproducible by deterministic schemes. (Craig)

Parameterization of cold pools needs to consider: entrainment, cloud size, organization, and randomness. Simulation of orographic convective systems highlighted the importance of cold pools in convective organization. An outline for a particle model for cold pools was proposed (Boeing).

Tools Permutation Entropy can be used to identify dynamical signals prior to MJO initiation. The process of MJO initiation is broken down into local dynamical chnages and MJO lifecycle. The Nonlinear Laplacian Analysis (NLSA) framework is extended to the MJO mode and explained in term of SVD by analogy to EOF analaysis. A time lagged embedding procedure was integrated into NLASA and the resulting model was applied to local low-level winds and moisture changes. Moisture buildup prior to initiation was observed. Perturbation TKE and vertical integrated moisture budget partitioned time series were used. The permutation entropy metric was used on the partitioned data to assess convection organization under the assumption the whole MJO sequence is a Markov according to the partitioned time series. (Tung)

NLSA recovers MJO and BSISO modes without bandpass-filtering or spatial orthognality. Kernel analog forecasts of the MJO and BSISO based on NLSA show skill up to 45 50 days. (Giannakis)

Effects of moisture in convective boundary layer: buoyancy flux peaked at smaller scales, more overlap between forcing and dissipation spectra, shallow cloud moistening. (Waite)

A multi-scale model with intraseasonal and diurnal components demonstrates effects from the latter on the former. Multiple scale asymptotic analysis was used to derive model for interaction of intraseasonal disturbances and the diurnal cycle. It is found that tilted heating is important for diurnal cycle to have impact on intra-seasonal scale. The modulation of diurnal cycle by intra-seasonal variations is also evidenced from the model (Yang).

A Low-Order Nonlinear Stochastic Model is built to predict the two MJO modes derived from the nonlinear Laplacian spectral analysis with skills up to 40, 25 and 18 days in strong, moderate and weak MJO years. (N. Chen)

Based on LES, direct entrainment is sensitive to buoyancy, vertical velocity, but independent of cloud size; detrainment depends on critical mixing fraction in shallow convection. (Austin)

Insentropic stream function analysis was extended to moist dynamics and convective motions. It is based on a stream function calculation in the (θ_e,z) space so that parcels move along constant surfaces in this space. This allows among other things a direct diagnostic of diabatic heating; the velocity component in the $theta_e$ direction is obtained by a simple derivative with respect to z of this stream function. Comparison with Eulerian formulation demonstrates ability of the new procedure to capture mesoscale and synoptic scale flows in the Hadley circulation which are otherwise filtered out in the Eulerian framework (Pauluis).

A clustering approach applied to isentropic analysis identifies the patiotemporal evolution of 7 different convective regimes in OLR, precipitable water and precipitation rate. (Slawinska)

Breakout group discussion sessions and recommendations

In addition to the 35 talks, the workshop included breakout sessions where participants were separated into small groups of about 7 people each, in separate rooms. Each group was assigned a specific topic to discuss and asked to address specific questions and prepare a report to be presented by the a group designate on the last day of the workshop, i.e, Friday morning. There were asked to break up their discussion into the following three themes.

I. Scientific and practical issues associated with their topic

II. State-of-art and recent advances

III. Challenges and recommendations

All the groups were asked to address the following questions as they apply to their specific topic:

a. How should convective stochasticity and organization be quantified in observations and model simulations so they can be directly compared to each other?

b. How does the perception or definition of convective stochasticity and organization depend on time and spatial scales?

c. How do convective stochasticity and organization depend on the large-scale environment?

d. How different are interactions of the large-scale environment with organized convection vs. non-organized convection?

e. To what degree can organized and non-organized convection be parameterized in global climate models?

f. Can satellite and ground radar observations provide consistent information of the degree of stochasticity of convection?

The following groups were formed and met during breakout sessions of 2 to 3 hour duration, on Monday, Wednesday and Thursday afternoons. At the end of each breakout session there was a general discussion session in the big lecture room and the groups were asked to summarize their respective group sessions and feedback was given from other groups accordingly.

The following groups were formed and their summary reports are attached in Appendix B.

I. Stochastic Modeling and Observation Tools: Gianakis, Ajayamohan, De La Chevrotirere, Ogrosky, Schumacher, C. Zhang, N. Chen, Yang, Brenowitz, Kiladis,

II. Parameterization: G. Zhang, Mukhopadhyay, Christensen, Park, Siebesma, Craig, Tung, Boing,

III. LES/CRM simulations: Austin, Waite, Moncrieff, Hagos, Bellenger, Pauluis, Khouider,

IV. MJO initiation: Shuyi Chen, Johnson, Goswami, Stachnik, Majda, Shengqian Chen, Thual, Stechmann,

Outcome of the Meeting

Many collaborations among individual workshop participants were forged during the workshop. Collectively, the workshop decided to pursue the following:

(a) Recommend a special issue on Stochasticity and Organization of Tropical Convection to the online open-access journal Mathematics of Climate and Weather Forecasting. The recommendation has since been accepted. The description of the special issue is posted at http://degruyteropen.com/mcwfsotc/. Nine workshop participants have expressed interest to contribute to the special issue.

(b) Write a workshop summary article for the Bulletin of the American Meteorological Society. The organizers of the workshop will draft this article.

(c) Write an article to document and compare stochasticity of the MJO and its associated perturbations using different MJO indices to assess our knowledge on this subject and the uncertainties associated with it. About 6 workshop participants will collaborate on this.

Appendix A: Workshop Agenda

Monday, May 27
8:45 9:00 Welcome remarks by BIRS manager and organizers
9:0010:30 Talks
Chair: Boualem Khouider
9:00-9:30 Mitch Moncrieff: Supercluster-like Organization & Inertial-Gravity Wave Interaction during the Year of Tropical Convection (YOTC)
9:30-10:00 Richard Johnson: MJO Initiation: Multiscale Processes Inferred from DYNAMO
10:00-10:15 Samson Hagos: Cloud Permitting Modeling of Shallow-to-Deep Convection Transitions during the Initiation and Propagation of Madden-Julian Oscillation
10:15-10:30 Hugo Bellenger: Processes of MJO Preconditionning Shallow Convection and Clear Air Turbulence
10:30-11:00 Coffee break
11:00 12:00 Talks
Chair: Boualem Khouider
11:00-11:30 Qiang Deng: The Role of Stratiform Heating in Simulating MJOs in a Stochastic Multicloud GCM
11:30-12:00 Ajaya Mohan Ravindran: MJO/MISO in a coarse resolution aquaplanet General Circulation Model
12:00 13:30 Lunch
13:00 Tour of Banff Centre Campus (May need to have quick lunch to join the tour)
14:00 Group photo
14:00–15:00 Breakout sessionsgroups will meet and start discussing their themes.
15:00–15:30 Coffee break
15:30 17:00 Breakout discussions continue (possibility short 5 min presentations within breakout sessions to put the groups up to speed)
17:00 17:30 General discussion: Remarks and comments, possible consultations across groups.

Tuesday, May 28
9:0010:30 Talks
Chair: Mitch Moncrieff

9:00-9:30 Courtney Schumacher: Mesoscale Organization from an Observational Perspective
9:30-10:00 Sam Stechmann: A Spatiotemporal Stochastic Model for Tropical Precipitation & Water Vapor Dynamics
10:00-10:15 S. Thual: A MJO Skeleton Model with Refined Vertical Structure
10:15-10:30 Nan Chen: Predicting the Cloud Patterns of the Madden-Julian Oscillation through a Low-Order Nonlinear Stochastic Model
10:30-11:00 Coffee break
11:00 12:30 Talks
Chair: Moncrieff
11:00-11:30 Phil Austin: Cloud entrainment and detrainment in high resolution simulations of convection
11:30-12:00 Pier Siebsma: Using Conditional Markov Chains (CMCs) in convection parameterizations
12:00-12:15 Michael Waite: The spectral kinetic energy budget in LES of convective turbulence
12:15-12:30 Boeing: Convective cold pools and their effects on cloud formation
12:30 13:30 Lunch
13:30–15:00 Breakout sessions
15:00–15:30 Coffee break
15:30 16:00 Breakout sessions (continue)
16:00 17:00 Group preliminary reports (10 min each)
Chair: Chidong Zhang
17:00-17:30 General discussion, comments/feedback on prelim reports
Chair: Boualem Khouider

Wednesday, May 29
8:3010:30 Talks
Chair: Chidong Zhang
8:30-9:00 Shuyi Chen: Stochastic Ensemble Modeling of Scale-Dependent Error Growth and Multiscale Interaction in Tropical Convective Systems
9:00-9:30 George Kiladis: Stochastic Aspects of Convective Organization within the MJO
9:30-9:45 Shengqian Chen: Multiscale Asymptotics for MJO Skeleton and Tropical-Extratropical Interactions
9:45-10:00 Reed Ogrosky: Identifying the MJO Using the Skeleton Model
10:00-10:15 Noah Brenowitz: Enhanced Persistence of Equatorial Waves via Convergence Coupling in the Stochastic Multicloud Model
10:15-10:45 Coffee break
10:45 12:45 Talks
Chair: Chidong Zhang
10:45-11:15 Guang Zhang: Examination of Convective Parameterization Schemes and ?Their Scale-Awareness Using Observations and Cloud-Resolving Model Simulations
11:15-11:45 P. Mukhopadhyay: Modification of Sub-grid Scale and Grid Scale Cloud and Convective Parameterization in CFSv2 and Its Impact on Organized Convection and Improving Model Fidelity
11:45-12:15 Hannah Christensen: Stochastic Parameterization: Representing Model Uncertainty in Earth-System Modelling
12:15-12:30 SeungBu Park: A unified Parameterization of Dry and Moist Convection for General Circulation Models
12:30-12:45 Joanna Slawinska: Convective Regimes as Revealed by Isentropic Analysis
12:30 14:00 Lunch
Afternoon: Free activities.

Thursday, May 30
9:0010:30 Talks
Chair: Mitch Moncrieff
9:00-9:30 Pauluis: Isentropic analysis for convective motions
9:30-10:00 George Craig: Convective Fluctuations and Stochastic Parameterization - Revisited

10:00-10:15 B. Goswami: Is SuperparameterizationCapable of Breaking The Deadlock ? - Seeking the Answer in Superparameterized CFSv2 664-Day Climate
10:15-10:30 Qiu Yang: A Multi-Scale Model for the Intraseasonal Impact of the Diurnal Cycle of Tropical Convection
10:30-11:00 Coffee break
11:00 12:30 Talks
Chair: Chidong Zhang
11:00-11:30 Dimitris Gianakis: Kernel Analog Forecasting of Intraseasonal Oscillations
11:30-12:00 Wen-Wen Tung: The Emerging States of MJO Convection Initiation
12:00-12:12:15 De La Chevrotiere: Bayesian Inference for the Stochastic Multicloud Model Using the Giga-LES Dataset
12:15-12:30 Justin Stachnik: Sensitivities of the MJO to the Shape and Strength of the Tropical Warm Pool in the Stochastic Skeleton Model
12:30 13:30 Lunch break
13:30–15:00 Breakout sessions
15:00–15:30 Coffee break
15:30 17:00 Breakout sessions: Concluding and Report preparation

Friday, June 1
9:0010:00 Group reportsfinal (20 min each)
Chair: Chidong Zhang
10:00-10:30 Coffee break
10:30 12:30 Concluding discussion and wrap up
Chair: Boualem Khouider
12:30 13:30 Lunch break
13:30 Workshop ends

Appendix B: Group reports

Group Discussion 1

Challenges, Issues
Definition and motivation of organization: Based on existing tracking methods, you can define scales for organization. Characterize in terms of area, lifetime, structure (e.g. linear), propagating/nonpropagating, large-scale impact (Q1, Q2, Q3).Under suppressed condition, it may not be organized but can have an impact?

Two important questions:
a) what is ”organization”? and
b) why do we care about organization”? Two levels of organization can be defined:

Level 1:
a) Defined by large area / long lifetime.
b) Important for the accompanying stratiform rainfall / vertical diabatic heating proles.

Level 2:
a) Defined by large area / long lifetime + linear organization (e.g. squall line);
b) Important for momentum transport (e.g. $u'w'$);

Conclusions:
a) Multiple ”Levels” of organization;
b) Important for different aspects of convective parameterization;
c) Potential for more predictability, including teleconnection patterns;

To the question how to compare models to observations?”:

RMM is appropriate for global MJO statistics. However, there are distinct alternatives to the RMM index that should be considered to be used in different contexts.

Here are some recommendations:
OMI: Convection only;
MJOSA: Convection and circulation, including off-equatorial circulation; No temporal filtering or empirical orthogonal functions; There are two outcome to the MJOSA, one form which gives an amplitude in every longitude in time, and another form which provides two indices. Because of the absence of temporal ltering, monitoring is in real-time.
NLSA: Convection only; It requires minimal preprocessing, no spatial-time pre-filtering or seasonal partition; It provides clean separation of Summer boreal ISO and MJO with favourable predictability; No spatial orthogonality required.

Revisit MJO metrics. Examples of MJO metrics that are of interest for organization: Rain rate on the 10E-10W band (comes in response to the fact that the RMM is circulation based), Longitude/propagation speed. Desired statistics: mean, variance, decorrelation should be included in standard diagnostics.

Recommendations:

How can the SMCM be leveraged to get more information? How realistically can it be used in operational GCM? Calibrate the SMCM for the cloud areas equilibrium distribution or transition time scales from data. Geographic and scale dependence is important. Dataset suggested: DYNAMO (Indian), DARWIN (Australia), KWAJEX (Central Pacific), TRMMVGPM.

Improve the stratiform representation: couple the stratiform formation from to the large-scale environment: important indicators are upper level shear, middle level relative humidity. (Add non-precipitating envelope?). Include local interactions between the microscopic convective elements;

Develop dynamical diagnostics/criteria for the SMCM to compare against observations: e.g. a localized Wheeler-Kiladis”-like criteria for the cloud area fractions that helps to validate and/or retune the SMCM.

Low order model: Use multiplicative noise or hidden Markov processes to approximate the cloud transitions. Generally, further development and applications of space-time localized spectra, indices, and conditional statistics for other models and observational studies;

We strongly suggest that codes, diagnostic tools, and actual indices be made available to the community.

Group 2: Report of the Parameterization Group.

State of the art. Problems with variability in deterministic parameterizations.

Deterministic convection parametrisation schemes aim to represent the average of the possible effects of unresolved, subgrid-scale convective clouds on the resolved scale state. This is a good approximation for large grid boxes, since the average is taken over many convective plumes, so the statistical relationship encoded in the parametrisation scheme will hold. However as model resolution increases, the validity of this approach is compromised. For smaller grid boxes, the issue of sampling becomes more important, and the variability in the unresolved forcing for a given resolved-scale state is expected to increase. There is therefore much interest in developing stochastic parametrisation schemes that represent the variability of unresolved convective processes and the impact that this unresolved variability has on the large scale state.

Approaches to stochastic parameterization

Pragmatic Approaches, Not scale aware, not related to physics

Two operational stochastic schemes have been developed by the European Centre for Medium-Range Weather Forecasts (ECMWF) that have been adapted and adopted by modelling centres worldwide. The first of these is the Stochastically Perturbed Parametrisation Tendencies scheme (SPPT), whereby a spatially and temporally correlated random number field is used to perturb the parametrisation tendencies from the models physics package. This pragmatic approach to representing uncertainty was initially proposed to improve the reliability of the Centres medium range forecasts by correcting the under-dispersive nature of the ensemble. The imposed spatial and temporal correlation scales have no physical basis, though coarse graining studies have been used to retrospectively justify the multiplicative nature of the noise and its magnitude. Despite the ad hoc choices in the formulation of the scheme, SPPT is surprisingly effective at improving ensemble reliability, reducing forecast error, and improving biases in the mean and variability of the models climate. However, the scheme comes under much criticism for the lack of physical justification in its approach.

The second scheme is the Stochastic Kinetic Energy Backscatter scheme (SKEB). This scheme represents a process that is otherwise missing due to the truncation of the model equations, namely the upscale transfer of kinetic energy from small to large scales. This is achieved by perturbing the stream function to inject energy directly into the largest scales. Despite being physically motivated, the scientific basis of the scheme is weak, as is reflected in the wide variety of implementations used in different weather centres. In addition, the scheme is not scale aware: the backscatter ratio must be re-tuned for each new model resolution. Recent work by Shutts (2015) suggests a physically motivated improvement to the scheme, whereby the interaction of convection with model dynamics near the grid scale is recognised as the principle source of model error represented by SKEB style schemes.

Stochastic multiplume, Scale-aware across

A physically motivated model, which has gained much attention in recent years, is the stochastic multi-plume model (Plant and Craig 2008). Convective variability is first characterised mathematically. With the assumptions of convective equilibrium and non-interacting plumes, an expression for the distribution of individual mass fluxes, and for the probability distribution of total mass flux, can be derived, where the large scale state is used to constrain the mean total convective mass flux. The variance of the convective mass flux scales inversely with the number of convective clouds in the ensemble, and the variability about the mean becomes increasingly important as the grid box size is reduced or in cases of weak forcing. The first step in implementing a parametrisation scheme based on this theory is estimating the large-scale environmental properties necessary for the convective equilibrium assumption: the atmospheric state is first averaged over neighbouring grid boxes within a specified area such that this region will contain many clouds. Importantly, this enables the scheme to scale well across model resolutions, with no retuning required, up until resolutions where the grid scale becomes comparable with the scales of convective clouds. Temporal correlations are introduced into the scheme by assuming that clouds have a finite lifetime.

Multicloud model, Temporal organisation, self-similar?

A second physically motivated model is the stochastic multi-cloud model, based on analysis of observations and theoretical understanding of tropical dynamics (Khouider et al. 2010). The parametrisation is centred around three cloud types observed over the warm pool and in convectively coupled waves, shallow congestus, stratiform and deep cumulus clouds, emphasizing the dynamic role of each of these cloud types. Each GCM grid box is first divided into a number of lattice sites, spaced 1-10km apart. As in the multi-plume model, this introduces a degree of scale awareness to the multi-cloud model, which subsequently performs well at a variety of resolutions. The scheme avoids introducing ad hoc parameters, as in other parametrisation schemes. Each lattice site switches from cloud type to cloud type following a set of probabilistic rules, conditioned on the large-scale state. The transition timescales can be estimated from observational data or LES simulations. The order of the transitions introduces temporal organisation to the system, which is important for simulating the MJO.

Issues

Stochastic parameterizations aim to take into account temporal and spatial organisation of clouds in a stochastic matter. In doing this a number of issues should be taken into consideration:

Upscale influence of small scale processes.

It is not a priori clear which processes and structures on the subgrid scale are relevant in the sense that they have implications on the larger resolved scale. Care should therefore be taken to i) only develop stochastic parameterizations for those subgrid phenomena that actually influence the larger scale dynamics and ii) make sure that the stochastic parameterization descriptions actually generates realistic perturbations on the resolved scales

Spatial versus temporal subgrid organisation

Most current schemes for stochastic convection parameterizations (.e. multicloud model, multiplume model, conditional markov chain (CMC) approaches) do introduce temporal correlations but have a lack of spatial organization. It is unclear to what extend it is necessary to introduce spatial correlation which would turn CMC type of parameterizations more into cellular automata type of descriptions. This would introduce new complexity but will also allow for explicit spatial organization on the subgrid scale.

Scale awareness

Stochastic parameterizations become increasingly important if the resolution is becoming comparable with the size of the phenomenon that requires parameterization. This implies that a stochastic parameterization should be scale aware since the variability should increase with higher resolution in a realistic manner. If the resolution becomes even higher than the phenomenon (i.e. convective plume) but is still not high enough that it resolves the process most of the presently presented new stochastic frameworks break down (multicloud model, multiplume model). The range of validity of a stochastic parameterization needs to be explicity recognized, and approaches that represent only the smaller clouds or convective motions could be explored.

Stochastic perturbations across different parameterizations

It is quite likely that when existing stochastic cloud models (i.e. the multicloud model) are implemented in existing operational GCMs it will not be possible to incorporate these into one single (i.e convection) parameterization. For instance the stratiform part of the multicloud model is in most GCMs (at least in part) represented in the cloud scheme. This creates the problem of how to practically implement recently developed off-line stochastic models in present day numerical weather prediction (NWP) and climate models.

Determination of parameters in stochastic parameterizations

At present many of the parameters in stochastic parameterizations (i.e. the multicloud model) are only beginning to be constrained by observations and turbulence resolving models. Especially if these models are to be used in operational models thay need to behave in a realistic manner over the whole phase space of possible realisations. This requires a serious investment in the training and optimalisation of the used parameters in these parametrizations. A further issue is the dependence of stochastic variability on weather regime, which has been found in mid-latitudes and may occur in the tropics as well, and needs to be accounted for in the choice of parameters or even the formulation of the scheme.

Trigger of convection

The rigger function for convection is mostly still done in a deterministic fashion. It may be more appropriate to incorporate the trigger function as an integral part of a stochastic convection framework.

Stochastic vs Numerical noise

Even when using deterministic parameterizations there is numerical noise at the grid scale. When introducing stochastic parameterizations one needs to keep track of the intended variability originating form stochastic parameterization vs the unintended numerical noise due to numerics, including potential interactions between the two.

Opportunities and recommendations:

Systematic analysis of coarse-graining CRMs/ LES

One prime opportunity is to make use of large-domain turbulence resolving simulations. By systematic coarse graining the simulation results as well as systematically varying the imposed resolution it will be possible to inform parameterizations on relationships between resolved and subgrid processes, the degree of stochasticity and scale-awareness, all as a function of the used resolution. Ideally such simulation studies could be based on recent tropical field experiments such as Cindy-Dynamo or upcoming programs such as YMC and could be organized in collaboration with the WGNEs Grey Zone Project.

Use Hierarchy of models in a consistent way: from cellular automata to GCMs

The use of hierarchy of models in terms of complexity is encouraged, starting from exploring new approaches in simplified models (Lorentz96) and test basic principles, further exploiting it in models of intermediate complexity and finally using it in operational models with the full operational complexity.

LES as convergence test

Operational mesoscale models should all have the option to be able to run in a LES-mode. This will allow more systematic exploitation of simulating periods during field campaigns are various resolutions. It also makes it easy to evaluate parameterizations in such models and investigate how/whether their behaviour is converging when approaching turbulence resolving resolutions.

Common metric to test and evaluate parameterizations in general.

We hope to see recommendation along these lines from WG1 and would certainly support those. In particular, process-oriented metrics relating to structure and organization of convective play an important role between bulk indices for large-scale phenomena such as MJO and direct measures of parameterization output such as Q1, Q2 and Q3.

Testing the stochastized versions in an ensemble environment

Stochastic parametrization represents a paradigm shift when compared to deterministic parametrization: a stochastic model represents one possible realization of the system, as opposed to the most likely evolution represented by a deterministic scheme. It is therefore important to assess stochastic schemes in an ensemble setting. How well does the scheme represent model uncertainty? What can be learned from the ensemble about predictability of the process in question? How does the ensemble mean compare to traditional deterministic forecasts?

How to incorporate existing stochastic parameterizations into current parameterizations in operational models?

An area of active research should be investigating ways of including state-of-the-art stochastic schemes into operational models. It is not necessarily obvious how new schemes should interact with the other physics schemes or, in the case of the multi-cloud model, how well the scheme will perform in mid-latitudes. Nevertheless, this is an important step to take to allow these schemes to be evaluated against existing forecast models and observations.

Determine Upscale Growth from CRMs

In order for a stochastic parametrisation to be useful, the stochasticity must have an impact on the resolved scale flow. An interesting area of research is determining exactly how stochastic parametrisations can have this impact. Determining upscale error growth using CRM simulations can inform the community as to which uncertain processes should be made stochastic, and which are unimportant in this respect.

Group III: LES/CRM Simulations

ISSUES

- Measurements of evaporation and fine-scale humidity for evaluation of LES/CRM simulations cold pools, PBL structure (ML depth evolution)
- Limitations of periodic boundary conditions; coupling of CRM/LES to large scales; validity of the weak-temperature gradient approximation.
- Cross-grid interaction in super-parameterized models: Periodic boundary conditions preclude propagation of organized convection (e.g., MCS) across the parent climate model grids. Propagation across climate grids requires interaction between vertical shear on the low-resolution climate grid with latent heating on high-resolution CRM grids. This can result in structural bias, e.g., absence of mesoscale downdrafts. A better understanding of the spatio-temporal evolution of convective activity and, in particular, of the nature of scale interactions.
- Microphysical parameters that are significant drivers for CRM/LES simulations.

B. STATE-OF-THE-ART

- 1-km-grid models are positioned to evaluate the statistics (e.g., clustering, isentropic analysis, mass flux) and stochastic aspects (e.g. conditional pdf, time-lag correlation) of organized convection in large computational domains.
- Use of high-res global NWP virtual global field campaign analysis for initial conditions and/or evaluation of MJO and convectively coupled equatorial wave simulations. Development of scaling laws for convection (Giannakis, Peters and Neelin, Stechman, Craig) that can be tested and evaluated in CRM/LES.
- New methods for analyzing thermodynamic cycles in numerical models can be used to assess the interactions between thermodynamics and dynamics in convection.
- Momentum transport by organized convection is distinct from CMT by unorganized cumulus, i.e., counter gradient momentum transport and upscale kinetic energy tendencies. Theory and simulation show significant effects on the MJO and convectively coupled equatorial waves. (few convective scheme taking this into account: Wu et al. discussion about super parameterization that do not re-inject CMT to the large-scale)

- Organized convection in shear has been approximated as Lagrangian multiscale coherent structures and has been represented in simplified global models by multicloud parameterization. It timely to examine aspects such as organized momentum transport in full GCMs (e.g., CSM and CCSM). (e.g., It has been shown that all CMIP5 models fail to represent the Tropical Easterly jet because they significantly underestimate upper-tropospheric shear.)

- The Vertical Structure and Diabatic Process intercomparison is an example of a multimodel intercomparison of MJO that compares models across time and space scales and could be employed for the Year of the Maritime Continent[1].

Four papers on a collaborative project between GASS, YOTC and the MJO Task Force Vertical Structure and Diabatic Processes of the MJO have been accepted for publication in JGR.
a) 20-year climate simulations;
b) 20-day hindcasts;
c) 2-day hindcasts;
d) analysis of the GCMs that contributed to all 3 projects.

C) RECOMMENDATIONS

- Employ virtual global field campaign data (10 km grid ECMWF IFS) to evaluate large-domain 1-km-grid CRMs, as a supplement to actual field-campaign data. (for example model precipitation and organized convection and momentum transport following Praveen et al. (2015) 10.1175/JCLI-D-14-00415.1) also cf. parameterization development

- Contribute to the Year of the Maritime Continent (YMC) actual and virtual field campaign for (2017 2019) Design a YMC Grey Zone project on organized tropical convection resembling the giga-EUCLIPSE cold-air outbreak study. CRM O(1km) vs LES . how good is 100m resolution?

- Use CRMs to inform convection initiation, stratiform/convective fraction. Suggest CRM modellers archive variables from simulations for standard diagnostics (isentropic analysis, cloud transitions).

- Adoption of new diagnostics to address the statistical behavior of convection (NLSA, isentropic analysis, lagrangian trajectories, information theory, Hidden Markov Model (Crommelin and Vanden-Eijnden) that could be systematically for model intercomparison

- The use of CRMs to quantify transitions between cloud types may be enhanced by the use of simplified microphysics parameterizations that could be more easily explored. The microphysics of evaporation is of particular interest. Comparison with disdrometers and dual-polarization radars measurements from field campaigns could provide an observational constraint. Need simple microphysics, easy to tune and compare in particular in terms of precipitation efficiency (). Test the sensitivity of microphysics to subgrid turbulence, vertical velocity

- Coupling CRM/LES with the ocean.
  (1) Precipitation leads to fresh water lenses. Convection initiates on their edge (Carbone et al.? link with cold pools?).
  (2) During calm weather (shallow convection, congestus) strong stratifications appear in the first tens cm of the ocean making it reactive on the scale of 1 hour. Impacts PBL and convection triggering (Bellenger et al 2010, Ruppert and Johnson 2015, Johnson et al 2015).

Response to Questions

[1] https://usclivar.org/sites/default/files/meetings/2014/summit-presentations/Zhang_Summit2014.pdf

a) and b): Whats the physical interpretation of the theoretical variables in multicloud parameterization, what do we need from CRM/LES and observations, e.g., cold-pool aspects, evaporation. Mesoscale scale-selection multicloud parmeterization/ multiscale coherent structures context need to be evaluated against large-domain LES/CRM and observations, including geostationary satellite observations.

d) and e): The effects of vertical shear on moist convection is given little attention compared to moistening/drying/heating contrary to results of theoretical, numerical and observational studies.

f): Forward operator (radar simulator) development for LES/CRM would facilitate comparison of models with radar measurements.

Group 4: Initiation and Dynamics of MJO/Physical Processes

Summary and Report

One of the most challenging aspects of identifying the initiation of a primary Madden-Julian oscillation (MJO) event is first defining the MJO. The MJO is highly stochastic and may initiate from any number (or combination) of mechanisms such as stochastic convection and internal atmospheric variability, recharge-discharge, air-sea coupling, upscale energy transport, moisture modes and cloud-radiation interactions, extratropical forcing (e.g., Rossby waves or equatorially propagating cold air outbreaks), among others that are documented in recent and historical literature. The MJO also displays a strong seasonality and interannual variability; each MJO event is different and unique.

Nevertheless, we see a need for a more unified definition of the MJO. Likewise, there is the pressing challenge for the development of revised MJO indices that can better identify local MJO activity and event initiation. Such an index would be most successful if it could be computed from both observational and model data. Current popular indices such as the Real-time Multivariate MJO (RMM) index (Wheeler and Hendon 2004) struggle with identifying the initiation of primary MJO events when occurring at scales smaller than the zonal wavenumber-1 modes that comprise the leading EOFs of this index. Similarly, the RMM requires global wind data and thus cannot be computed from limited domain simulations such as regional climate or cloud resolving models (CRMs). Global climate models (GCMs) will be largely unsuccessful at tracking the MJO via precipitation or simulated OLR anomalies while low-order models may be limited to a few variables like convective heating. As such, we recommend the development of an MJO index for initiation based on our identifying and characteristic traits of the phenomenon, potentially including the following qualities:

- The presence of large-scale convective anomalies along or near the equator and grouped within 1000 km (i.e., the convective envelope)
- The development of mesoscale organization within the convective envelope and the appearance of multiscale interactions
- The start and maintenance of eastward propagation at 5 m s-1 item The onset of zonal circulation anomalies that eventually grow to scales of wavenumber-1 to wavenumber-3

To better elucidate the dynamics and mechanisms responsible for MJO initiation, we also recommend additional research that attempts the following:

- Synthesis and climatology research that documents the distribution of mechanisms considered most important for observed and modeled MJOs (i.e., statistical analysis of mechanisms associated with MJO initiation)
- Model intercomparison studies for all scales of numerical analysis (i.e., GCMs, CRMs, LES, and low-order analytic models) that focus on the successful initiation of primary MJO events rather than maintenance or propagation (similar to the second objective of the GASS-YoTC Vertical Structure and Physical Processes model evaluation project with high-frequency model output)
- Additional studies examining the failure of large-scale MJO initiation

- Improved understanding of MJO event termination and/or the corresponding initiation of long-term quiescent periods (e.g., why do some years have little to no MJO activity?)

- Better understanding the continuum of Kelvin waves and the MJO in addition to the role of wave interactions on influencing MJO organization and event initiation

- Discovering the role of stochasticity, organization, and multiscale processes that contribute towards MJO initiation

Additional work is needed to incorporate these findings into numerical models. For example, recent work has shown that the mid-level moistening in models is often insufficient compared to observations and provides a reference for tuning multiscale interactions and accounting for the upscale effects of clouds and convection in convective parameterizations. It is our hope that the above work may help identify those large-scale environmental conditions and dynamical mechanisms that are considered necessary and sufficient for MJO initiation over the Indian Ocean and other locations in addition to improving mid-range global weather forecasts.

CLIFF's NOTES:

1. Scientific and practical issues associated with MJO initiation

a. What is the MJO/How to define the MJO?

b. The MJO experiences strong seasonality and interannual variability

c. Challenges associated with common MJO indices
   i. Do we use regional or global indices?
   ii. What metrics best describe the local onset of an MJO? Convective organization, or just any sort of OLR and energetic anomaly?

d. Can we list different conditions and situations that initiate MJOs in the Indian Ocean?

2. Quantifying stochasticity and organization

a. What is organization?

b. What is stochasticity?

c. Stochasticity and organization are both scale-dependent in models and observations

3. Merging stochasticity and organization

a. What are the large-scale environmental effects on stochasticity?

b. Do we need an index to measure convective organization for the MJO?
   i. Is this useful? Would it help to better predict the initiation of primary MJOs?
   ii. What convective organizations are most common prior to the large-scale onset of an MJO?

Long Record

1. Scientific and practical issues associated with MJO initiation

a. What is the MJO/How to define the MJO?

i. Is the MJO regional or global? Do events originate solely in the Indian Ocean basin or do we consider activity that develops over the west Pacific also an MJO?

ii. Convective vs. circulation anomalies

iii. Eastward propagating vs. stationary waves

iv. Characteristic traits of an MJO event

1. We define the MJO as an equatorially envelope (grouped within 1000 km) of convective disturbances over the Indian Ocean that excites planetary scale zonal wind anomalies and travels eastward at roughly 5 m s^{-1}

2. No two MJOs are the same, it is highly stochastic unlike other mode constrained modes of tropical convection (e.g., tropical cyclones)

v. Possibility of dry modes, though question remains as to initial source of deep heating and temperature anomalies without convection

Unable to rely on SST anomalies and sensible heat fluxes; resultant heating is too shallow

vi. Do MJO events always exist in some background state or are they only present once reaching some critical amplitude?

vii. Consideration of other large-scale variability

1. Weickmann and Berry (2008) global wind oscillation

2. Straus and Lindzen (2000) growth of baroclinic modes on planetary scales

b. The MJO experiences strong seasonality and interannual variability

i. The Indian Ocean basin has strong seasonality in the zonal winds and varies with ENSO and IOD phases

ii. Do we more broadly consider monsoons and northward propagating modes in the boreal summer part of the MJO continuum?

Thual et al. (2015): The stochastic skeleton model can produce northward propagating ISO modes with a seasonally migrating warm pool, though eastward propagating mode is still preferred in the model

c. Challenges associated with common MJO indices

i. The RMM index (Wheeler and Hendon 2004) is reliable with pre-existing MJOs and MJO maintenance yet may miss the initiation of a primary event if occurring at scales smaller than the wavenumber-1 circulation component within the RMM EOFs

ii. The OMI (Kiladis) uses 2D EOFs (no meridional averaging) of OLR without consideration of circulation anomalies

iii. In short, we struggle to define local indices, especially when we consider that the MJO has global implications

d. Can we list different conditions and situations that initiate MJOs in the Indian Ocean?

i. This depends highly on you define and explain the MJO

ii. Our field campaign observations are statistically insignificant and we need a much longer record of MJOs

2. Challenges and issues associated with convective organization

a. What is organization?

i. Development of cold pools and organized momentum transport within the convective system

ii. Should appear continuously precipitating in satellite data using certain OLR thresholds, though threshold choice is subjective

b. Organization is scale-dependent

i. Typically see mesoscale organization within the active phase of the MJO

ii. Some folks may talk about coherent vortices as organization in turbulence
iii. Suppressed phase conditions, which are highly stochastic, still can self-aggregate and generate organized cold pools

c. The organization of convection will he highly driven by shear, among other environmental factors
i. Self-aggregation is likely less important than shear
ii. Should this be included in our metrics to predict MJO initiation?

3. Challenges and issues associated with quantifying stochasticity

a. What is stochasticity for tropical convection?
i. Dictionary says stochastic is random
ii. Stochasticity implies probabilistic distributions that can be analyzed statistically, though not explicitly predicted without the use of ensembles
iii. In a model, the same deterministic variables can give rise to any number of stochastic states, though the distribution should still be deterministic

b. Stochasticity is scale-dependent
i. Synoptic waves can be relatively deterministic, but downscaling to convective scales becomes more stochastic in models and observations
ii. What averaging and filtering need to be done for our context (Earth science) to bring out the stochastic and deterministic parts of our system?
iii. How many scales do we need?

c. Stochasticity in models is dependent on grid size and resolution

4.Merging stochasticity and organization

a. What are the large-scale environmental effects on stochasticity?
i. Say you have a very dry vs. moist environment. Which is more stochastic?
1. Are Bernard cells and cumulus clouds organized?
2. Will convective always self-aggregate due to radiative instabilities and interactive radiation in models?
ii. Self-aggregated shallow convection over long times may demonstrate organization, but would not call that deterministic

b. Do we need an index to measure convective organization for the MJO?
i. Is this useful? Would it help to better predict the initiation of primary MJOs?
ii. What convective organizations are most common prior to the large-scale onset of an MJO?

DISCUSSION FROM WEDNESDAY, APRIL 29, 2015

1. Scientific and Practical Issues/Statement of Problem

a. Defining the MJO
i. Each MJO (and initiation) is very different

b. Measurements
i. Indices capture different features
ii. RMM index is not well suited for the Indian Ocean
1. Cannot track active convective component over those domains (evidenced by DYNAMO data)
2. RMM is much better suited for a global index
iii. Separating primary from secondary MJOs varies by index

c. Mechanisms and Modeling
    i. Tropical internal atmospheric variability
    ii. Global instabilities and linear modes (both tropics and extratropics)
    iii. External forcing
    1. Extratropical influences
    2. Upscale energy transport, waves, and aggregation
    iv. Air-sea coupling
    v. Role of stochastics
    vi. Primary and secondary MJOs

2. State-of-Science and Recent Advances

a. Defining MJO features and characteristic traits (visual)
    i. Equatorially envelope of clouds and convection (TRMM precip, OLR)
    ii. Wavenumber-1 circulation anomalies
    iii. Large-scale organization (multiscale)
    iv. Eastward propagating at 5 m s^{-1}

b. Metrics and Indices
    i. CLIVAR Working Group/Waliser et al. (2009); Gottschalk et al. (2010)
    1. Variance, single variable EOFs, temporal spectrum, lag-longitude, east-west power, wavenumber-frequency filtering, etc.
    ii. WH04 RMM, OMI, Skeleton index, etc.

c. Mechanisms
    i. Charge-Discharge
    1. Not present in DYNAMO, saw perpetual shallow convection
    ii. Air-sea interactions
    iii. Role of cloud populations
    1. MSE increases from radiation interactions and reduced LW cooling
    iv. MJO initiation mechanisms are specific to each event
    v. Initiation will also be dependent on low frequency variability
    vi. May have differences in winter and summer MJO events

3. Challenges and Recommendations

a. We need a new index for local aspects of the MJO and convective initiation

b. Should include and test all mechanisms in tractable models
    i. Recommend a model intercomparison study of primary MJO initiation (separate for GCMs, CRMs, LES, idealized models), perhaps similar to the second objective of the GASS-YoTC Vertical Structure and Physical Processes (secondary MJOs) with high-frequency output at each timestep
    ii. Synthesis study documenting frequencies of most common initiation mechanisms (climatological frequency)
    1. Mechanism identifiability is a challenge in incomplete data
    2. Would need to set up a hierarchy for determining primary mechanisms
    iii. Studies on MJO initiation failure

c. Multiscale interactions for initiation remains a challenge

d. Need comparison observations for models of different complexities

Discussion from Thursday, April 30, 2015

1. State of the science/Potential mechanisms for MJO initiation

a. Moisture mode and cloud-radiation interactions

b. Global mode

c. Recharge/discharge

d. Air-sea coupling

e. Upscale energy transport

f. Extratropical forcing

g. High-frequency stochastic synoptic forcing

2. Concerns with indices

1. Recommend additional research into the development of initiation indices

2. Different models will have different features based on their resolution
   a. Some may have heating variables
   b. CRMs allow you to track the convection by rain rate or precipitation
   c. Regional models cannot do the RMM index because they require information on the winds and variance over a global domain
   d. GCMs will not be able to do precipitation
   e. Common observed MJO features
   i. Need to have propagation, near equator, and be based on physical data

3. Potential for future work

a. MJO initiation with several known mechanisms
   i. Comparison of relative importance
   ii. How to determine primary mechanism?
   1. Likely cannot do with observations alone (arguing over what initiated DYNAMO MJOs, which are among most well observed)
   2. Would need to do statistically in models (have controlled variables)
   iii. Use of virtual field campaigns (YOTC) and high-resolution datasets (existing and forthcoming global high resolution datasets)
   iv. Instead of determining the one mechanism that is most important, can we better synthesize all of our work and get a statistical picture of the conditions associated with MJO initiation?
   v. Do we have well-defined necessary or sufficient conditions

b. Initiation of suppressed phase and long-term quiescent MJO conditions
   i. We can detect gradual deepening of the mixed layers (from 400-500 m to 700 m) during the suppressed phase, resulting in more plumes hitting the LCL and cloud layer

c. Case studies of failed MJO event initiation
   i. Basic question: is the MJO always present or does it only organize into discrete events with significant amplitude and/or projection on our indices?
   ii. 2014: The Year without an MJO.
   iii. Saw lots of rainfall, but didnt see any propagation in precipitation features from Hovmöller plots

d. Better understanding of MJO termination
   i. Often see deeper cold pools with longer recovery time (up to 30 hrs) during suppressed phase (drier air makes cold pools even stronger)
   ii. Dependencies on large-scale environment such as cold SSTs, large-scale and long-term drying, or even MJO strength as to whether it can maintain itself when propagating over the maritime continent

e. Continuum of Kelvin wave and MJOs needs to be realized/sorted
   i. Also have a continuum of westward moving waves
   ii. Typically only see one well-defined Rossby gyre event per year
   iii. Seasonality of the MJO means usually only one of the two Rossby waves is favored (i.e., tilted wave structures)

f. Additional comments about stochasticity and organization

4. Reviewing motivations

a. Why do we care about initiation?
   i. Operational community/forecast perspective: Models struggle most with the initiation aspect of MJO events and do ok with its maintenance and propagation once developed (similar to tropical cyclones)
   ii. If we improve MJO prediction, we improve mid-range weather forecasts
   iii. MJO generates tropical cyclones along the way in the Indian Ocean in addition to affecting TC activity in other basins (i.e., modulation of Atlantic hurricane activity)

b. How do we improve numerical models?
   i. How to implement MJO initiation mechanisms and put them into models?
   ii. Likely need to adjust multiscale interactions and parameterization (e.g., research has shown that the mid-level moistening in models is often insufficient compared to observations)
   iii. Essentially, need to better address upscaling from convection

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Chapter 13

Groups and Geometries (15w5017)

May 3 - 8, 2015

Organizer(s): Inna Capdeboscq (University of Warwick), Martin Liebeck (Imperial College London), Bernhard Mühlherr (Giessen)

Overview of the field, and recent developments

As groups are just the mathematical way to investigate symmetries, it is not surprising that a significant number of problems from various areas of mathematics can be translated into specialized problems about permutation groups, linear groups, algebraic groups, and so on. In order to go about solving these problems a good understanding of the finite and algebraic groups, especially the simple ones, is necessary. Examples of this procedure can be found in questions arising from algebraic geometry, in applications to the study of algebraic curves, in communication theory, in arithmetic groups, model theory, computational algebra and random walks in Markov theory. Hence it is important to improve our understanding of groups in order to be able to answer the questions raised by all these areas of application.

The research areas covered at the meeting fall into three main inter-related topics.

Fusion systems and finite simple groups

The subject of fusion systems has its origins in representation theory, but it has recently become a fast growing area within group theory. The notion was originally introduced in the work of L. Puig in the late 1970s; Puig later formalized this concept and provided a category-theoretical definition of fusion systems. He was drawn to create this new tool in part because of his interest in the work of Alperin and Broué, and modular representation theory was its first berth. It was then used in the field of homotopy theory to derive results about the p-completed classifying spaces of finite groups. Lately, fusion systems have been very successfully adopted by finite group theorists. When finite groups are considered in a category of (saturated) fusion systems, it turns out the proofs of some results in this field can be obtained in a more direct fashion.

Among all recent applications of the theory of fusion systems, the program laid out by Michael Aschbacher stands out as one of the most promising. The goal of this program is to simplify the existing proof of the classification of finite simple groups. A large part of the original proof of the Classification Theorem addresses the so-called finite simple groups of component type. One of the main difficulties that arises here is rooted in the possible existence of 'cores' of the 2-local subgroups of an ambient finite simple group. Aschbacher's program proposes instead to study saturated simple 2-fusion systems of component type, where the issues associated with cores do not exist. The aim is then to use this new classification to in turn obtain a novel, more straightforward method of classifying finite simple groups of component type.

Buildings and groups

Group theory studies the symmetries of objects of various kinds. Historically, one of the main sources for such objects are geometric and combinatorial structures. Indeed, in his famous *Erlanger Programm* Felix Klein proposed to investigate geometries via their group of symmetries. This principle has been most fruitful over the last century and led to deep insights. People then began to realise that it works also in the opposite direction: in order to understand a given abstract group it is helpful to construct combinatorial objects on which the group acts in a natural way. One of the most beautiful examples of an application of this reversed form of the Erlanger Programm is Tits' notion of a building. To any semi-simple, isotropic algebraic group Tits associates a building on which the group of rational points acts strongly transitively. For this reason 'most' of the finite simple groups act strongly transitively on buildings which explains the special link between the finite simple groups and this outstanding class of combinatorial objects. But this is by far not the only bridge between finite group theory and geometry. There are questions about infinite geometries which can only be settled by using deep results from finite group theory and in particular the classification of finite simple groups. Questions about generalizing a known result about finite groups to the infinite case often have a geometric flavor. In these cases one can hope that combinatorial techniques also apply in a more general context. Thus, there is already a long history of fruitful interactions between group theory and geometry, in particular buildings, which continues to provide new inspiration and challenges for both areas.

During the last decade there have been several major developments which are based on the interplay of the theory of buildings and finite groups. One of them is the investigation of Moufang sets. The classification of finite Moufang sets was accomplished by Hering, Kantor and Seitz in 1972 [20]. It is one of the most challenging questions whether all proper infinite Moufang sets are of algebraic origin. The fact that one has to require 'proper' was already suspected for a long time, but definitively confirmed only recently by the construction of non-split infinite sharply 2-transitive groups by Rips, Segev and Tent. Although there has been a lot of activity in the area, the question is still wide open. However, this problem inspired Caprace, De Medts and Grüninger to investigate Moufang sets from the perspective of locally compact groups which culminated in the result that a group which acts on a locally finite tree with a abelian Moufang set at infinity is provided by Bruhat-Tits theory and hence of algebraic origin.

Another mainstream topic in the area of buildings and groups is provided by Kac-Moody theory. Indeed, it was observed by Rémy [25] and independently by Carbone and Garland [6] that the buildings associated to Kac-Moody groups over a finite field provide most interesting examples of irreducible lattices in the automorphism group of the product of two locally finite buildings. Using this fact, Caprace and Rémy have been able to show that Kac-Moody groups over finite fields are simple groups, and it is still open whether this is also true over infinite fields. Currently this direction is a very lively area of research, because there many results about groups of Lie type which might be extendable to the Kac-Moody situation, but also new ones which are relevant for the theory of locally compact groups.

A somewhat different mainstream topic in the area over the recent decade has its origin in Serre's notion of complete reducibility in spherical buildings. Through this notion Tits' center conjecture from the 1960s became prominent and its proof was accomplished by Ramos-Cuevas in 2012. Ramos-Cuevas' arguments are based on a sophisticated analysis of buildings of exceptional type E_8 and the attempt to simplify it led to whole new understanding of exceptional Moufang buildings from geometric point of view. But there are also other important directions which aim for a better understanding of exceptional groups and their geometries. From a combinatorial point of view there is Van Maldeghem's program to study the algebraic varieties related to the Freudenthal-Ttis magic square, and from the point of view of Lie algebras there is Cohen's program to study Lie algebras via extremal elements.

Finite groups and related algebras

There are several strands of modern research involving finite groups. One is the theory surrounding the classification of finite simple groups, some of which was discussed in the previous sections. Others involve studying properties of the finite simple groups themselves, particularly their representations and their subgroup structure. For the latter, the maximal subgroups of the finite simple groups are of special interest, since any primitive permutation action of a finite group is just an action on the coset space of a maximal subgroup. Many applications of the classification of finite simple groups have been achieved using the theory of maximal subgroups, since automorphism groups of interesting combinatorial structures, Galois groups of interesting field extensions, and so on, can often be shown to act as primitive permutation groups.

Another fruitful way to study finite simple groups is via various algebras on which they act. For example, many of the groups of Lie type act as automorphism groups of simple Lie algebras. As for the 26 sporadic simple groups, 21 of them are contained in the largest one – the Monster – which was constructed as the automorphism group of the Monster Algebra, a non-associative real algebra of dimension 196883. A great deal of recent effort has gone into creating a general theory of such algebras, into which the Monster Algebra will fit as a particular case.

Presentation Highlights and Scientific Progress

Fusion systems and finite simple groups

There were a number of interesting talks given on the subject of fusion systems and their applications to the study of finite groups and on the continuing study of finite simple groups.

Michael Aschbacher talked about his ongoing project on the use of fusion systems to simplify the existing proof of the classification of finite simple groups. The focus of his lecture was a study and classification of the so-called *quaternion fusion packets*: the pairs $(\mathcal{F}, \Omega)$ where $\mathcal{F}$ is a saturated 2-fusion system and Ω is an $\mathcal{F}$-invariant set of subgroups that satisfy conditions that are quite similar to the hypotheses in his celebrated work on classical involutions [1, 2]. Among the examples of quaternion fusion packets are those coming from the fusion systems of finite groups of Lie type over the fields of odd order.

Ellen Henke discussed *linking systems*. This notion was originally introduced in the work of Broto, Levi and Oliver; it proved a useful tool to allow them to study the classifying spaces of fusion systems. The original notion was modified in the subsequent works of Broto, Castellana, Grodal, Levi and Oliver. In her talk Henke proposed a new notion of a linking system that allows her to show that there is a unique linking system associated to each fusion system whose objects are the subcentric subgroups, and that the nerve of such a subcentric linking system is homotopy equivalent to the nerve of the centric linking system. The existence of subcentric linking systems seems to be of interest for a classification of fusion systems of characteristic p-type. Linking systems also featured in Robert Oliver's lecture. In his presentation, he discussed his work on the automorphisms of fusion and linking systems of finite groups of Lie type.

Sejong Park talked about the cohomology of fusion systems. A celebrated theorem of Mislin shows that an isomorphism on mod-p cohomology implies control of p-fusion among compact Lie groups, and in particular among finite groups. Park discussed how this result can be generalised and proved in the setting of saturated fusion systems.

Gernot Stroth spoke about an application of his recent paper [24] with U. Meierfrankenfeld and R. Weiss. The highlight of the talk was showing that if a finite group G of parabolic characteristic 2 contains a subgroup H of odd index and $F^*(H) \cong \Omega_8^+(2)$, then either $F^*(G) \cong \Omega_8^+(2)$ or $\Omega_8^+(3)$.

Buildings and groups

The presentations concerning the interplay between buildings and groups can be roughly subdivided into three directions.

Kac-Moody groups and locally finite trees

As mentioned before, it is a prominent open question whether Kac-Moody groups over infinite fields are simple. A Kac-Moody algebra over the real numbers has a so-called 'compact form'. In [10] and [11] physicists discovered that the compact form of E_{10} admits a finite-dimensional representation, a result which was somewhat surprising. Max Horn explained this phenomenon in his talk. In fact, it is possible to produce finite-dimensional representations of the compact form of any symmetrizable real Kac-Moody algebra. As a special instance one can produce generalized spin representations for algebras of type E_n for any n. The resulting quotients are compact, whence reductive and often even semisimple. Cartan-Bott periodicity enables one to determine the isomorphism types of these quotients in this special case. As a result, this leads to a more conceptual approach to the representations discovered in the references above. It turns out that that Kac-Moody buildings play a key role in the development of the theory of spin representations. Indeed, one knows that there is a Curtis-Tits presentation for 2-spherical Kac-Moody groups and their compact forms. In order to construct the desired representations one only has to check the Levi-factors of rank smaller than 2. It turns out that one has to work with central covers of the Levi-factors in order to make things work and this is settled with Tits' theory of extended Weyl groups [27]. Thus the theory of these generalized spin representations is based on a local to global principle for Kac-Moody groups where one takes advantage of the fact that the local information is provided by the classical theory. A similar idea provides also the motivation of the theory of locally grouped spaces presented by Andrew Chermak. The fusion systems of finite groups of Lie type are well understood and therefore it is natural ask whether Kac-Moody groups over finite fields might provide other interesting examples of fusion systems which would provide new insights. Chermak's program aims for a better understanding of the unipotent radicals of the Borel subgroups in Kac-Moody groups. As already mentioned, Kac-Moody groups over finite fields provide interesting examples of irreducible lattices in the automorphism group of a product of two locally finite buildings. In this context, these unipotent radicals have been intensively studied but they still remain rather mysterious. Any new approach to improve our understanding of these groups is most welcome. The ideas presented by Chermak provide a new link between finite group theory and Kac-Moody theory which is very promising. Kac-Moody groups and their buildings played also a central role in the talk of Matthias Grüninger in which he presented a result about groups acting on locally finite trees which induce an abelian Moufang set at infinity. His result can be seen as an analogue of an earlier result in [5] of Caprace and De Medts. In [5] the characteristic 0 case was settled, and this work relies on heavy machinery from the theory of p-adic analytic groups. In his talk, Grüninger described a completely different strategy in positive characteristic. The idea is to produce an RGD-system inside the group in question. RGD-systems were introduced by Tits in [28] in order to describe the systems of root-groups inside a group of Kac-Moody type. In general there is a Moufang twin building associated to such an RGD-system which is a Moufang twin tree in the case considered here. The end-game now consists of checking which Moufang twin trees provide an abelian Moufang set at infinity. This is not easy but can be achieved by elementary arguments. Groups acting on trees where also considered in the the joint talk of Pierre-Emmanuel Caprace and Nicolas Radu from a different perspective. The main goal of their presentation is motivated by the structure theory of simple locally compact groups, of which a large class is provided by groups acting faithfully on locally finite trees. For a given tree they introduce a topology on a the set of isomorphism classes of a large family of simple groups acting on it and raise several natural questions. One among those is whether on can describe limit points of known isomorphism classes of such groups, and they give a beautiful answer to that question for rank 1 groups over local fields. It turns out that by increasing the ramification index in characteristic 0 one obtains the groups over the Laurent series as a limit.

Exceptional groups and algebras

The interplay between buildings and algebraic structures is particularly fruitful in the exceptional cases G_2, F_4, E_6, E_7, E_8. This is not at all surprising because the theory of buildings was created by Tits in order to have an additional tool for investigating the Lie algebras and groups of exceptional type. Several lectures given at conference underlined that the theory of buildings plays a central role for exceptional structures. Richard Weiss presented in his talk a uniform approach to several classes of exceptional Moufang buildings by constructing them as fixed point buildings of Galois-involutions of higher rank buildings. This is an application of a more general theory of descent for Moufang buildings. A remarkable aspect of this construction is the fact, that it provides a natural link between the theory of algebraic groups and the classification of Moufang polygons. More precisely, the existence proofs for the exceptional Moufang buildings that have been known up until now were based on Galois cohomology of algebraic groups on the one hand, and on explicit calculations in their coordinatizing structures described obtained in [29] on the one the other. Through this new approach to exceptional geometries one has a better understanding of how results from the theory of algebraic groups have to be interpreted in the context of Moufang buildings, and there is reasonable hope that these new insights will be useful in the investigation of Moufang sets. The geometries constructed in Weiss' talk actually represent the most interesting entries in the Freudenthal-Tits magic square. The latter was also considered in the talk of Hendrik Van Maldeghem from a different perspective. A couple of years ago he and his collaborators started a promising program aiming for a uniform characterization of several algebraic varieties related to the magic square. Crucial for this program is a functor which associates to a class of quadratic spaces over a field a Veronese representation. In his talk he focused on the case where the quadratic forms in question are defective. The outcome is that the functor yields several buildings of mixed type. The latter are no longer associated to reductive, but to pseudo-reductive groups which have been recently studied intensively in [7]. The Veronese representations in Van Maldeghem's program yield characterizations of several geometries of exceptional type. An alternative characterization of exceptional buildings is provided by the theory of root filtration spaces and a beautiful application of those has been provided in the talk of Arjeh Cohen. In his talk he considered Lie algebras generated by extremal elements. The latter play a central role in the classification of simple Lie algebras in positive characteristic $p \geq 5$. As an application of root filtration spaces he obtains a characterization of the classical Lie algebras among those which are generated by extremal elements. The idea is to construct out of simple Lie algebra a line-space whose points are the one-dimensional subspaces generated by extremal elements. If this space satisfies the axioms of a root filtration space, then one can reconstruct the building and deduce that the algebra is classical. Apart from the three contributions described so far, there were two further talks on groups and algebras of exceptional type given by David Craven and David Stewart which did not have any immediate connection to buildings and which will be mentioned in more detail in the following section.

Groups and graphs

As buildings can be interpreted as edge-colored graphs, the latter represent a more general class of combinatorial objects which provide a powerful tool to study groups. Indeed, graph-theoretical methods played a major role in the talks presented by Cohen, Van Maldeghem and Weiss which have been already mentioned. One prominent open question in the area of groups and graphs is a conjecture stated by Weiss in 1978 [30]. Luke Morgan gave an overview about the recent developments concerning Weiss' conjecture and presented a new result about locally semiprimitive arc-transitive graphs. A purely combinatorial result about graphs was the subject of Jeroen Schillewaert's talk. He described a probabilistic approach for proving the existence of certain substructures (partial spreads and ovoids) of classical geometries which are usually obtained by algebraic methods. The surprising aspect of his result is that – at least asymptotically – the bounds obtained by these methods are much better than the ones obtained so far.

Finite groups and related algebras

A wide range of topics involving finite groups was covered. First, as mentioned in Section 1.3, maximal subgroups of finite simple groups are a particular focus, and David Craven gave a lecture *Maximal subgroups of exceptional groups of Lie type*. The theory of maximal subgroups of the alternating and classical finite simple groups is is a reasonably complete state, but the same cannot be said for the exceptional groups of Lie type. For the latter, results of Liebeck and Seitz [22, 23] reduce the study to the maximal subgroups which are themselves simple groups, and moreover provide an absolute bound on the order of those potential simple subgroups that need to be considered. Craven announced some new results that determine many of these small maximal subgroups, the first substantial progress in this area for a number of years. Following the lecture, discussions with Praeger, Morgan, Giudici and Liebeck led Craven to the proof of an additional result that, together with a theorem of Lusztig, shows that the alternating group of degree 5 is never a maximal subgroup of an exceptional group, a result which has now been used in an application to the study of multiply arc-transitive graphs. Cheryl Praeger's lecture *Classifying the finite 3/2-transitive permutation groups* announced the completion of a long-term project to determine the 3/2-transitive groups – that is, the transitive permutation groups for which a point-stabilizer has orbits of equal size on the remaining points. This class of permutation groups includes Frobenius groups and 2-transitive groups, and the techniques for their classification again involve the theory of maximal subgroups, together with a substantial amount of representation theory.

As mentioned in Section 1, a fruitful way to study groups is via various structures and algebras on which they act, and there were quite a number of lectures along these lines. In his lecture *Finite subgroups of diffeomorphism groups of a compact manifold*, Laci Pyber discussed a 20-year old conjecture of Ghys (see [12]) which states that if M is a compact smooth manifold, then every finite subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ has an abelian subgroup of index at most $f(M)$, where $f(M)$ depends only on the manifold M. The conjecture was motivated by the famous theorem of Jordan showing that it is true when M is complex n-space. Over the years it has been proved in several special cases, so it came as quite a surprise when Pyber announced a counterexample to the conjecture. This has led to some weaker positive results, and a modification of Ghys's conjecture which still remains open. Another variation on the theme of groups acting on combinatorial structures came in the lecture of Nick Gill, *Constructing groupoids using designs*. This built on a famous example of John Conway [9] in which he used the projective plane of order 3 to construct the Mathieu group M_{12} and also a related groupoid that he called M_{13}. By considering the same construction, replacing the projective plane by more general Steiner systems with blocks having 4 points, Gill showed how to construct whole families of groupoids that generalize M_{13} and are related to classical groups over small fields. Concerning the representation theory of finite groups, Ron Solomon (*Recognizing abelian and nilpotent Hall subgroups from the character table*) answered part of an old question of Richard Brauer, showing how the character table of a finite group can be used to determine whether or not it has abelian Sylow p-subgroups for some prime p.

The Monster sporadic group was constructed by Griess [19] as the automorphism group of a 196883-dimensional real algebra now known as the Monster algebra. In the last decade or so, Ivanov and others have introduced the theory of Majorana algebras [21] which attempts to develop a general theory of algebras in which the Monster algebra is a special case. In their lectures, Sergey Shpectorov (*Axial algebras and groups of 3-transpositions*) and Tom de

Medts (*Jordan algebras and 3-transposition groups*) discussed variations and generalizations on this theme, showing the potential richness of this line of investigation by demonstrating beautiful connections with the theories of Jordan algebras and 3-transposition groups. On the more classical topic of Lie algebras, David Stewart (*Maximal subalgebras of the exceptional Lie algebras in good characteristic*) announced new results on simple subalgebras of exceptional Lie algebras in positive characteristic, showing that even though there are many non-classical types of simple Lie algebras in such characteristics (see [26]), only the Witt algebra among these can occur in an exceptional Lie algebra in good characteristic. This potentially opens the way to a new detailed study of subalgebras that previously seemed intractable. Arjeh Cohen's lecture *Lie algebras generated by extremal elements* provided a link with the theory of buildings, showing that simple Lie algebras generated by extremal elements have an embedded geometry that is the shadow of a building, and moreover that this geometry determines the Lie algebra uniquely.

Outcome of the Meeting

Fusion Systems

Fusion systems play an increasing role in our understanding of the finite simple groups. This was already visible in the Banff 'Groups and Geometries' conference in 2012 and was confirmed by the multiple talks on this subject given during this conference. Following the presentation of Aschbacher and of his new groundbreaking results, we can be optimistic that the study of fusion systems will lead to more straightforward proofs of substantial parts of the classification of finite simple groups.

The new results of Henke are a very promising fresh tool in the program introduced by Meierfrankenfeld, Stellmacher and Stroth, of which the goal is to classify the finite simple groups of characteristic p-type. Fusion systems are also of course interesting to study in their own right, and the talks of Oliver and Park demonstrated new developments in the area itself.

Chermak's program appears to be a promising new direction in the theory of fusion systems. This program focuses on the investigation the unipotent radicals of Borel subgroups in Kac-Moody groups over finite fields from the point of view of fusion systems. These unipotent radicals naturally belong to the theory of buildings. This program hints at the possibility of the use of buildings in a new context, thus providing a novel perspective on fusion systems associated with finite groups of Lie type.

A discussion of Stroth and and Parker followed as a development of Stroth's talk. It allowed both to improve the results presented in Stroth's talk. Furthermore, Aschbacher and Stroth had some discussions on a possible fusion system classification.

Revision of the classification

Now that the first generation proof of the classification of finite simple groups is complete (cf. [3]), there are currently two very active mainstream areas of research focused on producing new proofs of the classification theorem. One is the Gorenstein-Lyons-Solomon program (GLS): a number of the participants of this meeting are involved at various degrees in this effort, under the leadership of R. Lyons and R. Solomon. The aim of this program is to provide a self-contained proof of the classification in a series of eleven monographs. The first six volumes have been published in the AMS monograph series ([13], [14], [15], [16], [17], [18]), and substantial progress has been made towards the completion of several of the other volumes. Both Lyons and Solomon were present at the meeting: such interactions were extremely beneficial, particularly to the progress of volume 7. Some parts of this volume, and also some of the later ones, require numerous results on the identification of finite simple groups of Lie type. The most recent approach to this is the geometrically oriented recognition theorems of Phan-Curtis-Tits type. Numerous discussions involving Lyons and Shpectorov centred around results of this type took place during the meeting. The remaining "special even case" is currently under close scrutiny. Some partial results have already been published, notably the ones dealing with the subcase of finite simple groups of mixed characteristics. Volume 8 of the GLS-series is planned to be devoted to this part of the classification, and to contain this work. The remainder of the $e(G) = 3$ case is planned to be presented in volume 9 according to the current projections. The state of the art on this part of the project was discussed by Lyons and Solomon during the meeting together with other participants (I. Capdeboscq, C. Parker, K. Magaard).

Groups acting on trees

Bruhat-Tits buildings have been a focus of many previous conferences, in particular the Banff 2012 'Groups and Geometries' meeting. A new direction which was already initiated in the meeting of 2012 is the investigation of the one-dimensional case, i.e. the case of Bruhat-Tits trees. It is motivated by the attempt to improve our understanding of Moufang sets and important classes of locally compact groups. There is the prominent question of whether all proper Moufang sets are of algebraic origin. By the result presented in Grüninger's talk, it is reasonable to conjecture that this is indeed the case for any Moufang set at infinity of a locally finite tree, because it is known to be true if the root groups at infinity are abelian. Assuming that this conjecture holds one has the interesting phenomenon that the non-uniform irreducible lattices in the automorphism group of the product of two locally finite trees (see [25]) are arithmetic if and only if they induce a Moufang set at infinity. It would be most exciting to have a direct proof of this.

Geometric structures in positive characteristic

Several talks at the conference were concerned with a geometric approach to exceptional groups and algebras. The big advantage of using combinatorial arguments in this context is the fact that they apply equally well in positive characteristic, which often a major obstacle when using algebraic methods. The talks of Cohen, Van Maldeghem and Weiss highlighted this in context of the classification of simple Lie algebras, the 'degenerate cases' of the Freudenthal-Tits magic square and the construction of exceptional geometries. It would be most desirable and a perspective for future research to relate espcecially the latter two contributions with the work of Conrad, Gabber and Prasad on pseudo-reductive groups in [7] and [8].

Finite groups and algebras

Some very promising themes for future reaserch emerged from the talks of Shpectorov and de Medts. The theory of Majorana algebras [21] originated in vertex operator algebra theory, and was originally designed mainly to provide an axiomatic setting for the Monster algebra in which that group could be investigated systematically. By introducing the more general theory of axial algebras, and showing their surprising and beautiful connections with Jordan algebras and 3-transposition groups, Shpectorov, de Medts and their collaborators have uncovered a potentially very rich new area of research in the area of groups and algebras.

Final remarks

One of the foremost objectives of this meeting was to bring together junior and well established researchers working in the various different fields closely related to finite and algebraic simple groups. The structure of the conference was based on the assumption that developing, maintaining scientific exchanges between these connected areas was best achieved by creating new perspectives in research and stimulating scientific collaboration.

The lectures and the time schedule were designed by the organizers in accordance with these objectives. In the selection of the lectures, preference was given to subjects which offered the participants the possibility to learn about novel developments in an area. To foster productive interaction, the timetable was such that significant breaks were introduced between the talks.

The feedback of many participants to the organizers was very positive. The outstanding quality of the talks was often mentioned. Several new collaborations were started during the meeting, and ongoing ones found it the perfect setting to be continued. Beside the outstanding scientific level, the attendees particularly enjoyed the clarity of the lectures. The speakers paid special attention to explaining clearly the main ideas and avoiding technical details, which made these lectures profitable to all. It was also remarked positively, that there was a comparatively high number of young speakers and that all of them gave beautiful lectures.

Participants

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Capdebosq, Inna (University of Warwick)
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Chermak, Andrew (Kansas State University)
Cohen, Arjeh (Eindhoven)
Craven, David (University of Birmingham)
De Medts, Tom (Ghent University)
Devillers, Alice (University of Western Australia)
Gill, Nick (University of South Wales)
Giudici, Michael (The University of Western Australia)

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Kassabov, Martin (Cornell University)
Khl, Ralf (UniversitŁt Gieen)
Liebeck, Martin (Imperial College)
Lynd, Justin (Rutgers University)
Lyons, Richard (Rutgers University)
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Seitz, Gary (University of Oregon)
Shpectorov, Sergey (University of Birmingham)
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Chapter 14

Dispersive Hydrodynamics: the Mathematics of Dispersive Shock Waves and Applications (15w5045)

May 17 - 22, 2015

Organizer(s): Mark Ablowitz (University of Colorado - Boulder), Gino Biondini (State University of New York - Buffalo), Gennady El (Loughborough University), Mark Hoefer (University of Colorado - Boulder)

Overview of the Field

Dispersive hydrodynamics is the domain of applied mathematics and physics concerned with fluid motion in which internal friction, e.g., viscosity, is negligible relative to wave dispersion. In conservative media such as superfluids, optical materials, and water waves, nonlinearity has the tendency to engender wavebreaking that is mitigated by dispersion. The mathematical framework for such media can often be described by hyperbolic systems of partial differential equations with conservative, dispersive corrections that play a fundamental role in the dynamics. Generically, the result of nonlinearity and dispersion is a multiscale, unsteady, coherent wave structure called a dispersive shock wave or DSW. Over long time scales, multiple wavebreaking events or, in focusing media, modulational instability can lead to the development of soliton (strong) turbulence. This meeting brought together an international collection of mathematicians and physicists in order to identify common interests and emerging problems involving DSWs, soliton turbulence, and their mathematical description.

This field of research has origins in soliton theory, conservation laws, and fluid dynamics. In 1965, the seminal computational work of Zabusky and Kruskal [1] demonstrated the existence of soliton solutions to the Korteweg-de Vries (KdV) equation through a process of nonlinear wavebreaking. That same year, Whitham introduced a general asymptotic approach to study modulated periodic nonlinear dispersive waves [2, 3]. Both of these works considered conservative, nonlinear, dispersive wave problems. Again in 1965, although within the context of a different field of research, Glimm's fundamental work on hyperbolic conservation laws [4] contributed to the rapid growth in understanding of this field, see, e.g. [5]. The marriage of dispersive nonlinear waves and hyperbolic conservation laws in the context of dispersive hydrodynamics was initiated in 1974 at the hands of Gurevich and Pitaevskii [6] through their study of a Riemann problem regularized by dispersion in the KdV equation. The resulting dispersive shock waves were understood utilizing Whitham's modulation equations, later shown to describe the weak, zero dispersion limit of the KdV equation by Lax, Levermore, and Venakides [8, 9]. The KdV Whitham modulation equations are now known to be strictly hyperbolic and genuinely nonlinear [10], highlighting the deep connections between conservation laws and dispersive nonlinear waves in dispersive hydrodynamics. Another fundamental work was on averaging of multiphase solutions to the KdV equation utilizing finite gap theory [7].

An essential aspect of dispersive hydrodynamics is its physical realization. Laboratory measurements of DSWs were undertaken in the context of undular bores in shallow water waves by Favre in 1935 [11]. Applications to collisionless plasma in the 1960s [12, 13] motivated the early theoretical works [1, 6]. Additional geophysical applications include internal waves in the ocean and atmosphere (e.g., the Morning Glory). More recently, experiments in ultracold atoms [14, 15] and nonlinear photonics [16, 17] have inspired further mathematical study of dispersive hydrodynamics.

The unique challenges presented by nonlinear wavebreaking and dispersion have since been explored in a variety of ways. Methods include the Inverse Scattering Transform, finite gap theory, matched asymptotic expansions, Whitham modulation theory, PDE analysis, numerical simulation, and experiment. Fifty years after the seminal year 1965, the Dispersive Hydrodynamics workshop held at BIRS May 17-22 brought together practitioners from the dispersive waves, hyperbolic conservation laws, and experimental communities with the aim of addressing recent physical and mathematical developments in the field and to identify open problems, which are now described.

Recent Developments and Open Problems

There are many fundamental mathematical problems emerging in the study of multiscale, dispersive hydrodynamic phenomena. Some of these are listed below.

1. Analytical and numerical description of multidimensional DSWs in KP or other integrable and nonintegrable equations. An extension of the existing theory for one-dimensional DSWs to two spatial dimensions is a long-standing problem posing a number of mathematical challenges. Despite recent progress in the understanding of pre-breaking KP dynamics as well as careful numerical simulations of some multidimensional dispersive regularization problems, breakthroughs in this area still lie ahead. A closely related direction which has been under active development in recent years is the theory of integrable hydrodynamic type systems in higher dimensions.

2. Modulation theory for hydrodynamic systems with non-strict hyperbolicity/linear degeneracy regularized by dispersive terms. The majority of existing DSW studies are related to systems which can be characterized as genuinely nonlinear, strictly hyperbolic conservation laws modified by small dispersion terms. The DSWs generated in such systems represent dispersive counterparts of classical, Lax shocks. At the same time, there exists a rich mathematical theory of nonclassical shock waves in hydrodynamic systems lacking genuine nonlinearity and/or strict hyperbolicity. The study of purely dispersive counterparts of nonclassical, dissipative shock resolutions is one of the outstanding problems in dispersive hydrodynamics. There have been several recent important works in this direction but a unified mathematical framework is yet to be developed.

3. Rigorous analysis of DSWs and their interactions via the inverse scattering transform (IST). Along with providing rigorous justification of the results from modulation theory, the IST method yields the description of many subtle details not captured by the formal modulation analysis. A number of fundamental results have recently been obtained in the rigorous description of DSW generation and interaction problems for the KdV and defocusing NLS equations. The extension to other integrable systems (modified KdV, vector NLS, etc.) is needed.

4. Incoherent wave structures in dispersive hydrodynamics. This is a new, rapidly developing direction involving the mathematical description of a range of macroscopically incoherent wave structures in dispersive hydrodynamics. The novel lines of investigation include integrable turbulence, soliton gases, incoherent DSWs, effectively viscous fluid dynamics in multidimensional dispersive hydrodynamics systems, and nonlinear wave counterparts of quantum mechanical effects in disordered media (e.g. Anderson localization). A closely related range of dispersive-hydrodynamic problems attracting recent interest concerns the formation of rogue waves in shallow and deep water as well as in nonlinear optical systems.

5. Universality of wave breaking in dispersive hydrodynamics. The rigorous mathematical description of the emergence of a DSW in the space-time vicinity of a gradient catastrophe is of fundamental importance for the foundations of dispersive hydrodynamics. There has been notable progress in proving Dubrovin's Universality conjecture for a number of integrable systems, with many deep, fascinating results obtained in this direction. There are still many unanswered questions, especially related to Universality in non-integrable (including viscously modified) systems.

6. DSWs in focusing media, in nonlocal media and in systems with higher order and nonlinear dispersion. DSW phenomena are not limited to hyperbolic systems modified by weak dispersive terms. The dispersive resolutions of gradient catastrophes can also occur in focusing media as well as in media characterised by nonlocal nonlinearities and higher order dispersive mechanisms. The relevant mathematical models include the focusing

NLS equation and its modifications due to non-locality, as well as systems with higher order/nonlinear dispersion arising in nonlinear optics, shallow-water dynamics and other areas. The main mathematical approach in the analysis of the dispersive regularization of gradient catastrophe in the integrable focusing NLS equation is the powerful Riemann-Hilbert steepest descent method, which has enabled a number of recent significant advances in the analytical description of semi-classical "elliptic" dynamics. The outstanding problems in this area include the investigation of higher-order breaking dynamics for analytical data and the evolution of non-analytic data. Of special interest are the recently proposed possible connections between the generation of multiperiodic breather type structures in the resolution of focusing singularities and rogue wave formation. The description of DSWs in nonlocal media (e.g. in liquid crystals) and in systems with higher order/nonlinear dispersion (e.g. fully nonlinear shallow water, nonlinear optics with higher order dispersion) is of great importance for applications and has only recently begun to be explored, posing a number of challenging mathematical questions.

7. Quantitative comparison of results of DSW experiments with existing theories. Despite the numerous existing theoretical results in the area of DSWs, the quantitative experimental verification of existing theory is presently lacking. There are only a handful of experimental results confirming certain features of DSW dynamics but no detailed comparison is available. Such a comparison, however, is vital for the further development of dispersive hydrodynamics as a fundamental discipline that stands on its own much like classical, viscous fluid dynamics.

Presentation Highlights

The meeting was organized around a general theme each day. A discussion of the highlights of the presentations on each day follows.

Day 1: Inverse Scattering Transform and analysis related to DSWs

The presentations on Day 1 mostly encompassed rigorous approaches to the study of dispersive hydrodynamics. These works can be divided into two classes, general methods encompassing non-integrable equations and methods applicable to integrable systems. First, the general methods.

Boris Dubrovin began the meeting with an overview of his general approach to wavebreaking in perturbed Hamiltonian systems. This Universality Conjecture that gradient catastrophe is dispersively resolved for short times by Painlevé transcendants, is an important short-time result for dispersive hydrodynamics. The rigorous proof of this conjecture has been completed for the integrable KdV and NLS equations, while there is numerical evidence of its validity in a wider class of equations [18, 19]. Utilizing Whitham averaging, Sergey Gavrilyuk identifies regions of hyperbolicity in a generalized p-system. He finds it useful to utilize mass-Lagrangian coordinates and derives the Whitham equations in a general form useful for analysis. These are important results for advancing a general approach, e.g., not reliant on integrability, to Whitham theory applied to dispersive hydrodynamics. Dispersive shock waves are difficult to study rigorously. For example, there are no existence results in non-integrable systems. Sylvie Benzoni-Gavage presented her work on a general criterion for the stability of periodic traveling waves in the Euler-Korteweg system, an analog of the general soliton stability criterion due to Grillakis, Shatah, and Strauss [20, 21]. As demonstrated previously, there is a direct connection between the hyperbolicity of the Whitham modulation equations and the spectral stability of periodic traveling waves [22].

The other analytical approaches to dispersive hydrodynamics principally involved integrable systems. In an effort to understand the dynamics of the Benjamin-Ono (BO) equation in the small dispersion limit, Peter Miller presented his work on estimating the direct scattering data for the BO equation in the small dispersion regime [23]. Ted Johnson utilized characteristic coordinates in the zero dispersion Ostrovsky equations in order to obtain precise conditions for wavebreaking via the zero of a Jacobian [24, 25]. There were four talks on Nonlinear Schrödinger (NLS) equations, all related to problems with nonzero boundary conditions at infinity. Utilizing the Riemann-Hilbert steepest descent method, Robert Jenkins considers the problem of an initial step for the defocusing NLS equation. The direct scattering is computed explicitly and the steepest descent method is used to provide error estimates that show that the well-known Gurevich-Pitaevskii self similar solution of the Whitham equations does indeed describe the leading order DSW dynamics. Barbara Prinari described the inverse scattering theory for defocusing, three-component (vector) NLS equations with nonzero boundary conditions at infinity [26]. She constructed dark-bright soliton solutions that exhibit a non-trivial polarization shift in the bright components much like bright soliton solutions of the focusing vector NLS equations. Gino Biondini and Dionyssis Mantzavinos presented their work on the focusing NLS equation with nonzero boundary conditions at infinity. Utilizing inverse scattering transform techniques, they identify the long time asymptotics, i.e., the nonlinear stage of modulational instability or what could be called elliptic (subsonic) dispersive hydrodynamics in contrast to the defocusing NLS equation that exhibits hyperbolic (supersonic) dispersive hydrodynamics.

Day 2: DSW experiments and physical applications

The presentations on Day 2 were principally organized around experiments and applications in dispersive hydrodynamics.

Stefano Trillo, one of the leading experimentalists in optics and dispersive hydrodynamics, began the day with a number of experimental results including laboratory observations of DSWs in water waves, optical DSWs exhibiting linear resonance [27], and incoherent optics involving a nonlocal NLS equation that exhibits a DSW in its Fourier transform [28]. In the water waves problem, Trillo showed comparisons between experiment and simulations of the KdV (long waves) and Whitham (full water waves dispersion) equations. The features of the small amplitude trailing edge of the DSW are particularly affected by the details of the dispersion with better agreement between experiment and the Whitham equation. The DSW-linear wave resonance originates from an inflection point in the dispersion relation, an important theme that will arise in other talks. In particular, Peter Engels presented experiments involving Bose-Einstein condensates (BECs), another medium that supports dispersive hydrodynamics. He demonstrated how, utilizing spin-orbit coupling, the dispersion relation can be engineered to have a more general form than the standard parabolic dispersion of "standard" BECs [29].

The remaining talks involved applications. There were four talks involving applications with NLS type equations. Two-dimensional dispersive hydrodynamics were investigated numerically by Arnaldo Gammal in the context of supersonic NLS flow past an obstacle, leading to the generation of oblique solitons, vortices, and linear bow waves [30]. Line dark solitons in the 2D defocusing NLS equation are known to exhibit a transverse instability. Boaz Ilan presented numerical computations involving supersonic flow in a confined channel where dark solitons are shown to be stable when sufficiently confined, having implications to confined, multi-dimensional dispersive hydrodynamics. Luca Salasnich presented his work on a generalized NLS model of a unitary Fermi gas that exhibits DSWs and compared these results with experiment [31]. There is some debate as to whether or not the unitary Fermi gas is effectively regularized by dispersion or dissipation [32]. A sequence of two talks by Gennady El and Alex Tovbis considered the small dispersion regime for the focusing NLS equation with a localized, "box" initial condition [33]. They compare the results of Whitham theory, the Riemann-Hilbert approach to inverse scattering theory, and careful numerical simulations, identifying DSWs and the local generation of approximate breathers that resemble structures studied in the context of rogue waves. Some of these elliptic dispersive hydrodynamics have similarities to hyperbolic dispersive hydrodynamics during the initial wavebreaking, DSW generation processes. However, the long time dynamics result in the generation of more and more phases in the modulated, quasi-periodic solutions. This is in contrast to the typical reduction in phases for the long time dynamics of hyperbolic dispersive hydrodynamics, e.g., for KdV and defocusing NLS.

Discussed so far have been approaches to dispersive hydrodynamics involving Riemann-Hilbert/steepest descent and Whitham averaging. Mark Ablowitz described another approach involving matched asymptotic expansions for the long time dynamics of the KdV equation with differing, constant boundary conditions at infinity [34]. He showed that, no matter how many phases are generated during intermediate times, the long time dynamics of this problem result in, at most, a single-phase DSW at large time plus radiation and solitons. This demonstrates the importance of DSWs as coherent solution components, along with dispersive radiation and solitons, of dispersive hydrodynamics. A talk by Guo Deng presented WKB expansions for the Schrödinger scattering problem in order to "count" the number of solitons in the seminal Zabusky-Kruskal computations [1]. The result was nine solitons. The talks were wrapped up by Naum Gershenzon who spoke on an application of the sine-Gordon equation and modulation theory to macroscopic friction.

The talks on Day 2 demonstrated the ubiquity of dispersive hydrodynamics in physical applications.

Day 3: connections between dispersive hydrodynamics and conservation laws

Day 3 was a half day of talks chiefly centered upon the connections between hyperbolic conservation laws and dispersive hydrodynamics.

Leading off the day was Philippe LeFloch, who provided an overview of rigorous and numerical methods for conservation laws regularized by dispersion and diffusion [35]. Such regularizations lead to microstructure near singularities and can have a significant impact on the uniqueness of solutions to dissipationless and dispersionless conservation laws. The emphasis in this talk was on the diffusion dominated regime, i.e., when the zero diffusion/dispersion limit exists in a strong sense, as opposed to the existence of only a weak limit in the zero dispersion limit case. Conservation laws with non-convex flux exhibit particularly interesting behavior, including shocks that do not satisfy the standard Lax-Oleinik entropy conditions. Rather, LeFloch seeks Riemann problem solutions that satisfy a single entropy inequality as well as a compatible kinetic function. This allows for uniqueness of solutions. Solutions in these systems can include double wave structures and undercompressive shocks. Utilizing these ideas, LeFloch describes numerical schemes that satisfy these requirements to a given order in the grid spacing, yet do not need to resolve the microscale generated by the higher order dispersive terms [36].

An alternative method by which to resolve diffusive-dispersive models was described by Michael Shearer [37]. He focused upon the modified KdV-Burgers equation by considering traveling wave solutions. A complete classification of the Riemann problem can be carried out. The interest here is in the relationship between the diffusion dominated traveling wave analysis and the dispersion dominated regime resulting in unsteady DSWs. A complete classification of the Riemann problem for the dissipationless modified KdV equation can also be carried out using Whitham theory. By direct comparison, Shearer identifies kinks as the purely dispersive analog of diffusive-dispersive undercompressive shocks. A new type of DSW is also identified in the mKdV equation, termed a contact DSW (CDSW) due to the propagation of its soliton edge with the local characteristic wave speed. A multi-valued mapping between diffusive-dispersive and purely dispersive Riemann problems showcases the similarities and differences between the two.

Antonio Moro presented a novel approach to understanding phase transitions by interpreting the order parameter of thermodynamic theories as the solution of a dissipatively and/or dispersively regularized conservation law [38, 39]. This yields powerful, direct relationships between shock waves and phase transitions. For example, the triple point between a solid, liquid, and gas can be identified with the merger of two shock waves. Moro goes on to describe a number of specific models that can be analyzed using conservation law methods. He ends his talk with a discussion of complex/disordered systems, e.g., spin glasses, that may be understood utilizing dispersive or diffusive-dispersive regularizations of conservation laws.

The day was completed by Christian Klein with a talk on numerical studies of DSWs in 1D [40] and 2D equations that exhibit blow-up. DSWs can be viewed as the dispersive regularization of singularity formation (gradient catastrophe). A different type of blow-up singularity that is not regularized by dispersion can occur in critical and supercritical dispersive wave equations. The competition between these two mechanisms was described by numerical computations principally of the generalized KdV and generalized KP equations. A key result was the observation that, for the initial data considered, the blow-up time was always greater than the critical time for gradient catastrophe.

Day 4: soliton turbulence experiments in water waves, optics; dispersive hydrodynamic applications in classical fluids

The final day of the meeting was broken up into two themes. The morning talks emphasized experiments and theory for soliton turbulence. The afternoon talks mainly involved the dispersive hydrodynamics of classical fluids.

Soliton turbulence is the branch of dispersive hydrodynamics that seeks to characterize the statistical properties of a complex, incoherent collection of solitons in a dispersion dominated medium. Alfred Osborne led the morning talks with a presentation on ocean water wave observations that exhibit statistical features consistent with a soliton gas [41]. In particular a $1/f$ dependence of the spectrum on the frequency f is consistent with a KdV soliton gas. In order to carry out this analysis, Osborne has developed numerical tools to analyze time series data using finite gap theory. A novel feature that he has recently observed is that of a gas of focusing NLS breathers. Breathers are commonly identified with rogue wave applications. The next talk by Pierre Suret presented experiments and numerical simulations of random initial waves propagating in a single mode optical fiber accurately modeled by the focusing NLS equation [42]. The experimentally observed, heavy tailed probability density function is shown through numerical simulations to be related to the generation of coherent structures such as breathers. Stéphane Randoux followed with a presentation on intermittency in an optical fiber experiment modeled by the defocusing NLS equation [43]. In contrast to the previous talk, here the probability density function of the incoherent wave amplitude is found to decay faster than typical Gaussian statistics. This phenomenon is identified with intermittency. The final morning speaker presented observations in nonlinear optics of shallow water wave type dynamics. Here, the propagation of a chirped pulse in a normally dispersive fiber (modeled by the defocusing NLS equation) is dominated by the nonlinear, convective terms, i.e., dispersionless terms. The dynamics of the system are well captured by the dispersionless shallow water equations. This result identifies the pre-wavebreaking connection between conservation laws and dispersive hydrodynamics.

The afternoon session focused upon the dispersive hydrodynamics of classical fluids. The first pair of talks by Michelle Maiden and Mark Hoefer involved two Stokes fluids with high viscosity contrast. When a lighter, less viscous fluid is injected from below into a vessel filled with a heavier, more viscous fluid, a stable fluid conduit can form. The interfacial dynamics of this conduit are dominated by nonlinear self-steepening due to buoyancy and nonlinear, nonlocal dispersion due to viscous interfacial stresses. These two features are the key components for dispersive hydrodynamics. Maiden presented a live demonstration and laboratory experiments of solitons, DSWs, and their interactions in this viscous fluid conduit system. The results demonstrate the versatility of this model dispersive hydrodynamic system. Soliton refraction by a DSW and DSW merger/refraction by another DSW are experimentally demonstrated. Hoefer followed Maiden's talk with a theoretical description of the system [44, 45]. Depending on the wavenumber, either the defocusing or focusing NLS equations describe the envelope dynamics of weakly nonlinear, periodic waves. Periodic solutions of the long-wave conduit equation were considered in the context of Whitham theory whereby the modulation equations were shown to be hyperbolic or elliptic depending on the wavenumber and amplitude of the carrier wave. The sign of dispersion is shown to change dynamically in the process of DSW formation whereby a multi-phase, implosion region is generated. This demonstrates the role of dispersion convexity, building on previous talks where the flux convexity played an important role. The next talk by Gavin Esler provided an analysis of the Riemann problem to the Miyata-Choi-Camassa system of equations modeling a Boussinesq two-layer fluid [46]. Using Whitham-El DSW fitting theory, the internal wave dynamics are shown to exhibit solibores (dispersive analogs of undercompressive shocks) and double wave structures. This and the previous talks on viscous fluid conduits, provide examples of the utility of Whitham theory as applied to non-integrable equations. The peculiar features of non-convex flux and/or non-convex dispersion can lead to novel structures.

Karima Khusnutdinova presented work on stratified fluids related to the theme of multidimensional dispersive hydrodynamics [47]. Khusnutdinova derives a cylindrical KdV type equation for the internal wave amplitude of a piecewise constant shear flow. The geometry of these internal ring waves are shown to be different from those on the surface. The model equation can be used to study multidimensional dispersive hydrodynamics in the ocean and other fluids. A fundamental, important aspect of dispersive hydrodynamics is the existence of conservation laws. For example, they can be used to derive Whitham modulation equations. Dimitrios Mitsotakis presented a direct, asymptotic derivation of conservation of mass, momentum, and energy for shallow water flows governed by the Serre equations. Numerical computations of undular bores and shoaling solitary waves exhibit canonical dispersive hydrodynamics.

The final talk of the meeting by Nikola Stoilov was on the construction of an integrable, dispersive regularization of the integrable Witten-Dijkgraaf-Verlinde-Verlinde equations. As shown throughout the meeting, integrable, dispersive nonlinear wave equations allow for powerful analytical methods to describe dispersive hydrodynamics.

Scientific Progress Made

Some important themes that emerged during the course of this meeting include

- **Hyperbolicity versus ellipticity:** Strict hyperbolicity or convexity in the dispersionless equations and the Whitham modulation equations yield what may be termed classical dispersive hydrodynamics. Such behavior is well modeled in the KdV and defocusing NLS equations. Talks by Biondini, Mantzavinos, El, Tovbis, Osborne, and Suret showed the emerging problems in soliton turbulence and rogue waves as important examples of dispersive hydrodynamics modeled by elliptic modulation equations. Modulational instability enhances incoherence yet coherent structures play an important role. Additionally, a dynamic change in the type of the modulation equations, analogous to transonic gas dynamics, yield new dispersive hydrodynamic features as shown by Maiden and Hoefer in the viscous fluid conduit system.

- **Non-convex nonlinearity *or* dispersion:** Hyperbolic conservation law theory has been developed since the eighties to resolve non-classical shock structures such as undercompressive shocks. One of this meeting's main themes was to provide a bridge between the conservation law and dispersive nonlinear wave communities. Diffusive-dispersive models described by LeFloch and Shearer demonstrate the similarities and differences between diffusive-dispersive and purely dispersive regularizations when the modulation equations are non-convex or linearly degenerate. The analysis of non-classical DSWs such as kinks or solibores, contact DSWs (also called trigonometric DSWs), and double wave structures benefit both in physical interpretation and mathematically from the rich hyperbolic conservation theory previously developed. With potential applications to statistical mechanics, among others, as shown by Moro, it is clear that further study of these systems is warranted.

Another form of non-classical dispersive hydrodynamics can emerge when the linear dispersion is non-convex. This theme was captured by the talks of Trillo, Maiden, and Hoefer. Radiating DSWs and DSW multiphase implosion are examples of the dynamics that can result from a dispersion that changes sign. Experiments in ultracold atoms that can now engineer the dispersion relation as shown by Engels motivate further studies of these non-classical phenomena.

- **Soliton turbulence:** This emerging field of incoherent dispersive hydrodynamics is currently being examined in ocean water wave field studies, e.g., Osborne, and nonlinear optical fiber laboratory studies, e.g., Suret, Randoux. The theory of these systems is in its infancy and, motivated by compelling experimental results, will undoubtedly continue to grow.

- **Application areas:** While "traditional" application areas of dispersive hydrodynamics such as nonlinear optics, superfluids, and water waves were well represented at the meeting, broader applications to friction (Gershenzon), viscous fluid conduits (Maiden, Hoefer), spin-orbit coupled BECs (Engels), thermodynamic phase transitions (Moro), and degenerate Fermi gasses (Salasnich) are promising avenues for further development.

- **General mathematical methods:** The mathematical theory of dispersive hydrodynamics has grown from canonical integrable models (KdV and NLS) to non-integrable systems. Talks by Dubrovin, Gavrilyuk, Benzoni-Gavage, Klein, Hoefer, and Esler reveal important general methods. Short time dynamics and the universality of wavebreaking are active areas of research. Whitham theory is not tied to integrable systems so its use more generally paves the way for further descriptions of non-integrable dispersive hydrodynamics. Basic analytical questions of existence and stability are wide open. For example, is hyperbolicity of the Whitham modulation equations sufficient for DSW stability? More mathematical and numerical tools to study dispersive hydrodynamics are needed.

- **Multidimensional dispersive hydrodynamics:** This area of research is wide open as there are very few results. The meeting exhibited talks by Klein, Gammal, Ilan (numerical studies) and Khusnutdinova (derivation of reduced equations).

- **Revisiting classical results:** Even models as fundamental as KdV and NLS continue to receive new analysis and hence new insights into classical dispersive hydrodynamics. One benefit of these and other integrable models is the detailed analysis enabled by Riemann-Hilbert, steepest descent (Jenkins, Mantzavinos, Tovbis), inverse scattering (Miller, Biondini, Prinari, Deng) and matched asymptotics (Ablowitz).

- **Dispersive hydrodynamics experiments:** There were seven experimental talks (Trillo, Engels, Osborne, Suret, Randoux, Wetzel, and Maiden) representing an expansion of physical interest in the field. Further quantitative comparisons between the mathematical modeling and analysis will serve to heighten interest in the field.

Outcome of the Meeting

As demonstrated by the diversity of talks during this meeting, dispersive hydrodynamics represents a broad set of physical and mathematical wave problems. The interplay between physics and mathematics was clear throughout the meeting. Applications in classical fluids, optics, ultracold atoms, and other areas motivate further mathematical developments. The strong attendance by almost all participants to each talk plainly demonstrated the broad interests and connections between the participating scientists. An important outcome of this meeting was the initial step to connect the conservation law and dispersive nonlinear waves communities. Another outcome was the bridging of rigorous analysis, applications, and experiments. This meeting may be a springboard to continued growth in the emerging field of dispersive hydrodynamics.

Participants

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Chapter 15

Applied Probability Frontiers: Computational and Modeling Challenges (15w5160)

May 31 - June 5, 2015

Organizer(s): Jose Blanchet (Columbia University), Shane Henderson (Cornell University), Donald Iglehart (Stanford University), Thomas Kurtz (University of Wisconsin), Pierre L'Ecuyer (Universite de Montreal), Amy Ward (University of Southern California), Assaf Zeevi (Columbia University)

General Overview

The workshop concentrated on applied probability, broadly defined, but with an emphasis in engineering and statistical applications, and having in mind both algorithmic approaches and theoretical insights. The workshop included two sessions of open problems, which, as we shall discuss, provide a good indication of research directions that the field is taking. These directions are centered at the core of contemporary computational advances and challenges in applied probability.

The areas that were covered by the workshop and which we singled out during our proposal submission are the following:

1) Stability and steady-state performance measures (including, but not restricted to, Markov chain Monte Carlo techniques),

2) Numerical methods for SDEs (stochastic differential equations) and related continuous time continuous space models,

3) Extremes and rare events,

4) Stochastic optimization and control,

5) Robust performance analysis of stochastic systems.

We will provide a brief overview of the current research interests and advances in these areas and also explain how the workshop's content covered some of these advances.

In addition to the presentations and posters, the open problem sessions were also related to these four areas and were key to expose some questions of current interest.

Computation of steady-state performance measures

Simulation methodology is the primary computational tool for steady-state analysis of stochastic systems. (In this document we use the terms stochastic simulation and Monte Carlo interchangeably.) There are literally thousands of documented applications of the Monte Carlo method for steady-state computations, motivated by different scientific areas such as Computer Science, Operations Research, Physics, and Statistics.

Some of the main challenges here includes: acceleration of simulation schemes for exploration of a high dimensional, multi-modal, target steady state distribution, a problem that often arises in the fields of Physics and Statistics. **Professor Gareth Roberts** delivered a lecture which discusses precisely these types challenges in the context of Bayesian analysis for very large data sets, giving rise to a likelihood which cannot be evaluated practically. **Professor Roberts'** approach, based on so-called pseudo-likelihood methods, successfully exploits unbiased estimation techniques for continuous time processes, which we shall discuss in the next section.

Another challenge in the area involves quantification of rates of convergence to stationary of specific Markov chains. This topic is often studied in Statistics and Computer Science. Motivated once again by applications to Bayesian statistics, **Professor Eric Molines**' talk concentrated precisely on the estimation of rates of convergence to stationarity for the continuous-time Metropolis chain (also known as Langevin diffusion) assuming a high dimensional, strongly convex, potential. He demonstrated a positive result (i.e. gracious rate of convergence scaling as the dimension increases) for such types of Metropolis chains.

Yet another type of challenge in the area involves the detection and deletion of the initial transient bias in the steady-state simulation of semi-Markov models, arising frequently in Operations Research, and for which typically very little is known about the equilibrium distribution. **Professor Soren Asmussen** discussed advances and open problems in areas such as exact simulation (i.e. simulation without bias) for regenerative processes observing that generic approaches might yield infinite expected termination time. In this vein, **Professor Karl Sigman** presented the first class of exact simulation algorithms for semi-Markov multiserver queues, using coupling techniques which guarantee, under natural stability assumptions, finite expected termination time for the procedure.

The talk delivered by **Professor Jim Dai** introduced the use of Stein's method for steady-state approximations of queues in heavy traffic. In his talk, **Professor Dai** demonstrated how Stein's method can be used to address rates of convergence in the so-called "limit interchange" problem discussed in work of **Professor David Gamarnik** and **Professor Assaf Zeevi**. Such problem concerns heavy-traffic approximations of steady-state distributions by using the stationary distribution of the finite-time heavy-traffic limit and has attracted a significant attention in the Operations Research literature.

Now, although steady-state distributions of complex systems can be quite challenging to compute, sometimes it is possible to obtain quite precise characterizations of steady-state distributions. This was illustrated by the talk given by **Professor Onno Boxma** who discussed closed form expressions involving various classical special functions in problems in the context of inventory, queueing, and insurance problems.

Finally, **Professor Thorisson's** talk which concentrated on fundamental theoretical questions in the context of two sided stationarity processes. For instance, consider a two-sided Brownian motion centered at zero. Characterize the random times that, after centering the process at one of such times, one obtains a two-sided Brownian motion. He provided non-trivial examples of such times involving a construction based on the local time at zero and suitable randomizations. Although his talk was not computational, his results gave rise to discussions connected to the topic of perfect simulation.

Numerical methods for Stochastic Differential Equations (SDEs) and Related Continuous Time Continuous Space Models

Continuous-time models, such as stochastic (partial or even ordinary) differential equations (driven by Brownian motion or other type of noise such as Levy or long range dependent processes) are ubiquitous in scientific applications primarily because, owing to the theory of analysis (both classical and stochastic), these models are amenable to mathematical manipulation. Nevertheless, computational tasks associated with these models are often challenging given the fact that they ideally require a continuum of information to be stored in a computer's memory. Moreover, basic probabilistic quantities, such as for example, transition probability functions, are virtually impossible to compute analytically in almost all cases.

In recent years, there have been a number of major advances in the theory of simulation that have enhanced our ability to access the solution to SDEs with an increasingly accurate assessment of the control error. For example, the work of [7] has allowed us to simulate, essentially without any bias, a range of diffusion processes. This work has enabled the possibility of providing estimates for the underlying transition density and, consequently, the direct application of explicit likelihood ratio methods in statistical analysis of diffusion models ([8]). In fact, an application of these methods was highlighted, as mentioned earlier, by the talk of **Professor Gareth Roberts.** The approach of ([8]) is also applicable as a means to implement the class of algorithms analyzed by **Professor Moulines**.

Another important recent advance in numerical methods for SDEs corresponds to multilevel Monte Carlo methods, (see [21]). This approach has enabled the estimation of a large class of expectations with an optimal rate of convergence. **Professor Mike Giles** delivered a comprehensive lecture which explained how the multilevel Monte Carlo method can be applied not only to SDEs, but also to nested simulation problems, optimal stopping problems, and chemical reaction simulations.

More recently, there has also been work that allows the simulation of piecewise linear processes which are strong sample path approximations to SDEs, in the sense they are subject to a prescribed (deterministic) error with probability one ([10]). These constructions enable the estimation of sample path expectations of complex objects such as multidimensional local-time-like processes, for example those arising in the solution of the so-called Skorokhod problem studied in stochastic networks ([32], [17]). The lecture delivered by **Professor Kavita Ramanan**, which discussed sample path derivatives of the solutions to the Skorokhod problem, brought up a potential avenue for using path approximations for numerical evaluation of unbiased estimators of derivatives of expectations of SDEs.

There are still many outstanding challenges in analyzing multidimensional diffusions, in SPDEs, and in SDEs driven by multidimensional Levy and long-range dependent processes. A problem related to local times and maxima of Levy process was posed by **Professor Mike Giles** at the beginning of the workshop, involving the existence of a density under certain conditions, and motivated by the development of efficient Monte Carlo methods for SDEs driven by Levy processes. Significant progress towards the solution of the problem (including a partial solution which is applicable to the examples of interest to **Professor Giles**) was reported by **Professor Jose Blanchet** during the last day of the workshop.

While, as mentioned before, there remain many outstanding challenges in the area, it was also evident from the discussions in the workshop that are also opportunities for cross fertilization among communities that tend to have little intersection, e.g., statistical inference (exact sampling of diffusions, [23]), mathematical finance and chemical reaction networks (multilevel Monte Carlo, [22], [2]), and stochastic queueing networks (strong couplings, [10]). As an additional example of such cross fertilization, we mention the poster presented by **Dr. Chang-han Rhee**, which explained how to use an extra randomization step on top of multilevel Monte Carlo ideas in order to estimate without bias steady-state expectations of a positive recurrent Harris Markov chain (see [36] and also [34]).

Extremes and rare events

This area has been traditionally investigated in the context of communication networks, finance, and insurance. **Professor Soren Asmussen** lecture and an open problem posed discussed some recent problems involving loss rates (i.e. the probability of packages being dropped) in the context of communication-transmission model involving potentially long range dependence.

Analysis of extreme events traditionally have relied on the classical theory of large deviations theory for light tailed systems, see for example [20]. More recently, in part motivated by some of the applications described earlier, there have been advances in the development of techniques for systems with heavy-tailed characteristics ([9], [5], [30]). Some of these methods are cross disciplinary, relying, for example, on the application of statistical extreme value theory, [20], combined with the theory of stability of Markov chains and Lyapunov inequalities (typically used in applications involving Markov chain Monte Carlo). **Professor Sigman**'s lecture gave an example of a perfect sampling algorithm which uses large deviations techniques for steady-state analysis.

A new set of applications, in the context of page rank algorithms, which require the use of heavy-tailed approximations arising from the analysis of tree-like models with light-tailed characteristics was illustrated by the talk of **Professor Mariana Olvera-Cravioto**.

Finally, we mention the talk of **Professor Ton Dieker,** who described the first exact sampling algorithm for so-called max stable random fields. These processes characterize the extreme behavior of a sequence of suitably scaled spatial processes and are natural in applications such as climate modeling. **Professor Dieker**'s talk combined ideas from large deviations and stationary concepts and also connected to topics simulation of continuous processes and large deviations ideas for Gaussian fields (see [1]).

Important open problems which were discussed, for example towards the end of **Professor Olvera-Cravioto**'s talk include the fact that the majority of the Monte Carlo estimators for rare events that are shown to achieve desirable optimality properties (in terms of achieving an optimal convergence rate) require the existence of asymptotic approximations. A challenge in the area is to design estimators that are both efficient in some sense and that are applicable in environments that do not require these types of approximations.

In addition, it remains to be seen how the theory and algorithms developed in the rare event simulation literature can aid the development of statistical inference procedures for the likelihood of rare events, for example, using a Bayesian perspective.

Stochastic Optimization and Control

Stochastic optimization entails the minimization of a function that is represented as an expectation, possibly subject to constraints that might also involve expectations. These problems can quickly become quite challenging from a computational standpoint and Monte Carlo methods are often used to find approximate solutions in a reasonable amount of time; this approach is known as simulation optimization. Examples in which simulation optimization arises in practical applications include the design of emergency-service systems [33] and service-system staffing [13].

The area is advancing rapidly, partly due to advances in hardware, and partly through advances in software enabled by recent research. In addition, the trend towards parallel computing, as evidenced through multicore architectures, the use of graphical processing units in scientific computing, cloud computing, and the increasing availability of high-performance computing (supercomputing), provides a rich environment for the development of new and exciting work in this area. **Professor Barry Nelson**'s poster presented an algorithm aimed at exploiting meta models for large scale optimization problems of discrete systems. **Professor Sandeep Juneja**'s lecture discussed lower bounds related to the best possible performance that can be expected from an optimal simulation scheme for optimization of discrete simulation optimization problems. And adaptive optimal allocation policies for such discrete optimization problems was discussed in **Professor Zeevi'**s poster.

Another new an exciting application involving a stochastic model for optimal allocation of bikes across New York City was given in the poster of **Professor Shane Henderson**; the optimal policy in his poster combines coupling of Markov chains and combinatorial optimization ideas in a novel way.

Another type of application in the context of stochastic control for staffing problems was given in the talk of **Professor Amy Ward**, where an asymptotically optimal solution to a complex stochastic control problem was provided. These types of approximate solutions have been studied in the last decade in heavy-traffic theory of queueing systems.

Finally, yet another application to the context of discrete stochastic optimization, in the context of assortment selection to maximize the revenue of a vendor, was illustrated by an open problem posed by **Professor Jose Blanchet**.

We mentioned problems involving discrete / ordinal optimization. In the setting of stochastic optimization problems with convex / differentiable structures, one must cope with noisy estimates of function values and derivatives. There is a substantial literature on methods for attacking such problems; see, e.g., [4], [35], Chapters 17-21 of [26]. The poster of **Professor Shanbhag** related to this problem via efficient use stochastic approximation methods. Also in this context, advances in stochastic optimization by means of sample average approximation methods with constraints were presented in the talk of **Professor Sigrun Andradottir**. **Professor Jose Blanchet** presented a poster showing the first class of estimators which achieve complete deletion of the systematic bias arising in virtually any application of sample average approximations.

A particular outstanding challenge is the design of adaptive procedures that adjust tuning sequences that calibrate the stochastic approximation algorithm, to various information that is a prior unknown concerning the target function that is to be optimized, and the environment in which it is observed/simulated (see, e.g., [14]). A related open problem involving adaptive root finding algorithms was posed by **Professor Shane Henderson.** These adaptive techniques have a strong connection with the theory of stochastic stability in Markov chains, analysis of Markov chain Monte Carlo algorithms, and stochastic recursions (discrete time analogues of SDE's). Given these, and other links stated above, we believe there is ample potential for cross fertilization and synergies to be realized in this area as well.

Robust Performance Analysis of Stochastic Systems

The last that was covered during the workshop relates to the issue of inaccurate model assumptions in calculating expectations of interest. From a methodological standpoint, robust performance analysis is closely related to stochastic optimization since a natural approach consists in solving a optimization problem to obtain a worst-case bounds among a class of models of interest.

Such worst-case analysis approach, subject to constraints, is popular in methodological areas such as robust control [25], and robust optimization, [28], [6]. And the approach has been used in application domains such as economics and finance [24], among others.

From a computational standpoint, this area brings a wide number of challenges and opportunities. For example, the choice of feasible regions from which to optimize often gives rise to infinite dimensional optimization problems. Some times those problems can be tractable if one ignores natural constraints (such as stochastic independence or Markovianity), but tractability comes at the expense of ending up with too pessimistic upper bounds. This is one of several open problems in the area, which were addressed in some of the talks in the workshop. **Professor Henry Lam** discussed recent advances involving techniques to approximate the solution to such optimization problems (including non-convex constraints) assuming that the feasible region becomes a small neighborhood around a baseline model. His talk also included model uncertainty considerations for rare events and steady-state analysis. Finally, in the context of robustness, **Professor Sean Meyn** posed an open problem involving the stability of robust non-linear filtering.

Conclusions

The workshop promoted the identification unresolved problems that, thanks to recent advances in other areas, are within reach of what we currently know. At the same time we believe that the workshop planted seeds for significant cross fertilization among the different allied areas. In fact, we discussed already some of this potential in the body of this report.

Participants

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Chapter 16

Advances and Challenges in ProteinRNA: Recognition, Regulation and Prediction (15w5063)

June 7 12, 2015

Organizer(s): Yael MandelGutfreund (TechnionIsrael Institute of Technology), Gabriele Varani (University of Washington)

Overview of the field and open questions

RNAprotein recognition is central to all biological events in both normal and disease cellular states. The meeting focused on a common discussion of recent development and challenges in the computational, structural and experimental investigation of this important class of biological recognition events.

Immediately following transcription, in fact while a gene is still transcribed, RNAbinding proteins (RBPs) associated with primary transcripts and regulate the fate, cellular localization, stability and translational efficiency for that gene product. The efficiency and location of RNA processing events (splicing, 3end processing, editing) is dictated by RBPs alone and in competition with RNA structure. Not surprisingly, many genetic diseases map to the disruption of RNAprotein recognition events. Both coding RNAs (e.g. gene products that contain an open reading sequence that us translated on the ribosome) and noncoding RNAs (sequences that function as primary transcripts and are not translated) associate with a myriad of cellular proteins, many of which are known and more that are still being identified. Understanding the molecular mechanism of proteinRNA recognition and regulation to the point of predicting and rationally altering proteinRNA interactions is a major challenge of experimental and computational biology. In fact, these major questions are only beginning to be addressed quantitatively through recent progress in structure determination and through the development of experimental and computational methods to study RNAprotein complexes in a high throughput manner.

A significant fraction of the genome, at least 5% and perhaps more than 10%, of all eukaryotes code for RBPs and many more genes code for RNAs which are not translated (non coding RNAs) but function by acting on other RNAs or chromatin and perform functions analogous to those of RNAbinding proteins in regulating RNA metabolism. As organisms and cell types become more complex and evolved (e.g. neurons; higher eukaryotes), regulation of gene expression at the posttranscriptional stage becomes comparatively more prominent. In these tissues and organisms, complex gene regulatory networks depend on the RNA/protein interplay, where an RNA-binding proteins act upon an RNA and/or vice versa. Modeling, predicting and altering these cellular networks require a quantitative understanding of individual proteinRNA interaction events, but also a broader genomewide catalogue of the activity and concentration of RBPs. A significant activity within the meeting was dedicated to new approaches to quantitative mathematical modeling of the activity of these proteins.

Many RBPs can be identified directly from their sequence and are wellannotated. Typically they belong to distinct structural classes that are reasonably well known (e.g. the RRM, the dsRND, Puf proteins etc), but there are potentially many more proteins which could bind to RNA (e.g. many metabolic enzymes). Identifying the complete complement of RNAbinding proteins in any eukaryotic organism remains a significant challenge of computational and experimental methods, yet the identification of a protein as belonging to the RBP class is not sufficient to address its function, because it does not provide any information on which RNAs it acts upon. Furthermore, the activity of RBPs can be highly specific (e.g. one protein acting on a small number of RNA targets) or diffuse, when a protein can act on a large variety of cellular RNAs. An understanding of the specificity of RBPs is central to biology and requires the joint application of both computational and experimental techniques. Determining RNAbinding specificity was a major theme of the meeting.

The first structural and biochemical information on RNAbinding proteins, at atomic detail, dates from the late 1980s. The last 20 years have seen very rapid progress in the structural (NMR and xray primarily) and biochemical characterization of RNAprotein complexes. These studies have investigated in considerable detail how many classes of RNAbinding proteins recognize RNA and have dissected the contribution of different molecular forces (e.g. electrostatics) to binding and specificity. However, as highlighted at the meeting, significant gaps in knowledge remain. Most significantly, even when the specificity of a protein is established structurally and/or biochemically, the effect of sequence variation cannot generally be predicted from this knowledge, yet are critical to understand the full complement of RNAs targeted by a protein in the cell and to understand how sequence variation might cause disease or drive species evolution. Despite this limitation, the traditional structural/biochemical approach to understanding RNAprotein interactions remain a central tool to interpret genome wide studies, but structure determination of RNPs still lags the comparable problem of proteinDNA or proteinprotein specificity considerably.

More recently, since approximately 2000 but especially since 2005, genomewide methods have been introduced to interrogate the specificity of RBPs not one sequence at a time, as was traditionally done, but by sampling the complete sequence landscape recognized by an RNAbinding protein in vitro and in vivo. A major component of the meeting was dedicated to the discussion, presentation and critical examination of high throughput experimental methods to address this important problem. The application of these methods, and their interpretation through structural principles on the one hand, and the subsequent generation of mathematical models of these interfaces promise to generate much better understanding of the causes and consequences of variation in RNAbased gene expression on organisms and disease.

The concept behind the workshop arose from the realization that the integration of different experimental and computational approaches is needed to understand RNAprotein recognition with the level of sophistication and depth required to answer fundamental biological processes. It is not sufficient for structural biologist or biochemists to understand in depth what other structural biologists are doing: the greatest opportunity for progress lies in the merging of different experimental and computational approaches to tackle this problem. Thus, the workshop was designed to bring together structural biologists/biochemists that focus on individual RNAprotein complexes, with genome biologists who have developed powerful experimental methods to investigate RNAprotein interaction across genomes, as well as computational biologists who seek to model and develop predictive tools based on the confluence of these experimental advances. The workshop was designed to foster the exchange of ideas between experimental and computational biologists and catalyze the development of new and improved technologies that merge experimental analysis with novel mathematical and computational techniques to better understand the rules of proteinRNA recognition with the ultimate goal of generating a better quantitative understanding of RNAbased biological regulation.

Presentation highlights

High throughput approaches to study proteinRNA interactions and the impact on downstream genes

The most significant advance in the field of RNAprotein recognition in the last 1015 years has been the development of genomewide approaches to investigate the RNA population targeted by an RBP and to establish its specificity. The objective of all of these methods is to derive an RBP code, i.e. the system of RNA determinants and protein partners that instruct gene expression at the RNA level. Exhaustive arraybased methods and related approaches interrogate the specificity of RBPs in vitro in purified form, while methods such as CLIP (crosslinking and immunoprecipitation) isolate the RNAbinding sites of the RBP in a specific cellular context. The two classes of methods are of course related but investigate RBPs in a different context. While the latter method would naturally seem to be more valuable and closer to provide physiologically relevant results, it also suffers from limitations of the experiment (RNA structural context; over or underrepresentation of expressed targets in the cell; false positives due to nonspecific interactions or overexpression; false negatives from underexpression; etc). An important and very fruitful theme of the meeting was the open and frank exchange of information and debate about the different limitations and strengths of these methods and how to best interpret and analyze the results and compare the outcomes of different experimental approaches.

Cellbased methods to address the question of which RNA a given protein associates with were introduced about 10 years ago by Ule, building upon earlier methods developed by Keene and Steitz. In short, these approaches use crosslinking of a specific protein to all the possible RNA it is associated with under particular cellular conditions, followed by deep sequencing to identify the target RNA. Keene provided a very interesting historical survey, starting with older biochemical methods that first identified the RRM as an RBP over 25 years ago, following with description of update highthroughput methods to address the same questions in a cellular context. Landthaler described in considerable detail an example of one particular regulatory system and highlighted the remarkable number of sites targeted by any specific RBP (>3,500 in that example) and the structural complexity of binding sites that can be identified by a systematic investigation of these motives.

Keene and Friedersdorf presented their more recent approach (RIPSeq) to identity the RNA binding sites for RBPs and measure quantitative binding strength. An emphasis of their presentation was the presence of overlapping binding site for different RBPs, which raised the issue of cooperativity and anticooperativity in RBP function. These are important questions that remain to be addressed satisfactorily in the current paradigm of one protein/one binding site used by essentially all biochemical and genome wide approaches.

Ule presented recent advances in the CLIP technology that provides nucleotide resolution, as opposed to broader mapping of targeted sequences that was possible in the past, and that addresses the issue of repetitive sequences (e.g. polypyrimidine tract binding protein binding to pyridimidine rich splicing signals) as well as proteins that bind to highly structured RNAs (e.g. Staufen). This last problem is particularly important because addressing structural context (i.e. which RNA secondary structure provides a binding framework for sequence specific and even more for structurespecific RBPs) remains a challenging and highly pursued problem for arraybased methods that investigate protein specificity in vitro. Yeo presented an update on efforts connected to the ENCODE project to generate a genomewide analysis of RNAbinding protein networks. He descried highly standardized approaches to map the targets of >300 RBPs using a combination of CLIPrelated approaches (CLIPSEQ; ChiPSeq; BindNSeq) in cells. He openly described efforts to remove artifacts from the data, improved positive sensitivity (e.g. normalization for RNA copy number, validation of antibodies) and obtain maps at nucleotide resolution. These data will be provided worldwide through a widely accessible server. In a departure from the focus on eukaryotic proteins of most of the meeting, Margalit focused instead on the equally complex problem of mapping at the transcriptomewide level the universe of proteinRNA interactions involving small non coding RNAs in bacterial organisms.

In vitro and computational approaches to assigning RNA binding motifs

The different high throughput approaches to study proteinRNA interactions can generally be broken down into invitro and invivo methods. While the invivo approaches, which were extensively discussed at the meeting (described above in section 2.1) can give a snapshot of the binding sites of a given RBP at a given cell type in a given condition, invitro approaches are equally important for determining the specificity of RBPs while controlling for cellular parameters which could influence the binding such as interacting proteins and other cellular factors. Given the wellestablished knowledge that the RNA structure plays an important role in determining the binding specificity of RBPs there has been a great effort in the field to detect the combined sequence and structural binding preferences of RBPs. At the meeting, Hughes presented a recent collaborative effort between his group and the group of Morris to determine the binding specificities of large cohort of RBPs. In a published study in Nature (2013), the two groups reported systematic analysis of the RNA motifs of 205 RNAbinding proteins that were extracted from a high throughput invitro selection experiment. At the meeting Hughes presented the main computational challenges in extracting binding motifs from the large scale experiments, specifically referring to long motifs which may represent cobinding of several proteins or different binding preferences of the same protein. In addition Hughes discussed the challenges and approaches that should be employed for detecting the combined sequence and structural preferences of RBPs. Morris then presented an overview of computational approaches developed over the years by his group and others for extracting combined sequence and structural RBP motif preferences. Morris concentrated on the RNAcontext algorithm, which is a probabilistic model that uses both sequence and structure parameters inferred from the data to extract the most probable motifs which reflect both the structural and sequence preferences of the RBP. Backofen presented a graphkernel based algorithm named GraphProt which uses an advanced machine learning approach to predict the combined sequence and structural binding preferences of RBPs extracted from invivo data and employs it to predict missing binding sites. He presented an example from collaborative work with the Landthaler group where they employed the GraphProt algorithm to identify the composite structuresequence motif recognized of a zinc finger RBP, which could not be detected by standard computational methods for motif finding.

Extracting the binding preferences from invivo experiments adds many different challenges, such as predicting true binding sites which have been miss identified by the experimental tests. Eyras presented an original computational approach for predicting binding motifs of novel RBPs by correlating the differential gene expression of RBPs in cancer vs normal tissues to alternative splicing events altered in the same tissues. By employing this approach on data from the TCGA they were able to recapitulate the binding motif of the well characterized RBP QKI. Dror addressed a major computational challenge: how to distinguish true binding sites from all sites that contain the binding motif of a nucleic acid binding, that yet are not bound by the protein. She presented her studies of DNA binding motifs, emphasizing that very similar protocols can be employed for identifying cognate DNA or RNA binding sites, and showed that the main features contributing to predicting true binding sites are the sequence content around the motif and the similarity of the motif to its neighborhood.

Approaches for detecting novel RBPs and defining their function

Great advances have recently been made in the development of highthroughput screens to identify RBPs in cultured cell lines. Such methods, known broadly as the interactome capture, take advantage of the poly
-A tails of primary transcripts RNAs as a bait to be captured by magnetic polyT beads. The interactome capture technology has contributed dramatically to the field of RNA
-protein interactions, increasing the number of experimentally identified RBPs as well as suggesting novel RBPs and specifically new, yet un explored, cellular mechanisms for these proteins. Milek from the Landthaler group, which were among the first groups to develop the interactome capture methodology, presented an advance study where they employed the methodology to specifically identify RBPs that bind RNA transcript upon ionizing irradiation in MCF
-7 cells. This study demonstrated the great advantage of the invivo interactome capture approach over the standard approaches for predicting RBPs in vivo and in
-silico, enabling the presenter to show the dynamics of RBPs in the cell and specifically to quantify the differential binding of RBPs to mRNAs in the cells under different conditions. Among the great advantages of the new technology to capture RBPs in cells is the ability to conduct comparative experiments to reveal the evolution of the RNAbinding proteome across species to better understand the origin of RNA regulation. Beckmann presented exciting results from his postdoctoral work in the Hentze group, together with computational approaches employed for discovering the commonalities and differences in the RNA binding proteome of human and yeast. Among the unexpected findings presented were inherited differences in the sequences and predicted structures of the RBPs. Gerber presented a comparative interactome study of the RBPs in yeast S. cerevisiae and in the nematode C. elegans. RBPs were detected again with similar techniques but, in both cases, RBPs were identified from living organisms and not from cultured cell lines. One of the most exciting and unexpected findings presented by both

speakers was that among the novel RBPs found in both human and yeast were several well characterized enzymes involved in central metabolic pathways in the cell, such as the carbon metabolism. The discovery that some of the basic enzymes involved in the most conserved and essential metabolic pathways in the cell have also RNA binding capacity may suggest a novel mechanism by which cells sense their metabolic status and provide finetuned feedback to the gene expression regulation.

An interesting discussion was conducted concerning the limitations of high throughput methods, which suffer from false detection rate which is in many cases hard to evaluate. One of the main aims of the meeting was to bring together people who develop these technologies with computational experts, to ponder together ways to both evaluate the reliability of the results and propose way for improving the technologies. MandelGutfreund presented new computational advances for predicting RBPs solely based on the physiochemical and electrostatic properties without relying on sequence homology to known RBPs. The algorithm combines modeling the domain structure from the protein sequence with a machine learning approach to define whether the domain binds to RNA was tested on data extracted from the interactome capture experiments and yielded promising results. The main challenges yet to be overcome are related to predicting the reliably of the high throughput experimental results.

RBPs have many diverse function in the cell, and understanding the function of all different RBPs is probably a nonrealistic task, but considerable efforts continue to be dedicated to this necessary task. At the meeting, Maquat presented fascinating evidence for the role of the RBP Staufen, the funding member of a class of proteins involved in subcellular RNA localization, in mediating mRNA
-mRNA cross talk via binding to Alu repeats at the 3UTRs of the mRNAs. As is true for the majority of proteins in the cell, many RBPs undergo alternative splicing generating different protein isoforms, adding further complexity to the problem of predicting RBP function. At the meeting, Fagg presented various biochemical approaches to determine the function of specific isoforms of the wellstudied RBP Quaking, demonstrating interesting autoregulation between the different protein isoform, fine tuning the expression level of the different functional isoforms in the different compartments of the cell.

Structural and biochemical approaches to study proteinRNA interactions and binding specificities

Structural and biochemical analysis of the structure and specificity of RNAbinding proteins remain a mainstay of the field; a requirement to interpret and analyze the results of highthroughput methods, and to translate that knowledge into computational models that are not fully dependent on sequence analysis. Although the complete RNAbinding proteins proteome is large and structurally diverse, the majority of RBPs in all eukaryotic organisms belong to relatively few structural classes that are well characterized structurally and whose mode of binding to RNA has been established in most cases. These proteins are also the most common subjects of highthroughput genomic investigations described before and provide the best subject for computational modeling of these interfaces. A complication, however, is that very often individual domains bind to RNA with only poor sequence specificity and modest affinity, and biological targeting is achieved by either utilizing multiple domains on the same protein or, combinatorially, by forming complexes containing multiple proteins that cooperate to bind to a specific RNA sequence and structure. The analysis and structural investigation of these modes of recognition were a major theme of the workshop and coincides with the state of the art and frontier of understanding of paradigmatic and very abundant RNAbinding domains.

The most common RNAbinding domain is the RRM, which is found in approximately 300 human proteins and represented by $>$10,000 sequences in Pfam. While the structural basis for recognition of RNA by single RRMs is understood (although not specificity), how multiple domains within a protein combine to generate sequence specific recognition is much less well understood, and very difficult to analyze with highthroughput methods for technical reasons. Allain provided a comprehensive review of the structures solved from his laboratory focusing on RBPs with multiple RRM domains bound to RNA. Rather than providing a unifying theme, it was clear that the binding modes of tandem RRMs on RNAs are extraordinarily diverse, a theme that builds upon and reinforces similar observations made on single RRMs by the same group in the course of the last 15 years. A similar theme was discussed by Tolbert who talked about regulation of HIV splice site by the tandem RRMs of hnRNP A1, where the role of one of the domain was purely structural, yet essential to define the specificity of targeting of these proteins, and dependent on the formation of a specific RNA structural context that mediated interdomain interactions, raising the possibility that RNA structure itself regulates allosterically the protein binding and its downstream biological effects. Clearly, atomic models of these interfaces must somehow account for these observations, the diversity of orientation and recognition principles, and the role of RNA structure on recognition, all features that make it much more difficult to predict the specificity of an RRM from its sequence alone.

A complementary problem was discussed by Sattler, who employs a combination of different structural methods to investigate how specific regulatory complexes assemble on premRNAs. Themes that were observed were cooperativity in binding between different proteins mediated by proteinprotein interactions; dynamic rearrangements of the protein and RNA structure; multiregister binding. Not only were these phenomena essential to understand molecular recognition, but also correlated to the biological function of these complexes, as illustrated in the most spectacular way in the study of the complex of proteins responsible for recognition of 3splice sites during premRNA splicing. How are these complexes to be modeled? How are highthroughput experiments designed to capture the true specificity of these proteins in the correct functional, multiprotein complex? These are outstanding questions that will require new experimental and computational developments in the near future.

Ramos discussed a paradigmatic KH domain protein, a second very common class of RBPs, called KSRP, that contains four such domains that are used to bind to RNA. Each domain has its own specificity, which was mapped, yet they cooperatively target a specific RNA in a manner that is as of yet unclear. Murn presented structural data on a less abundant binding domain, the CCCH zinc finger domain. Based on xray structural analysis, he showed how six CCCHzf domains recognize a bipartite recognition site on a mRNA, forming a unique topology of interactions between the protein and the RNA which is highly crucial for the protein function. Similarly, Leeper presented studies of the interaction of a multidomain RRM protein containing four domains and a long non coding RNA, a scarcely studied but biologically very important class of RNAprotein recognition events that deserve much greater attention in the near future.

An important question to be addressed is how specific is an RBP. Sattler illustrated the case of Roquin, which binds conserved stemloop structures regardless of their sequence and in a manner that depends only on certain structural features (the size of the loop, the length of the base paired region). Most spectacularly, Jankowsky presented his investigation of the C5 E.coli protein, that binds pretRNA within a conserved structural context in a manner that is independent of sequence, or so it was believed. Using a clever highthroughput method that couple selection for binding/processing with deep sequencing, he was able to define not just a few high affinity sequences for this protein, but to map the complete landscape, the complete thermodynamic profile for all sequences of five nucleotides bound by this protein. Although functionally the protein is nonspecific, the profile is not that dissimilar from that of highly specific transcription factors, with certain sequences in the high affinity tail of the distribution recognized with high specificity. Thus, there is not such a thing as unique sequence preference or nonspecific proteins, but a continuum of affinity or affinity distribution. It just so happens that biology does not utilize those sequences, because naturally occurring tRNA substrates are found only in the nonspecific center of the distribution. How common is this phenomenon of hidden specificity? How many RBPs utilize suboptimal sequences that do not coincide with the specific tail of the distribution of affinities? A more subtle point allowed by the generation of such an extensive profile was whether each position was recognized independently and, not surprising; it was found that there were correlation especially involving neighboring nucleotides. Given this cooperativity/anticooperativity, how likely to be successful are computational models based on the independent recognition of each nucleotide within a sequence?

Novel approaches for designing new RBPs and RNAbinding ligands

The reverse problem of specificity prediction is the redesign of specificity. In fact, a physicist would state that a problem is not satisfactorily understood until successful predictions are made and experimentally verified. While this task has been accomplished successfully for zinc finger proteins binding to DNA, it has been far more difficult to do the same with RBPs.

So far, the only significant success has been obtained with Pumilio proteins. These are multidomain RBPs, each containing typically eight structurally identical repeats that recognize a single nucleotide, A, G, U or C, in a manner that can be specified by changing 23 amino acids within each domain. Hall has demonstrated the recognition principle eloquently in the span of about 10 years and how this can be applied to design proteins that bind to single stranded RNA specifically. At the workshop, she illustrated the expansion of her structural analysis to other less canonical members of this protein family, unusual proteins that are more distantly related to the classical Puf motifs, while Henn provided a thorough biophysical analysis of a particular classical human Puf proteins, to thoroughly understand the thermodynamic and biophysical signature of this protein class, suggesting the presence of a nonclassical mode of binding to even the classical protein.

Progress in structure determination promises nonetheless to allow other classes of proteins to be designed, which would generate more diverse and interesting tools to interrogate biological processes but, most importantly from the perspective of the workshop, would provide an exacting test of our understanding of the molecular basis of specificity and of computational programs aimed at predicting, calculating or controlling RBPs. Thus, Ramos shows that a single base change in an RNA can be compensated by a single amino acid change in the protein, in a manner that affects the biological activity of this protein. Similarly, Varani showed the successful redesign of the specificity of an RRM protein, a long sought after goal that has so far escaped successful execution, based on the structural and computational analysis of two binding pockets that resulted in the generation of a protein with altered biological activity. These two examples are idiosyncratic and lack, so far, the systematic power of Puf proteins, but they are necessary steps to expand our understanding of this important protein family and our ability to utilize RBPs as tools.

A related approach that also addresses the issue of specificity (or not) of RBPs deals with the design of socalled argininerich motif proteins. This is a common class of domains, containing short, 710 amino acids stretches of Arg and Lys residues often found in phage and viral RNA binding proteins. Typically, this protein bind nonspecifically, in the absence of cellular factors, but Varani showed how by rigidifying the peptide and providing a cyclic framework, he was able to obtain very high (pM) affinity and specificity. These design projects would be considerable facilitated by better molecular modeling of these complexes, based on atomistic models, as illustrated by Carloni.

Computational approaches for predicting RNA binding interfaces and proteinRNA docking

The technological advances discussed at the meeting have greatly enhanced our ability to identify new RBPs and find probable RNA targets of selected proteins, but high throughput approached cannot provide the details on the specific mode of interaction between a given protein and its target. As extensively discussed at the meeting, the critical information regarding the pairwise interactions between proteins and RNA can only be derived from structural methods (currently low throughput). However, due to the enormous amount of effort, high cost and time needed to solve the structures of proteinRNA complexes, computational methods have been developed to bridge the gap between the extensive information derived from the high throughput experimental technology and the detailed highly desired but rare structural information.

At the meeting Eric Westhof discussed the different RNA structural features (defined as RNA modules) that can be predicted from sequence alone and a new computational approach for predicting the DNA and RNA pairwise probabilities in proteins, which are directly derived from physicochemical properties (learnt from low throughput structural methods) and evolutionary features (learnt from high throughput sequencing methods). Dobbs presented machine learning and homology based approaches for predicting RNAprotein pairs as well as methods for predicting the specific protein residues which are probable to be involved in the direct interaction with the RNA.

The next extremely challenging computational tasks in the field of proteinRNA interactions discussed at the meeting is predicting the detailed interaction between a protein and an RNA, even when the protein structure is known or can be predicted from close homology. Bujnicki gave an overview of the different strategies and computational methods employed for modeling proteins and RNA and for docking nucleic acids (DNA and RNA) on proteins. While docking methods are widely employed to study proteinprotein interactions, very few methods are currently available to model proteinnucleic acid complexes and most are sparsely tested. Bujnicki presented some examples of successful docking predictions, while Tuszynska demonstrated a dedicated software for proteinnucleic acid docking (NPDOCK) developed by the same group. Clearly the field of proteinRNA docking is at early stages and many challenges remain to be overcome before computational modeling can provide nearly atomic resolution structures of proteinRNA complexes.

The interplay between coding and noncoding RNPs

Cellular RNAs can be partitioned between coding (mRNAs) and noncoding RNAs, including rRNA, tRNA, snoRNA and a very large group of mainly uncharacterized long non coding RNAs (lncRNAs), promoter associated RNAs, antisense transcripts etc. One of the topics discussed at the meeting is the regulation of different classes of RNAs by RBPs. Ohler and Neelanman showed that lncRNAs are generally less stable RNAs and undergo more posttranscription regulations, yet much less is known, compared to mRNAs, about the RBPs which regulated these posttranscriptional events. Interestingly, it was discussed that lncRNA contain short coding region (Open Reading Frames; ORFs), but accurate detection of these short ORFs is a considerable computational challenge. Ohler presented an algorithm to analyses high throughput data derived from a relatively new experimental methodology developed by the Weissman group for mapping ribosome footprints, known as Ribosomal Profiling. The algorithm which is based on Fourier transform approach, evaluates the likelihood that a region codes for a peptide based on the 3nucleotide periodicity of the signal. Employing this algorithm, they were able to identify hundreds of new putative ORFs in lncRNA. These results again demonstrate the power of combining computational methodologies with high throughput experimental data.

Outcome of the meeting

The meeting was an astounding scientific success, according to all speakers, for three reasons.

1. Its format, intimacy and small number of speaker, which provided the meeting with the feel of a workshop, almost group meetinglike, where problems with techniques, approaches and ideas were openly addressed and discussed without hesitation, in a context that was not dominated by any group of speakers

2. The presence of several young speakers, graduate students and postdocs, about 1/3 of all attendants, who, by virtue of giving full presentations, were fully integrated in the community without subjection to more senior speakers

3. The design of the meeting to bring together speakers coming from different communities (structural biologists and biochemists; computational biologists and modelers; genome scientists who apply highthroughput methods), who know of each other and their work, but do not often communicate so closely in such a small workshop setting and with the opportunity to engage freely and extensively with members of the other communities.

As a result of these positive elements, as summarized brilliantly by a set of closing remarks and discussion led by Ares, the meeting highlighted several essential elements that the community believe would push the field further forward.

1. There is a continuing need to increase the number of structures of proteinRNA complexes, which provide the absolutely necessary basis to interpret genomewide dataset and inform computational prediction methods. The number and quality of structure of RBPs lags significantly behind the equivalent problem of proteinDNA recognition and this severely hinders progress in the field.

2. Computational prediction of the RNA target of an RBP and of RNA structure and the interplay between structuresequence and recognition lag behind comparable advances in proteinDNA recognition as well. Exploiting the database generated by highthroughput methods and the growth in structures would undoubtedly provide progress in the next few years, but close communications between the communities will be key to advances. Sequence based models should find significant increase in importance in the next few years as high throughout methods grow in scope. Progress in protein design would provide more exacting tests of computational atomic models of interfaces.

3. In addition to technical challenges with the reduction of false positive and false negatives, there is a need to further improve highthroughput methods to better account for the interplay of RNA structure and sequence in RNA recognition as well as the role of multiple protein (and RNA) domains in dictating specificity. These methods need to be expanded to biological systems other than traditional cultured cells. Methods that investigate the landscape of RBP in vitro should be expanded and more closely connected with in cell high throughput methods.

4. The role of new RNAbinding proteins, especially metabolic enzymes in cellular function must be better understood. How many unknown RBPs still exist? What is their functional role and which RNAs they interact with? Conversely, the new universe of noncoding RNAs must be characterized with regards to its association with RBPs.

5. Ultimately, this information should be fed into computational models of cellular regulatory circuits. Although the combinatorial complexity of RNAbased regulation is stunning and daunting, efforts should be initiated to establish programs to mathematically model these circuitry.

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Chapter 17

Hybrid Methods of Imaging (15w5012)

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Overview of the Field

Since its invention in the 1960s, computed tomography (CT) has become an indispensable technique of biomedical imaging. Numerous modalities have been introduced since then, including the traditional X-ray CT scan, SPECT, MRI, Optical-, Ultrasound-, and Electrical Impedance Tomography, with many others being currently developed. All these techniques are used to obtain images of the internal structure of the living tissue in humans or animals. They differ by the underlying physics, by costs (sometimes very significant) and health hazards to the patient (also significant in some modalities). But most importantly, they differ by the biomedical features they can (or cannot) detect. In spite of the wide variety of existing modalities, such tasks as detection of cancerous tumors in soft tissues still present a significant challenge. Thus, the search continues for new, more capable, more sensitive, and more affordable imaging techniques.

Recently, several "hybrid" of "coupled physics" modalities have been introduced. They remain a subject of intensive research activity since then, due to the great promises they hold for medical imaging. By combining two or three different types of waves (or physical fields) these methods overcome limitations of classical tomography techniques and deliver otherwise unavailable, potentially life-saving diagnostic information — at a lesser cost and with less harm to the patient. Among these methods are the Thermoacoustic Tomography, Photo-Acoustic Tomography, Ultrasound Modulated Optical and Impedance Tomographies, Magneto-Acousto-Electric Tomography (MAET) and several other modalities combining magnetic fields with ultrasound scanning of the tissue. Closely related to these methods are so-called combined physics modalities such as Current Density Imaging and Elastography.

As a rule, the images in these modalities are obtained by complex mathematical procedures, rather than through direct acquisition. In most cases, the necessary mathematics involves sophisticated techniques of integral geometry, contemporary theory of partial differential equations, spectral theory, microlocal analysis, numerical analysis, etc. Each time a new modality is introduced, researchers face new mathematical challenges, ranging from the theoretical questions about the existence, uniqueness and stability of the solutions to the equations (representing the images that need to be reconstructed), to the more practical tasks of developing computer algorithms and programs capable of computing the images fast and in high resolution, as required by modern medical practice. These tasks are not trivial; obtaining these results requires collaboration of theoretical and applied mathematicians, as well as an interdisciplinary dialog between the mathematicians and physicists, engineers, and medical practitioners who build and use this state-of-the-art equipment.

Recent Developments and Open Problems

Thermo-, photo-, and opto- acoustic tomography

Thermoacoustic Tomography (TAT) [61, 111] and Photoacoustic (or Optoacoustic) Tomography (PAT) [60, 89, 35] are the most developed of the novel "hybrid" methods of medical imaging. These hybrid (or "coupled physics") modalities combine different physical types of waves in such a way that the resolution and contrast of the resulting method are much higher than those achievable using only acoustic or electromagnetic measurements. In the case of TAT and PAT, these waves are electromagnetic (EM) waves (infrared laser radiation in PAT and microwaves in TAT) and high-frequency acoustic waves. In these modalities, a short pulse of EM waves irradiates the biological object of interest, thus causing small levels of heating. The resulting thermoelastic expansion generates a pressure wave that starts propagating through the object. The absorbed EM energy and the initial pressure it creates are much higher in the cancerous cells than in healthy tissues (see the discussion of this effect in [110, 112, 111]). Thus, if one could reconstruct the initial pressure, the resulting TAT/PAT tomogram would contain highly useful diagnostic information. The data for such a reconstruction are obtained by measuring time-dependent pressure on a surface completely or partially surrounding the object. Although the initial irradiation is electro-magnetic, the actual reconstruction is based on acoustic measurements. Therefore these methods yield high contrast because of the higher absorption of EM energy by cancerous cells, while also achieving good (sub-millimeter) resolution because of the ultrasound measurements. (The radio frequency EM waves are too long for high-resolution imaging). Thus, TAT/PAT combine the advantages of two types of waves by using them in tandem, while eliminating their individual deficiencies.

TAT and PAT began to be developed as a viable medical imaging technique in the mid 1990s [89, 60]. Some of the mathematical foundations of these imaging modalities were originally developed starting in the 1990s for the purposes of the approximation theory, integral geometry, and sonar and radar (see [3, 73, 2, 63, 48] for references and extensive reviews of the resulting developments). Physical, biological and mathematical aspects of TAT/PAT have been recently reviewed in [63, 2, 48, 47, 90, 107, 109, 111, 112, 99].

The inverse source problem

The first step in solving the inversion problem of TAT/PAT is finding the initial pressure in the tissues originating from the EM excitation; this step is frequently called the inverse source problem.

Explicit inversion formulas The simplest situation to treat is when the speed of sound in the tissues can be approximated by a constant, as is the case, for example, in breast imaging. Under the constant sound speed assumption, a solution of the inverse source problem can be represented by explicit closed-form formulas, for certain simple acquisition surfaces. Such formulas are similar to the well-known filtration/backprojection formula in X-ray tomography; they allow one to compute the initial pressure by evaluating an explicit integro-differential operator at each node of a reconstruction grid. The existence and form of explicit inversion formulas are closely related to the shape of the data acquisition surface. For the simplest case of a planar surface, explicit formulas have been known for several decades [14, 43, 41]. The authors of [113] found a filtration/backprojection formula that is valid for a plane, a 3D sphere and an infinite 3D cylinder. In [46, 44, 68, 85] several different inversion formulas were derived for spherical acquisition surfaces in spaces of various dimensions. In [65] explicit reconstruction formulas were obtained for detectors placed on a surface of a cube (or square, in 2D) or on surfaces of certain other polygons and polyhedra. Several authors [98, 91, 84, 51] recently found explicit inversion formulas for the data measured from surfaces of an ellipse (in 2D) or an ellipsoid (in 3D) surrounding the object. These formulas can be further extended by continuity to an elliptic paraboloid or parabolic cylinder [53, 52]. Certain more complicated polynomial surfaces (including a paraboloid) were considered in [91], although not all of these surfaces are attractive from a practical point of view, as they would require surrounding the object by several layers of detectors. In addition to the explicit formulas, there exist several reconstruction algorithms (for closed acquisition surfaces) based on various series expansions [86, 87, 69, 66, 54]. These techniques sometimes lead to very fast implementations (e.g. [69, 66]); however, their efficient numerical implementation may require certain non-trivial computational skills.

Operator solution In the more general case of variable (but known) speed of sound in the tissues, the solution of the inverse source problem is more complicated. In [1], a general operator inversion formula is obtained, which applies to arbitrary geometry of the observation surface, variable sound speed under a non-trapping condition and functions not necessarily supported inside the observation surface. The formula is written in terms of operator functions of of the Dirichlet Laplacian on the domain surrounded by the acquisition surface, and thus it is not easy to apply. However, it also leads to an eigenfunction expansion method that generalizes the series algorithm of [69] to the case of a variable sound speed, which is more tractable computationally.

Time reversal Time reversal was successfully used by multiple authors, both theoretically and as a computational technique, to solve the inverse source problem of TAT/PAT ([46, 58, 103, 57, 97, 104, 56]). This method consists in solving the wave equation backwards in time, within the domain Ω surrounded by the detectors, using the measured data as the Dirichlet boundary conditions on the boundary $\partial\Omega$ of Ω. Theoretical foundations of the method are the simplest in the case of the constant speed of sound in 3D space. Under these assumptions, the Huygens principle guarantees that the pressure within the domain and on its boundary vanishes after a finite time T_0. Therefore, one can impose zero initial conditions at $t = T \geq T_0$, and solve the wave equation backwards in time in the domain $(0, T) \times \Omega$. Since the wave equation with such initial and boundary conditions admits a unique solution, at the time $t = 0$ the so-computed solution will coincide with the sought initial pressure in the direct problem.

In the 2D case and/or when the speed of sound is not constant and is non-trapping, the pressure vanishes only as $t \to \infty$. If only a finite amount of data is collected for $t \in [0, T]$ with sufficiently large T, time reversal still yields a good approximation. In order to reconstruct correctly the singularities of the solution, one has to truncate the data smoothly ([58, 57]), or to initialize the solution using the harmonic extension of the boundary data at $t = T$ ([103, 97, 104]). Alternatively, a theoretically exact solution can be obtained in the form of converging Neumann series ([103, 97, 104]), although in practice the first term in the series is close enough (i.e., the error in the first approximation is less than the errors arising from noise in the data, insufficient sampling, etc.).

Quantitative TAT/PAT

One of the active areas of current research is the so-called quantitative PAT (QPAT) [37, 96, 106] which aims to recover, in addition to the initial pressure, optical properties of the tissue (e.g., Grüneisen coefficient) and the fluency of electromagnetic radiation as it propagates through inhomogeneous tissue.

In the setting of quantitative PAT, the initial acoustic pressure reconstructed above takes the form $H(x) = \Gamma(x)\sigma(x)u(x)$, where $\Gamma(x)$ is the Grüneisen coefficient quantifying the photo-acoustic effect (how much acoustic pressure is generated per absorbed photon), $\sigma(x)$ is the optical absorption coefficient, and $u(x)$ is the light intensity reaching the point x. Note that $\Gamma(x)$ and $\sigma(x)$ are constitutive (optical) properties of the biological tissues. Thus reconstructing $\Gamma(x)$ and $\sigma(x)$ could provide information about the health of these biological tissues. In contrast, $u(x)$ is a quantity that depends on the experimental setting, namely how light is injected into the domain and how it propagates in it.

The quantitative step of hybrid inverse problems precisely aims at quantifying which constitutive parameters may be reconstructed and with which stability. This now requires a model for light propagation. Assuming a well-accepted diffusion model, which is accurate for optically sufficiently large objects, then $u(x)$ is the solution to the following elliptic equation:

$$-\nabla \cdot D(x)\nabla u(x) + \sigma(x)u(x) = 0,$$

in a domain $X \subset \mathbb{R}^n$ with, say, Dirichlet conditions $u = f$ on the boundary ∂X. Here $D(x)$ is an additional diffusion coefficient. Recall that we now know $H(x) = \Gamma(x)\sigma(x)u(x)$. What information on the unknown parameters $(D(x), \sigma(x), \Gamma(x))$ may we infer from knowledge of H and the above PDE constraint? Clearly two constraints cannot possibly help us reconstruct four parameters (since $u(x)$ is also unknown). If instead of considering a single experiment with boundary condition f we considered J different experiments with boundary conditions $\{f_j\}_{1\leq j\leq J}$, we would know $H_j = \Gamma(x)\sigma(x)u_j(x)$, for $1 \leq j \leq J$. What can we reconstruct for different values of J, and with which stability? How do such reconstructions depend on the choice of the boundary conditions $\{f_j\}$?

This kind of questions, for the modality QPAT as well as for many other hybrid inverse problems, was a central theme in the workshop. For QPAT modeled by a diffusion equation, we have a relatively complete theory available to us. The upshot is that independent of the value of J, there is no chance to reconstruct all three parameters $D(x)$, $\sigma(x)$ and $\Gamma(x)$ simultaneously. All one can reconstruct is two <u>independent</u> functionals of these three parameters. When one such parameters is known a priori, then the other two can be reconstructed. Moreover, for sufficiently large J, the reconstruction of the parameters may be formulated as an elliptic system of differential or pseudo-differential operators that provide optimal stability estimates, in the sense that we know how many times data need to be differentiated in the reconstruction of said parameters. For references on such results and generalizations of what is described above, see for instance [25, 30, 28, 40, 37, 64].

In the setting of quantitative TAT, the propagating radiation is much lower frequency and has to be modeled by the Maxwell equations instead of the above diffusion model. The initial pressure is then given by $H(x) = \Gamma(x)\gamma(x)|E|^2(x)$, where Γ is still the Grüneisen coefficient, γ is the electric conductivity and E is the electric field. For references on the results that have been obtained for this problem, see e.g. [13, 29, 33].

Open problems in TAT/PAT/QPAT

Problem with incomplete data In most practical situations the object of interest cannot be completely surrounded by the acoustic detectors. (Perhaps, the only exception is imaging of small animals.) Most of the theoretical and algorithmic results obtained for the inverse source problem in TAT/PAT are based on the assumption of the complete acquisition (from a closed acquisition surface). Theoretical analysis shows (see, e.g., [63]) that stable solution of such an inverse problem with incomplete data is only possible if the object and the acquisition surface satisfy the "visibility condition": every bi-characteristic of the wave equation originating at each point of the domain of interest must reach the detectors in finite time. The existing constructive approaches to treat the problem with incomplete data are either computational in nature [70], or are parametrix-based [13, 14, 92], i.e. aim to reconstruct accurately only the jumps of the function but not the quantitatively correct values. Numerical experimentation has shown [103, 97] that a satisfactory solution can be obtained by an iterative algorithm (Neumann series), in the case when the visibility condition is satisfied. However, in the case of open space propagation of acoustic waves convergence of such iterations has not been proven yet

Reflecting boundaries Practically all existing theory of TAT/PAT is based on the assumption that acoustic waves propagate in free space, and that reflections from detectors and the walls of the water tank can be either neglected or gated out. In this case, acoustic pressure within the object vanishes quite fast and the inverse problem of TAT/PAT can be solved by time reversal. Time reversal yields a theoretically exact reconstruction if the speed of sound is constant or if it satisfies the so-called non-trapping condition (see [1, 58, 103, 97]). This method can be implemented for a general closed acquisition surface and known speed of sound using finite differences [58, 97]; it can also be realized (for simple domains) using the method of separation of variables [69], or, for certain geometries, it can be replaced by equivalent explicit backprojection formulas [65]. Other reconstruction algorithms, although not related directly to time reversal, also require that the pressure vanishes sufficiently fast [63].

However, free space propagation cannot always be used as a valid model. For example, one of the most advanced PAT acquisition schemes (developed by researchers from the University College London [42]) uses optically scanned planar glass surfaces for the detection of acoustic signals. Such surfaces act as (almost) perfect acoustic mirrors. If the object is surrounded by such reflecting detectors (or by a combination of detectors and acoustic mirrors), wave propagation occurs in a resonant cavity. It involves multiple reflections of waves from the walls, and, if the dissipation of waves is neglected, the acoustic oscillations do not die out. Traditional time reversal and other existing techniques are not applicable in this case; new reconstruction algorithms need to be developed for TAT/PAT within resonant cavities. There are few works on this problem [38, 39, 67, 55]. Some of these techniques (e.g. [67]) work only in special geometries and with constant speed of sound. Others ([55]) are more general, but the analysis requires spectral information not easily available aside of several simple cases. This problem requires further investigation.

Magnetic Resonance Elastography

Magnetic Resonance Elastography (MRE) [80, 74, 101] is a non-invasive medical imaging technique that measures the mechanical properties (stiffness) of soft tissues by introducing shear waves and imaging their propagation using MRI. Pathological tissues are often stiffer than the surrounding normal tissue. For instance, malignant breast tumors are much harder than healthy fibro-glandular tissue. This characteristic has been used by physicians for screening and diagnosis of many diseases through palpation. MRE calculates the mechanical parameter as elicited by palpation, in a non-invasive and objective way.

In magnetic resonance elastography, shear waves are generated by an electro-mechanical transducer on the surface of the skin. (Shear waves are mechanical waves in elastic tissue in which oscillation occurs in the direction normal to the propagation direction). The propagation of the shear waves is then registered (as a function of time and space) by the MRI scan, with the scan modulated by the same frequency as the frequency of the shear waves. This encodes the amplitude of the shear wave in the tissue in the phase of the MRI image. The inverse problem arising in this modality is to reconstruct a quantitative measure of tissue stiffness (an elastogram) from the MRI images.

MRE is currently being investigated as a diagnostic tool for a multitude of diseases affecting tissue stiffness. This modality is being clinically used for the assessment of hepatic fibrosis, since it is well known that the liver stiffness increases with the progression of this disease. MRE is also being utilized for monitoring treatment efficacy of fibrosis and related diseases.

The micro-MRE (magnetic resonance elastography) and PVS (pendulum-type viscoelastic spectrometer) are measurement devices to measure the viscoelasticity of a medium. More precisely, micro-MRE provides an interior measurement of time harmonic waves inside a medium. PVS provides a boundary measurement, obtained by a laser, of the displacement of a specimen under time harmonic torsion or bending generated by Lorentz force (i.e. subjecting the specimen to an oscillating magnetic field). The frequency range of measurement for PVS is much larger than that of for micro-MRE, and the sample size for micro-MRE is much larger than that of for PVS. While MRE has been successful for diagnosing liver diseases, the study of micro-MRE is to provide a benchmark test for MRE.

Recent advances in MRE and micro-MRE were the subject of several talks at the workshop; see also below for additional details on the second, quantitative, step of elastography, which is shared by both magnetic resonance elastography as described above and ultrasound elastography [24, 50, 88].

Other hybrid modalities

Several other hybrid modalities beyond Photo-acoustic tomography and Elastography have been analyzed in the mathematical literature and have been discussed during the workshop. While such modalities involve a first modeling step that we are not going to describe here, their mathematical analysis primarily involves solving a quantitative inverse problem from knowledge of internal data, as was the case for the quantitative PAT and TAT problems described in section 17.

These modalities are all described by a partial differential constraint involving unknown coefficients and an internal constraint, also possibly involving unknown coefficients. Here is a partial list of examples. In the quantitative step of Elastography, one seeks to reconstruct elastic properties of biological tissues from knowledge of internal displacements, which were provided by MRE or Ultrasound Elastography as mentioned above. In a scalar model for such displacements, one aims to understand what may be reconstructed from (a, b, c) in an elliptic equation of the form

$$(-a_{ij}\partial_i\partial_j + b_i\partial i + c)u_k = 0, \qquad X, \quad u_k = f_k, \quad \partial X$$

(with Einstein convention of summation of repeated indices) from knowledge of u_k in X for $1 \le k \le K$. This problem, as well as generalizations to time dependent problems and systems of elasticity and applications to other fields such as hydraulics, are analyzed in, e.g., [7, 19, 31, 32, 34, 72, 75, 76].

In the application of Magnetic Resonance Electrical Impedance Tomography (MREIT), also known as Current Density Impedance Imaging (CDII), one seeks to reconstruct the (possibly tensor-valued) conductivity γ from knowledge of the current $J = \gamma\nabla u$ or its norm $|J| = \gamma|\nabla u|$, where u solves the elliptic model $\nabla \cdot \gamma\nabla u = 0$ on X with appropriate boundary conditions. This problem has been extensively studied recently, with a list of references that includes [22, 59, 83, 82, 81, 100].

In the modalities called Ultrasound Modulation (Electrical or Optical) Tomography or acousto-optics tomography, the electrical or optical properties of biological tissues are modified by ultrasound modulation. This results in problems with know internal functionals of the form $H = \gamma\nabla u \cdot \nabla u + \eta\sigma u^2$ for known constant η, where u is a solution of an elliptic equation of the form $-\nabla \cdot \gamma\nabla u + \sigma u = 0$ on a domain X with appropriate boundary conditions. The modeling and analysis of this mathematically difficult problem is addressed in a series of papers such as [11, 12, 17, 20, 23, 26, 27, 36, 49, 77, 78, 79].

Each of these problems display specific features that are not shared by other inverse problems. There are, however, common themes that most of these hybrid inverse problems share, including a modeling as a system of pseudo-differential or differential equations. General frameworks to analyze these hybrid inverse problems were developed in, e.g., [18, 64].

Also, in most hybrid inverse problems, specific qualitative properties of solutions to elliptic equations need to be satisfied, for instance that there be no critical points, or that the gradients of n solutions form a basis in $\mathbb{R}^n$ at each point in a domain $X \subset \mathbb{R}^n$. A great variety of results have been obtained in this direction, while many more interesting problems remain open. Several recent results were discussed in the workshop; see below. For a partial list of references on such results, the reader may consult [4, 5, 6, 8, 9, 16, 21, 30].

Additional information about hybrid inverse problems may also be found in the following review papers and books [10, 15, 16, 93, 99].

Presentation Highlights

Thermo- and photo- acoustic tomography

One of the first talks of the conference was by **Alexander Oraevsky** (TomoWave Laboratories, Inc.) on of the founders of the Optoacoustic tomography (OAT, also known as PAT). The speaker concentrated on advances made by this modality in two decades after its invention, and on present challenges arising in OAT. Currently, PAT is entering the real world of clinical applications, with diagnostic imaging of breast cancer being the first major market niche for this technology where existing modalities have apparent drawbacks. The main value of OAT is in its potential capability to provide functional and molecular information based on quantitatively accurate display of the optical absorption coefficient. However, quantitatively accurate OAT has not been demonstrated yet. The speaker emphasized the need for a full view three-dimensional tomography system that acquires complete set of forward data and uses rigorous solutions for inverse problem of image reconstruction have the potential for success in the breast cancer diagnostics. As a step in this direction the speaker and collaborators combined laser optoacoustics and ultrasound tomography systems in a single device. The optoacoustic sub-system provides images based on distribution of molecular chromophores in the body, while the ultrasound sub-system provides anatomical images of tissue structures and can also provide the speed of sound and acoustic attenuation images, which can be used for iterative reconstruction of more accurate optoacoustic images.

The topic of practical and computational challenges arising in practical applications of hybrid modalities was continued by **Mark A. Anastasio** (Washington University). The speaker has reviewed recent advancements in practical image reconstruction approaches for Photoacoustic Computed Tomography (PACT), including physics-based models of the measurement process and associated optimization-based inversion methods for reconstructing images from limited data sets in acoustically heterogeneous media. He also discussed applications of PACT to transcranial brain imaging and breast cancer detection.

An overview of several mathematical theories applying to photo-acoustic and thermo-acoustic tomography was presented in **Rakesh**'s lecture, including those with an integral geometric flavor [46, 48] as well as those based on the underlying wave equation for sound propagation [45, 103, 105].

Two talks by **Linh Nguyen** (University of Idaho) and by **Eric Todd Quinto** (Tufts University) addressed the problem of incomplete data in TAT and PAT. One of the frequently used practical approaches to this problem is to simply backproject (or backpropagate) the measured data while replacing the missing part of the data by zeroes. This leads to loss of certain features in the reconstructed image and sometimes generates spurious features, or artifacts. In his talk, L. Nguyen considered this problem in terms of the inversion of the spherical means Radon transform with incomplete data, and analyzed geometry and strength of the artifacts in the reconstruction. Continuing the topic, Todd Quinto presented a general paradigm to classify added artifacts in limited data tomography, developed jointly with J. Frikel that explains the locations and properties of added artifacts that appear in limited angle tomography. The speaker used microlocal analysis to understand the effect of data restriction, and provided reconstructions from real and simulated data for X-ray CT, photoacoustic tomography, and the circular transform.

Talks by **Yang Yang** (Purdue University) and **Sebastian Acosta** (Baylor College of Medicine) addressed another important open problem of TAT/PAT — reconstruction in the presence of reflecting boundaries. Y. Yang presented a study of the mathematical model of thermoacoustic tomography in bounded domains with perfect reflecting boundary conditions. The speaker presented an averaged sharp time reversal algorithm (developed jointly with P. Stefanov) which solves the problem with an exponentially converging Neumann series. The presented numerical reconstruction was implemented in both the full boundary and partial boundary data cases. Similarly, S. Acosta considered the problem of photoacoustic and thermoacoustic tomography in the presence of physical boundaries such as reflectors or interfaces, which reflect some wave energy back into the domain. The difference with the previous talk was that non-perfectly reflecting boundaries were considered. The resulting inverse problem was related with a statement in boundary observability and stabilization of waves. The speaker presented uniqueness and stability of the inverse problem and proposed two different reconstruction methods. It was shown that in both cases, if well-known geometrical conditions were satisfied, the inverse problem can be solved under the assumption of variable wave speed and in the case of measurements given on a subset of the boundary.

L. Kunyansky (University of Arizona) presented a talk on inversion formulas for the spherical means transform with centers lying on the boundaries of in corner-like domains, such as a quadrant in 2D or an octant in 3D. The presented formulas allow one to reduce the problem of inversion of the spherical transform to the similar problem formulated in term of the classical Radon transform. Methods and algorithms for solving the latter problems are well known.

As one can see in the previous talk outlines, integral geometry techniques are an essential ingredient in the first, qualitative step of photo-acoustic tomography. Integral geometry of course finds many applications in imaging sciences. This includes the imaging of materials, and at the forefront of current research, the reconstruction of anisotropic coefficients. **Victor Palamodov** presented recent results on the evaluation of the residual elastic strain in structural material that requires imaging a six-component tensor quantity ε in three dimensions by combining two imaging modalities. He first devised a method of reconstruction of small residual strain fields in a body based on data of diffraction pattern under penetrated X-ray or neutron radiation. The mathematical model is the longitudinal (axial) line transform of the tensor ε. These data are only sufficient for determination of the Saint-Venant tensor $V\varepsilon$ which is a 2×2- symmetric-skewsymmetric tensor field. A complete reconstruction of a strain field by this method is impossible, since the Saint-Venant tensor vanishes for any potential strain field.

As a second imaging modality, the method of polarization tomography is based on measurements of transformation of the polarization ellipse of the penetrating light through a weakly optically anisotropic material. The mathematical model is the line integral $T\varepsilon$ of the traceless normal (truncated transverse) part of the stress field ε. A simple method of reconstruction of the displacement form from Tuy-complete data of $T\varepsilon$ for any tensor whose axial line integrals vanish. As Victor Palamodov showed us, both methods can be combined for complete reconstruction of an arbitrary small strain tensor e from data of axial and traceless normal ray integrals of ε.

Quantitative Photo-acoustic Imaging (QPAT)

As indicated in section 17, QPAT aims at reconstructing optical coefficients from knowledge of an initial pressure pressure of the form $\Gamma(x)\sigma(x)u(x)$. Three talks, by Giovanni Alberti, Kui Ren, and Otmar Scherzer, analyzed various aspects of QPAT.

In his talk, **Kui Ren** considered the setting where $u(x)$ is modeled as the angular average of a phase space photon density $v(x, \theta)$, which then solves a linear Boltzmann equation. Kui Ren presented several new theoretical results on this problem, as well as applications to the related problem of fluorescence photo-acoustic tomography.

In the previous talk, as well as in many past works on QPAT, a precise model for $u(x)$ is necessary. **Giovanni Alberti**'s presentation focuses on situations where such model may not be known precisely, as is arguably the case in photo-acoustics in some settings. Let us write $f = \log(\Gamma\sigma)$ and $g_i = \log u_i$. The main focus of his talk was the reconstruction of the signals f and g_i, $i = 1, \ldots, N$, from the knowledge of their sums $h_i = f + g_i$, under the assumption that f and the g_is can be sparsely represented with respect to two different dictionaries A_f and A_g. This generalises the well-known "morphological component analysis" to a multi-measurement setting. The main result states that f and the g_is can be uniquely and stably reconstructed by finding sparse representations of h_i for every i with respect to the concatenated dictionary $[A_f, A_g]$, provided that enough incoherent measurements g_is are available. The incoherence is measured in terms of their mutual disjoint sparsity. Giovanni Alberti showed us how to apply such a disjoint sparsity approach in quantitative photoacoustic tomography, including in the case when the Grüneisen parameter, the optical absorption and the diffusion coefficient are all unknown.

In his presentation based on joint work with Elena Beretta (Milan), Markus Grasmair (Trondheim), Monika Muskieta (Wroclaw), and Wolf Naetar (Vienna), **Otmar Scherzer** further considered the reconstruction of diffusion, absorption, and Grüneisen parameters in QPAT. Recall from [28] that the three parameters cannot uniquely be reconstructed without prior information. Otmar Scherzer showed us that when one assumes piecewise constant diffusion, scattering, and Grüneisen parameters, respectively, then this problem can be decomposed into edge detection problem for the fluence $(u(x))$ and its derivatives and a parameter selection process based on the jump relations of the diffusion equation. Novel edge detection algorithms tuned to these problems have been presented.

Magnetic Resonance Elastography

Mayo Clinic is one of the birthplaces of Magnetic Resonance Elastography (MRE), and, currently, one of the leading institutions in the development of this modality. It was represented at the workshop by **Armando Manduca** who gave a talk "*Magnetic Resonance Elastography: A Signal Processing Perspective*". The speaker concentrated on strategies of improving the performance of MRE. He presented a statistical signal processing framework for steady-state MRE that enables rigorous characterization of the accuracy, precision, and uniqueness of harmonic motion information estimated from the raw data collected by an MRI scanner. After deriving and demonstrating the utility of this framework, the speaker discussed statistical strategies for optimally estimating MRE harmonic information directly from raw MRI data, overviewed several unique mathematical aspects of this problem, and presented a robust computational strategy for solving this problem.

Joyce R. McLaughlin (Rensselaer Polytechnic Institute) gave a talk on issues related to stability and statistics in shear stiffness imaging. She considered two different setups common in MRE. In the first measurement technique, the tissue is excited with a time harmonic oscillation and then sequences of magnetic resonance data are taken and processed to produce a movie of the oscillating tissue within the body. For this experiment the speaker presented stability results for a single elastic vector movie. In the second experiment one pulse or a sequence of pulses are imparted by focusing ultrasound; a wave with a front propagates away from the pulse position. The arrival time of one component of the wave is calculated from the movie created from a sequence of ultrasound data sets. (Technically, this modality is not MRE, but a closely related sonoelastography). The speaker investigated statistical properties of the noise in the image when using the direct algorithm, showing that even though the variance is infinite there are some favorable statistical properties.

Gen Nakamura (Hokkaido University) presented his latest results on micro-MRE and PVS. The speaker highlighted the importance of having a good model equation for the measurements and good inversion scheme to recover viscoelasticity from the measurements, in order to get some approximate true value of viscoelasticity of a medium. He introduced the background and motivation for the latest study, and presented the data analysis of these two rheological measurement devices.

Current Density Imaging, Acousto-Optic Imaging, Seismic Imaging, and Hybrid Inverse Problems features

Many other modalities were discussed at the workshop.

Amir Moradifam and Alexandru Tamasan gave two lectures on their recent work in Current Density Impedance Imaging (CDII), where the objective is to reconstruct γ from knowledge of $J = |\gamma\nabla u|$ where u is the solution to the elliptic equation $\nabla \cdot \gamma\nabla u = 0$ with appropriate boundary conditions.

Alexandru Tamasan (U of Central Florida) considered the inverse problem with boundary conditions given by the Complete Electrode Model introduced by Somersalo, Cheney and Isaacson [102]. As in the setting with Dirichlet boundary conditions [83], the inverse problem is modeled as a weighted minimum gradient problem. The main result obtained by Tamasan, joint with A. Nachman and J. Veras, is that knowledge at the boundary of the electrodes and their average input currents allows us to obtain the level sets of u but not their values. Additional knowledge of u on an appropriate line at the boundary then uniquely characterizes u and σ.

Amir Moradifam (U of California, Riverside), in a joint work with Robert L. Jerrard and Adrian Nachman, recast the CDII problem as an example of the general least gradient problem

$$\inf_{u\in BV_f(\Omega)} \int_\Omega \varphi(x, Du),$$

where $f : \partial\Omega \to \mathbb{R}$ is continuous, $BV_f(\Omega) := \{v \in BV(\Omega) : v|_{\partial\Omega} = f\}$, and $\varphi(x,\xi)$ is a function that, among other properties, is convex and homogeneous of degree 1 with respect to the ξ variable. The lecture covered existence, uniqueness, and comparison theorems for such minimizers. In particular, it was shown that if $a \in C^{1,1}(\Omega)$ was bounded away from zero, then minimizers of the weighted least gradient problem $\inf_{u\in BV_f} \int_\Omega a|Du|$ were unique in $BV_f(\Omega)$. Counterexamples showed that the regularity assumption $a \in C^{1,1}$ was sharp.

While hybrid inverse problems often find applications in medical imaging, some are concerned with seismic imaging. In fluid-saturated porous media, electromagnetic fields couple with seismic waves through the electroseismic conversion. In his talk, **Jie Chen** (Purdue University) presented recent results obtained with M. De Hoop on the mathematical analysis of the electroseismic conversion, and in particular showed us a stability result of recovering the seismic sources from the boundary seismic measurements followed by the inversion of a system of Maxwell's equations with internal data.

Alexander Mamonov (U of Houston) tackled the seismic imaging problem directly from boundary seismic data. In a joint work with Vladimir Druskin and Mikhail Zaslavsky, they introduced a novel nonlinear seismic imaging method based on model order reduction. The reduced order model (ROM) is an orthogonal projection of the wave equation propagator operator on the subspace of the snapshots of the solutions of the wave equation, which allows for the removal of multiple reflection artifacts and also enables us to estimate the magnitude of the reflectors similarly to the true amplitude migration algorithms. Numerical results for the standard Marmousi model and a synthetic high contrast hydraulic fracture example showed the usefulness of the new approach.

Acousto-optics was described earlier in these pages as an ultrasound modulation of the optical properties of a domain of interest. **John Schotland** (U of Michigan) presented a novel method to reconstruct the optical properties of a highly-scattering medium from acousto-optic measurements. The method is based on the solution to an inverse problem for the radiative transport equation with internal data. A stability estimate and a direct reconstruction procedure were described in this talk based on a joint work with G. Bal and F. Chen.

The essential reason for the ultrasound modulation in acousto-optics as well as ultrasound modulated electrical impedance tomography (EIT) is that these inverse problems are severely ill-posed in the absence of such a modulation. Their analysis nonetheless remains an essential aspect of theoretical inverse problems. Moreover, in spite of their limited resolution, such ill-posed inverse problems find important applications in medical imaging. Two talks were devoted to the analysis and the applications of variations of the standard Calderón problem [108].

In his presentation, **David Isaacson** (Rensselaer Polytechnic Institute) introduced a method to image the ventilation perfusion (VQ) ratio using EIT, which is the ratio of the volume of air entering a region of the lungs per breath divided by the volume of blood entering the same region per heart beat. This simple, macroscopic, quantity finds extremely useful applications in medical imaging. After providing means to define, measure, and image an approximation to this VQ ratio using electrical impedance imaging, David Isaacson showed us the relevance of such a test based on images, movies, and data from human subject studies obtained using the RPI and GE electrical impedance imaging systems.

The analysis of stability estimates for the variants of the Calderón problem remains a very active area of research. **Kaloyan Marinov** (Technical University of Denmark) presented recent joint results with Pedro Caro on the inverse boundary-value problems in an infinite slab with partial data. Generalizing results obtained by Li and Uhlmann [71] for the Schrödinger equation and by Krupchyk, Lassas and Uhlmann [62] for the magnetic Schrödinger equation, the obtained a log-log stability estimate for such inverse problems. Two settings of boundary measurements were considered: in the first inverse problem, the corresponding Dirichlet and Neumann data are known on different boundary hyperplanes of the slab; in the second inverse problem, they are known on the same boundary hyperplane of the slab.

Let us finally comment on the lecture given by **Yves Capdeboscq** (U of Oxford). We have mentioned earlier that many hybrid inverse problems required specific qualitative properties of solutions of elliptic equations. One such property is that the gradients of n solutions form a basis in $\mathbb{R}^n$ at each point inside a domain of interest X. It is relatively straightforward to construct such solutions (harmonic polynomials will do) for solutions of $\nabla\cdot\gamma\nabla u = 0$ when γ is sufficiently close to identity. Yves Capdeboscq presented a striking result essentially showing that in dimension $n \geq 3$, a choice of boundary conditions that works for a class of γ, in the sense that the n gradients are linearly independent inside X, does not work for another appropriate class of highly oscillatory conductivities γ. In other words, it may be the case that for certain hybrid inverse problems, the number of measurements that need to be performed depends on the structure of the unknown coefficients in order to afford reconstruction procedures with optimal stability. These results in dimension $n \geq 3$ are in sharp contrast to the two-dimensional setting; see [9].

Scientific Progress Made & Outcome of the Meeting

Hybrid inverse problems have emerged as a recent subfield of inverse problems theory to model, analyze, and compute, several coupled-physics imaging modalities recently introduced by biomedical engineers. This area of researche benefited from outstanding amount of work done by the applied mathematics community, as these pages hopefully convinced the reader. Its analysis involves many classical methods of inverse boundary value problems, such as integral geometric techniques, and inverse wave propagation problems that well studied in applications to medical and geophysical imaging.

This field also involves areas of mathematics that were not before of central interest in inverse problems. One main difference can be seen in the quantitative, second, step, of hybrid inverse problems, where reconstructions are performed from knowledge of internal functionals, which, as useful as they are, do not directly provide quantities of interest such as the elastic, electrical, or optical properties of biological tissues. Another new field of study came from the realization that hybrid inverse problems could optimally be solved provided that solutions of partial differential equations satisfied specific qualitative properties, such as not having any critical points, having gradients or Hessians of solutions that are of maximal rank, or light cones associated to some solutions with trivial intersection [18]. The aforementioned talk by Capdeboscq offered very interesting results on the problem of maximal rank of gradients of elliptic solutions.

More broadly, the field of hybrid inverse problems provides a wonderful platform for mathematicians and engineers to collaborate and foster a very promising area of research in biomedical imaging. The Banff International Research Station is an ideal environment for necessary discussions involving researchers with very different backgrounds to take place.

While it is early to assess the full impact of the meeting, here are a few points we can make on the scientific outcome of the meeting and its scientific value.

The two most successful hybrid imaging modalities at the moment are elastography and photo- and thermoacoustic imaging. These modalities were highly represented at the meeting, both from the engineering and mathematical points of view. Very lively discussions involving, among other researchers, Giovanni Alberti, Mark Anastasio, Guillaume Bal, Yves Capdeboscq, Amir Moradifam, and Alexandre Oraevsky took place to devise strategies in quantitative photo-acoustic tomography that would be both mathematically sound and feasible from an engineering perspective.

Elastography was also very well represented, mostly with researchers working on magnetic resonance tomography. Armandu Manduca, Joyce McLaughlin, and Gen Nakamura, will continue their long-standing collaborations to push the limits of this imaging modality, which currently performs extremely well in applications where resolution is not too demanding, and which is currently being developed to be used in more challenging applications such as brain imaging, that require much finer resolution.

From a mathematical perspective, the search for new methods that allow us to analyze the qualitative properties of PDE solutions remains very active. Giovanni Alberti and Guillaume Bal are currently collaborating on a method to prove that critical points of solutions to elliptic equations always exist for well chosen conductivities in dimension $n \geq 3$ independent of the imposed Dirichlet boundary condition. Several new results in this direction are expected to be announced in a near future, some of which being at least partially attributed to the discussions that took place during the meeting.

A set of lively conversations between Lin Nguyen and Leonid Kunyansky that took place during the workshop will result in a joint paper, currently being prepared for publication.

This, very partial, list of outcomes of the meeting is but one indication of the feeling shared by most participants, that the meeting had been an extremely useful one. A similar meeting "Mathematical Methods in Emerging Modalities of Medical Imaging" took place at BIRS in 2009. The list of references appended to the present report shows that an incredible amount of research in this area has been conducted since then. Some of that research can be directly linked to interactions among mathematicians and engineers that took place during that meeting. We are confident that the 2015 meeting will prove to be equally successful.

Participants

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Lai, Ru-Yu (University of Washington)
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Chapter 18

Advances in Combinatorial and Geometric Rigidity (15w5114)

July 12 - 17, 2015

Organizer(s): Robert Connelly (Cornell), Steven Gortler (Harvard University), Tibor Jordan (Eotvos University, Budapest), Brigitte Servatius (Worcester Polytechnic Institute), Meera Sitharam (University of Florida), Walter Whiteley (York University)

Overview of the Field

The mathematical study of rigidity dates back at least to Maxwell's study of structures in the mid 19th century, when this still unsolved problem was implicitly posed: which graphs form rigid bar-and-joint frameworks in 3-space, for almost all configurations of the joints? With a revival of the mathematical and computational field of rigidity starting about 40 years ago, there has been an accelerating pace of interest and results, decade by decade [21]. In the last decade which included three prior 5-day BIRS workshops, broad areas of geometric and combinatorial rigidity have seen: (i) major growth and consolidation in the theoretical results and in the techniques being used; (ii) expansion of the range of theoretical and applied areas where the theory is being applied and where new problems have arisen; (iii) additional researchers joining the work representing a range of ages, including a new generation of researchers from mathematics and computer science emerging to become major contributors; (iv) an increased visibility as an active research area in more universities and more countries, and (v) a recognized source of techniques in new areas where there are geometric constraints. As one sign of this current activity, several major collective works from recent workshops should appear in the next months [6].

In simple terms, the geometric and combinatorial rigidity community focuses on multiple approaches for detecting whether an input set of polynomial equations representing a geometric constraint system generically (a) has a solution (independence); (b) has locally isolated solutions (rigid), or (c) has exactly one solution up to a space of "trivial" transformations in the chosen geometry (globally rigid). One summary of the questions that inspire current work in rigidity is:

–whether the properties listed above hold generically, and are therefore combinatorial, for appropriate forms of genericity?

– when the properties are combinatorial, whether there exists a polynomial time decidable combinatorial characterization for testing these properties, or a combinatorial characterization of a particular type, for example matroidal related to submodular counting properties, or by graph decompositions and inductive graph constructions?

– what changes in the geometry, metric, dimension etc. of an initial solution preserve the key rigidity properties?

– what is the impact of initial symmetry, or initial periodic repetitive structure, and what are the key geometric conditions for singular (non-generic) behavior which shifts independence to dependence, rigidity to flexibility, and global rigidity to multiple realizations?

Recent Developments and Open Problems

We outline recent progress and key questions in a selection of topics chosen through input from active researchers from recent BIRS workshops and Workshop [15w114].

Inductive Construction, Decomposition, Matroid methods and Efficient Algorithms for characterizing/recognizing (local and global) rigidity and independence in dimension 3 and higher. These methods underpin core existing results, and provide examples and obstructions on the path towards solving the long standing open problem of combinatorial characterizations of the graphs which generically form rigid bar-and-joint frameworks in 3D and higher dimensions. There has been a revival of activity in this topic and this was the topic of our 2012 follow-on 2-Day BIRS workshop [4, 13, 8]. Even many known characterizations lack fast algorithms although the known cases of "pebble game" algorithms for sparse matroids and graded sparse matroids promise efficient extensions. As more computer scientists have entered the field, this topic is poised for a further surge of activity.

New developments on Global and Universal Rigidity Unlike in the case of local rigidity, it was only recently shown that global rigidity is generic in all dimensions [5]. The algebraic techniques employed are exciting, and carry over to a related characterization of universal rigidity. The results and the techniques have recognized applications in a variety of settings from constructing superstable tensegrity structures to sensor network localization, and determination of molecular structures from NMR data (distances) as well as dimension reduction and compression. Yet, there is a sense that the new techniques from convex analysis and semidefinite programming for universal rigidity (global rigidity in all higher dimensions) have not been fully exploited. Nor are there sufficient efficiently algorithmic constructions to prove some of the conjectured results. It would be useful to study and develop examples and extensions which clarify a number of aspects of global rigidity and maybe decide for which problems global rigidity is the most appropriate concept (e.g. the concept of "stability" of protein and nano molecular structures).

Extensions and generalizations of existing results to other expanded types of constraints, objects and underlying geometries

So far, the results are overwhelmingly for distance constraints and some are restricted to the Euclidean geometry and Euclidean metric. Recently, most of these now classical results have been generalized, from bar-and-joint structures to body-bar frameworks in all dimensions and sometimes extended to results not available for bar-and-joint structures [21, 4], and to classes of distance and direction constraints [21, 7]. Some additional very recent developments include the determination of some key combinatorial results with applications to structures in non-Euclidean normed spaces [10], as well as to 3D bar-and-joint frameworks supported on 2-dimensional manifolds [12]. The search for generalizations is motivated not only by recognized applications in CAD, additional constraints for control of robotic formations (with direction constraints), and emerging applications such as machine learning and graphics (see Item 5), but also by a need to deepen/broaden the existing theory. Step by step this requires a unification of algebraic and geometric techniques including duality, coning, lifting and projections, projective and affine geometry and synthetic geometry.

Extensions of existing results to symmetric and periodic settings (i.e., when the constraint systems, solutions, motions etc. are invariant under the action of various finite, periodic or crystallographic symmetry groups, and when their behavior changes from fully generic realizations).

One striking experience is that some symmetries can shift a generically rigid structure into a symmetry generic flexible structure and that these particular symmetries are common in protein structures (e.g. dimers for half-turn symmetry)[18] and periodic structures (e.g. inversive symmetry for crystals)[16]. Periodic structures have a mixed impact with some results that are more general than bar-and-joint frameworks [1, 13], and others which are more incomplete. There are starting to be algorithms that can test for this behavior under symmetry [17]. These are relatively new topics arising from crystallography (Zeolites and RUM Spectra [14]) and applications to material sciences such as the behavior of silica bilayers which show mirror symmetry between the layers. There has been an explosion of recent work and the connections to rigidity theory are still developing [6, 1].

Specific problems arising in various applications of independent mathematical interest

An example of this scenario is the recent work on a large repertoire of CAD constraints, using Grassmann-Cayley algebra for extracting geometric meaning within conditions under which efficient combinatorial methods can determine independence and sufficiency conditions for sparse graded graphs of an extended constraint system [11]. Following on prior work on direction-length constraints, each wider set of constraints opens up the full range of approaches: generic infinitesimal characterizations and fast combinatorial algorithms, the geometry of singularities, configuration spaces [19] and possible extensions to other metrics, inductive techniques, symmetry, global rigidity. Another example are results on independence, rigidity and global rigidity of certain types of incidence constraint systems promising provable bounds on a longstanding problem in Machine learning, called Dictionary learning [20]. More significantly, the above application examples require new types of constraints and geometries as well as a mix of existing and original approaches, that highlight rigidity theory's connections to diverse mathematical areas, and contribute to its increasing maturity and sophistication.

Presentation Highlights

We began the workshop with 5 minute presentations by all participants on problems they were working on and were looking for collaborations and feedback on. This established an immediate basis for engaged conversations from lunch on the first day on through to the last day when people were leaving, or were bridging into the companion 2 Day which started immediately after this workshop, with 25 of the participants.

A second key feature of our workshop was a substantial collection of model building materials which supported evening exchanges with sample models to encourage interdisciplinary communication, as well as a number of samples which were used to illustrate talks, including conjectures. This is a practice which our community has used effectively for our sequence of workshops, and one that supports lively evening conversations and development of conjectures, counter-examples, and alternative ways to place examples into various contexts for analysis.

We were able to support a sequence of survey talks which new-comers to the community report supported developing an initial overview of the state of the field as well as current unsolved problems. We then had a series of more focused presentations on recent results.

On the Thursday, an eminent Structural Engineer, William Baker from Skidmore, Owens, Merrill Ltd. (SOM), dropped by from a morning visit to a job site in Calgary to join conversations with four collaborators in a project to produce an annotated version of James Clerk Maxwell's 1864 (and following) works on reciprocal diagrams. On the same day, two of these collaborators presented related talks drawing on work of Maxwell as well as recent explorations of Structural Engineers interested in extending geometric tools to support insightful designs of buildings on all scales. We note that their work includes design and analysis for some of the tallest buildings in the world. These presentations and visits generated conversations and model building which extended late into the evening.

The opportunity to add on a 2 day workshop at the end of the 5 day workshop paid big dividends, by allowing developing collaborations to expand and consolidate. These collaborations will be a key theme below.

Scientific Progress Made

When asked for updates on the impact of the workshop, participants submitted the following brief descriptions. Overall, there is a strong desire for a future workshop in several years, and an appreciation for the impact of the two day extension which gave both an additional focus, and some further time to consolidate collaborations and progress, as mentioned below.

General responses

The following is typical of first-time participants Andy Vince: "Impact, sure: As a relative newcomer to the area, the conference gave me a great overview of current research on combinatorial rigidity theory. "

From an experienced participant (Patrick Fowler): The concentration of experts in a genuinely collaborative atmosphere made it hard *not* to have new ideas and get real work done. My thanks to BIRS and especially to the workshop organizers. I found the workshop very useful in getting an overview of the field and seeing the breadth of applications being made by mathematicians, engineers, computer scientists and physical scientists, meeting existing and potential collaborators and working with models brought along by other participants. The programme of talks and the opportunities for interaction with others gave me at least half a dozen directions for exploration in new research. The atmosphere was as always conducive to discussion, and time outside formal sessions was used to work intensively on a manuscript with two other participants.

Meera Sitharam reported that the graduate students in her research group benefited tremendously from the interaction with the larger international rigidity community. Walter Whiteley also reported a similar benefit for his Ph.D. student.

specific comments on collaborations and scientific progress

1. During the BIRS workshop, Patrick Fowler, Simon Guest and Bernd Schulze had several intensive research discussions over evenings with models, regarding classes of over-braced but typically flexible body-hinge

frameworks. These structures are based on polyhedra with rigid faces where an independent subset of faces has been replaced by a set of holes.

These symmetry calculations detect flexibility in these structures that is not revealed by counting alone. At the moment, we are exploring the rigidity and flexibility properties of symmetric 'block-and-hole structures' further. The process of revising the associated manuscript (see below) itself crystallised our thoughts on some further applications of symmetry to block-and-hole polyhedra, and a second ms is now in the early stages of preparation.

2. During the Banff workshop, Walter Whiteley, Yaser Eftekhari, Tony NixonShin-ichi Tanigawa and Bernd Schulze continued discussing the transfer of rigidity between spherical bar-joint frameworks (with points on the equator) and Euclidean point line frameworks, as well as connections to slider constraints from mechanical engineering and other recent rigidity work. The talk by Bill Jackson on Wednesday included examples that were quickly recognized as connected to this work and the discussion group broadened to include Bill.

   These explorations continued into the weekend, on the side of the Global Rigidity Workshop. At one point on Sunday, an ad hoc discussion included about 20 interested participants making connections among previously disjoint research results. This work has continued during various follow-up exchanes over the last few months. In the outcomes section we mention some manuscript(s) flowing from this ongoing work.

   A chapter on these connections is also proposed for the Handbook on Geometric Constraints, mentioned below.

3. Derek Kitson and Katie Clinch have been writing up a result regarding the existence of grid-like isostatic placements for graphs with specified edge-disjoint spanning trees. This work was initiated at the workshop.

4. Following observations during the workshop, Whiteley made some suggestions to Kitson after the workshop on direction length frameworks with a non-Euclidean norm. Kitson and Whiteley each made some notes and they will pick this up again later when time is available. Katie Clinch might like to be involved, having worked on both topics.

5. Bryan Chen and Derek Kitson found a corollary of the recent characterization by Cruickshank, Kitson and Power of the generic minimal rigidity of triangulated spheres with a single block (or hole). They have proven that the (3,6)-tightness condition on the "block and hole graph" (of a triangulated sphere with a single block or hole) is equivalent to (3,0)-tightness of a graph constructed from its "face graph" (terminology from the paper of Cruickshank, Kitson an Power). One advantage of the (3,0)-tightness condition is that it can be checked via the standard pebble game.

6. Wayne Lam and Bryan Chen have shown that the infinitesimal flexes of a certain family of periodic bar-joint frameworks (the "twisted kagome" graphs) solve discrete versions of the Cauchy-Riemann equations and thus provide a realization of discrete complex analysis in the setting of rigidity theory. This is inspired by earlier work by the physicists Sun, Souslov, Mao and Lubensky on conformal invariance in the continuum elasticity of such frameworks.

7. Research interactions at the workshop by Meera Sitharam and Brigitte Servatius settled a problem about body-pin structures in 2D (where multiple bodies meet at a pin), partially with the help of an old theorem by Walter Whiteley.

8. Based on interactions at the workshop, Andy Vince and Meera Sitharam improved a previous algorithmic characterization of an abstract rigidity matroid and completed the proof that it gives the best known upper bound on the rank of the 3D rigidity matroid.

9. During the workshop, revisions were made to three draft chapters for the Handbook of Discrete and Computational Geometry by Schulze and Whiteley (Symmetry and Rigidity, and Rigidity and Scene Analysis) and Jordan and Whiteley (on Global Rigidity). One of these is now submitted, and the other two are in next to final drafts.

10. Steven Gortler had conversations with Anthony Man-So at BIRs about the relationship between a max rank PSD equillbrium stress matrix for a universally rigid framework, and the concept of "singularity degree" in the field of semi-definite programming (the topic of Anthony's talk). See the outcomes section for the resulting publication.

11. Ideas that were discussed between Bob Connelly, Steven Gortler and Louis Theran at BIRS were instrumental to their recent (not yet published proof) that every graph that is generically globally rigid in R^d must have a generic d-dimensional framework that is universally rigid. This was an open question from the previous BIRs rigidity workshop.

12. The discussions of reciprocal diagrams resulted in continuing exchanges among four of the participants at the workshop. These will be followed up both in revisions to a Handbook Chapter mentioned below, and in a proposed chapter for the Handbook of Geometric Constrains mentioned in the next section.

    William Baker, a Civil Engineer from SOM submitted the following questions related to the rigidity of structures.

    Here is a list of things that structural engineers need to know that your group probably already knows.

    (a) How to tell when a 2D truss is stable or has mechanisms and how to find the mechanisms graphically.

    (b) An understanding of infinitesimal and finite mechanisms in 2D.

    (c) How quickly do infinitesimal mechanisms lock-up as they distort from the original geometry.

    (d) The effect of pre-stress on mechanisms not associated with the pre-stress in 2D.

    (e) How to tell when a 3D truss is stable or has mechanisms and how to find the mechanisms graphically.

    (f) An understanding of infinitesimal and finite mechanisms in 3D.

    (g) The effect of pre-stress on mechanisms not associated with the pre-stress in 3D.

    (h) What is the higher dimension version of Maxwells 2D reciprocal diagrams and polyhedral (i.e. 3D trusses and 4D polytope (polychora?), etc.).

Outcome of the Meeting

The following concrete outcomes have been reported..

1. A manuscript on 'Mobility of a class of perforated polyhedra' (Patrick Fowler, Simon Guest, Bernd Schulze) was drafted during the meeting, completed shortly afterwards and accepted by International Journal of Solids and Structures in January 2016.

2. Conversations mentioned on above on singularity degree lead Steven Gortler to realize that recent results due to Connelly and Gortler on second order universal rigidty might help characterize singularity degree. These ideas now appear in [2].

3. The workshop was crucial in initiating an NSF-Applied Math proposal on "Stability of Structures Large and Small", by 4 of the US participants, (Robert Connelly, Steven Gortler, Mike Thorpe, and Meera Sitheram) If awarded, will help partially support research as well as outreach meetings of this community.

4. The workshop was key in consolidating the chapter topics and authors for the new Handbook of Geometric Constraint Principles being edited by 3 members of this community, for CRC press. Most of the authors were at the meeting, and outlines were generated during and shortly after the meeting. See`http://tinyurl.com/HandbookGCSP-outline` for more details.

5. Other interactions at the workshop helped Meera Sitherams group at the University of Florida to (a) complete a paper that will appear in the journal Symmetry (concerning symmetries in sphere-based assembly configuration spaces), edited by another of the participants, Bernd Schulze.

6. Following some initial conversations in Banff, Shin-ichi Tanigawa, Viktoria Kaszanitzky and Bernd Schulze also started to work on the generic global rigidity of periodic frameworks (with a fixed lattice representation). A paper on this topic should be submitted soon to a high-level journal fairly soon.

7. The group of Eftkhari, Jackson, Schulze, Tanigawa and Whiteley are now completing a fourth (final?) draft of a paper that summarizes the first portion of our expanding findings connecting points on the equator on the sphere with lines in point-line configurations in the plane, and sliders. The paper includes a number of new or translated combinatorial results for classes of frameworks. The results also include extensions to point-hyperplane frameworks and points on the equator of hyperspheres. We expect to submit this paper to a high-level journal within the next few months. We are also working on a second paper.

   Other connections from this work are drafted as chapters to appear in the Ph.D. thesis of Yaser Eftekhara.

From the reports of the participants, this was an amazingly productive workshop. The collaborations and share continue, with a related follow up meeting will be held as an ICMS Workshop on "Geometric Rigidity Theory and Applications" in Edinburgh May 30-June 3, 2016 `http://icms.org.uk/workshop.php?id=383`. All of the organizers of this ICMS workshop were participants in this BIRS workshop.

Overall, there is a strong desire for a future workshop in several years, again with a two day extension to consolidate collaborations and give an additional focus. Our thanks to BIRS for your continuing financial support and strong organizational support.

Participants

Baker, Troy (University of Florida)
Bereg, Sergey (University of Texas - Dallas)
Chen, Bryan (Leiden University)
Clinch, Katie (Queen Mary London)
Connelly, Robert (Cornell University)
Cruickshank, James (National University of Ireland)
Eftekhari, Yaser (York University)
Fowler FRS, Patrick (University of Sheffield)
Gortler, Steven (Harvard University)
Guest, Simon (Cambridge University)
Guler, Hakan (Queen Mary, University of London)
Hempel, Maria (ETH Zurich)
Jackson, Bill (University of London)
Jordan, Tibor (Eotvos University)
Karpenkov, Oleg (University of Liverpool)
Katoh, Naoki (Kwansei Gakuin University)
Kirly, Csaba (Eotvos University)
Kitson, Derek (Lancaster University)
Lam, Wai Yeung (Technische UniversitŁt Berlin)
Lee St.John, Audrey (Mount Holyoke College)
McRobie, Allan (Cambridge University)
Mitchell, Toby (Skidmore Owings & Merril LLP)
Nixon, Anthony (Lancaster University)
Power, Stephen (University of Lancaster)
Ross, Elissa (MESH Consultants Inc.)
Schulze, Bernd (Lancaster University)
Serocold, Hattie (Lancaster University)
Servatius, Brigitte (Worcester Polytechnic Institute)
Servatius, Herman (Worcester Polytechnic Institute)

Sidman, Jessica (Mount Holyoke)
Sitharam, Meera (University of Florida)
Sljoka, Adnan (Kyoto University)
So, Anthony Man-Cho (The Chinese University of Hong Kong)
Tanigawa, Shin-ichi (Kyoto University)
Theran, Louis (Aalto University)
Thorpe, Michael (Arizona State University)
Trelford, Ryan (York University)
Vince, Andrew (University of Florida)
Wang, Menghan (University of Florida)
Whiteley, Walter (York University)
Willoughby, Joel (University of Florida)
Zhou, Shao (Worchester Polytechnic)

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Chapter 19

Lifting Problems and Galois Theory (15w5035)

August 16 - 21, 2015

Organizer(s): Frauke Bleher (University of Iowa), Ted Chinburg (University of Pennsylvania), Andrew Obus (University of Virginia), Rachel Pries (Colorado State University)

Objectives The aim of the BIRS workshop "Lifting Problems and Galois Theory" workshop was to bring together researchers and advanced Ph.D. students working on Galois theory in order to advance this field. The lifting problems for curves which were the focus of the workshop have been a central topic in algebraic geometry since the work of Grothendieck, Deligne and Mumford and others on étale fundamental groups and moduli spaces of curves. The subject has been undergoing rapid change due to the introduction of new techniques, as in the proof of the Oort conjecture.

The workshop had 41 participants. There were 8 one-hour talks, thirteen 45-minute talks, and one 95-minute problem session.

One-hour talks The one-hour talks of the workshop gave introductions to several aspects of lifting problems and Galois theory.

Frans Oort: Lifting questions. This talk gave an overview of the main questions and techniques used in the study of lifting problems. The speaker introduced liftings of algebraic curves (with automorphisms), liftings of higher dimensional varieties, and (CM-)liftings of abelian varieties.

Stefan Wewers: Swan conductors and differential obstructions. Swan conductors measure ramification of Galois extensions with respect to a valuation. They exist in many different forms. This talk explained how a certain Swan conductor with "differential value" due to Kazuya Kato can be used to define obstructions against lifting Galois covers from characteristic p to characteristic zero. Proving that this obstruction vanishes in certain cases was an important ingredient in the proof of the Oort conjecture on lifting cyclic Galois covers (by Obus, Pop and Wewers). By recent work of Andrew Obus, proving a more general vanishing result is the only obstacle left in proving a "generalized Oort conjecture" on lifting covers with cyclic Sylow p-subgroups.

David Harbater: Galois group schemes over arithmetic curves. Much of modern Galois theory takes place in the context of function fields of curves defined over complete discretely valued fields. A common strategy is to choose a projective model of the function field and consider Galois branched covers over the closed fiber, which one then attempts to lift to the whole model. This has been used for the inverse Galois problem, embedding problems, and lifting problems, often with the help of patching methods in order to work locally. Traditionally one considers finite Galois groups (or profinite groups in the limit), but one can also treat non-constant finite Galois group schemes via torsors, as in work of Moret-Bailly. This talk considered more general linear algebraic groups as Galois group schemes over such function fields, in two contexts: the inverse differential Galois problem, and obstructions to local-global principles.

Robert Guralnick: Groups and curves. This was a survey talk discussing how finite group theory is useful in studying various problems related to Brauer groups, coverings of curves, automorphism groups of curves and liftings of curves with group actions from positive characteristic to characteristic zero.

Pierre Dèbes: Specializations of covers and inverse Galois theory. This talk presented a series of problems and results from a program about the specializations of covers of the line in connection with inverse Galois theory. The main topics included Hilbert's irreducibility theorem, the Inverse Galois Problem and its regular version and the Malle conjecture.

Kiran Kedlaya: Combinatorial constraints on lifting problems via p-adic differential equations. This talk described an approach to recovering the standard combinatorial constraints arising in the study of local lifting problems as a corollary of the properties of convergence polygons of p-adic connections.

Irene Bouw: Computing L-functions of superelliptic curves. This talk reported on algorithmic results for computing the local L-factor and the conductor exponent of a cyclic cover of the projective line at the primes of bad reduction. As an application, the functional equation was verified numerically for a large class of examples.

Frans Oort: CM liftings. This talk explained the full story from Deuring (1941), via Weil, Tate, Honda-Tate, isogenies (1992) and finally results of the recent book (2014) "Complex multiplication and lifting problems" by Ching-Li Chai, Brian Conrad and Frans Oort, giving full answers to possible CM lifting questions. Several proofs were given and complete answers were formulated.

45-minute talks

The 45-minute talks of the workshop were given by senior researchers, postdocs and advanced graduate students.

Lior Bary-Soroker: Geometric versus arithmetic ramification.

Let $f : C \to \mathbb{P}^1$ be a branched covering defined over $\mathbb{Q}$. For $a \in \mathbb{Q}$, the fiber $f^{-1}(a)$ gives rise to a number field (in fact, étale algebra) which, loosely speaking, is generated by the coordinates of the points in the fiber. The main focus of this talk was the study of the number of ramified prime numbers in these number fields. Two results were presented:

(1) a central limit theorem, which answers the question what the typical number of ramification is, and

(2) sharp upper bounds.

The underlying idea behind these results is that the geometric branch locus "controls" the arithmetic one. If time permits, some applications, e.g. to the minimal ramification problem will be discussed.

Anna Cadoret: Structure of the image of the geometric étale fundamental group on étale cohomology with $\mathbb{F}_\ell$-coefficients.

When studying representations of the arithmetic étale fundamental group on étale cohomology, the knowledge of the structure of the image geometric étale fundamental group plays a crucial part. In particular, when the ring of coefficients is a field, it is conjectured that this image is semi simple. This is known for $\mathbb{Q}_\ell$-coefficients and in characteristic 0. This talk focused on the case of $\mathbb{F}_\ell$-coefficients in positive characteristic.

Rachel Davis: Galois theory of a quaternion origami.

Let X be equal to an elliptic curve over $\mathbb{Q}$ minus its origin. Let $f : Y \to X$ be an étale cover and let $\overline{x} \in X$ be a geometric point. Grothendieck and others consider Galois representations arising from the action of $G_{\mathbb{Q}}$ on $\{f^{-1}(\overline{x})\}$. In this talk a particular map f was studied with deck transformation group equal to the quaternion group.

Armin Holschbach: Étale contractible varieties in positive characteristic.

By Artin-Schreier theory, the affine line $\mathbb{A}^1_k$ over an algebraically closed field k of characteristic $p > 0$ has an infinite fundamental group. This is in contrast to the situation in characteristic 0, where the affine line can be thought of as an algebraic equivalent of the unit interval in topology: Not only is it simply connected, but it is indeed contractible in the sense of étale homotopy theory. Since the affine line as natural candidate does not work, the question arises whether there is any étale contractible variety in positive characteristic. In this talk, it was shown that there are no non-trivial smooth varieties over an algebraically closed field of characteristic p that are étale contractible, and some consequences for the decomposition theory of fundamental groups of varieties in positive characteristic were discussed.

Aristides Kontogeorgis: Representations of automorphisms and deformation of curves.
In this talk, applications of the representation theory of automorphism groups of curves were given to the theory of deformations of curves with automorphisms.

Christian Liedtke: Good reduction of $\mathrm{K3}$ surfaces.
By a classical theorem of Serre and Tate, extending previous results of Néron, Ogg, and Shafarevich, an Abelian variety over a p-adic field has good reduction if and only if the Galois action on its first ℓ-adic cohomology is unramified. In this talk, it was shown that if the Galois action on second ℓ-adic cohomology of a K3 surface over a p-adic field is unramified, then the surface admits an "RDP model" over that field, and good reduction (that is, a smooth model) after a finite and unramified extension. (Standing assumption: potential semi-stable reduction for K3's.) Moreover, examples were given where such an unramified extension is really needed. On the way, existence and termination of certain semistable flops were established, and group actions of models of varieties were studied.

Sophie Marques: Holomorphic differentials for Galois towers of function fields.
In this talk, the necessary conditions were recalled which permitted Boseck to obtain an explicit basis for the space of the holomorphic differentials for Kummer and Artin-Schreier extensions of a rational field. For this, the basics about Kummer and Artin-Schreier extensions were reviewed, particularly the existence of standard forms. Then, it was explained how it is possible to obtain a basis for a Galois tower of function fields of a rational field, provided the existence of a global standard form using Boseck's method. The Galois action on the basis was described and the Galois module structure of the holomorphic differentials for a cyclic function field over a perfect field was presented. This is a natural extension of the results done previously by Sotiris Karanikolopoulos and Aristides Kontogeorgis over an algebraically closed field. Finally, encountered problems for possible further developments/applications were presented.

Danny Neftin: Monodromy and ramification of rational functions.
The monodromy group and ramification type are two fundamental invariants associated to every rational function. This talk discussed the accumulating work towards describing all possibilities for both the monodromy group and the ramification type of an indecomposable rational function.

Jennifer Park: Faithful realizability of tropical curves.
Every algebraic curve over a nontrivially valued field has a corresponding tropical curve (through a process called "tropicalization"), where tropical curves are defined as balanced weighted 1-dimensional rational polyhedral complexes. It is then natural to ask whether tropical curves can be realized as the tropicalization of a smooth, complete and connected algebraic curve. Further, the question arises whether the tropicalization can be faithful. In this talk, the basics of the related topics in tropical geometry were first outlined, and then the above question of faithful realizability was answered for a large class of tropical curves.

Christalin Razafindramahatsiaro: Deuring's constant reductions theory and lifting problems.
Let X be a stable curve over a Dedekind scheme S, with smooth generic fiber X_η. It is well known (from Deligne and Mumford) that there exists a natural injective homomorphism between the full automorphism group of X_η and any special fibre of X. In this talk, first a generalization was given of this theorem in function fields of one variable version. Then, a solution to a "weak lifting problem" for cyclic curves was presented. In particular, the complete list of all full automorphism groups of hyperelliptic curves in odd prime characteristic was given that can be lifted to characteristic 0.

Zachary Scherr: Separated Belyi Maps.
Let C be a smooth, projective and geometrically irreducible algebraic curve defined over $\mathbb{C}$. In 1980, G. V. Belyi gave an unexpected necessary and sufficient condition for C to be isomorphic to a curve defined over $\overline{\mathbb{Q}}$. Namely, that there should exist a *Belyi map* $\varphi\colon C \to \mathbb{P}^1$. That is, a finite morphism which is ramified only over the three point set $\{0, 1, \infty\}$. This talk was concerned with how much flexibility there is in constructing Belyi maps. For a fixed curve $C/\overline{\mathbb{Q}}$, it is known that for each positive integer n there are, up to automorphism, finitely many n-element subsets of $C(\overline{\mathbb{Q}})$ occurring as the preimage of $\{0, 1, \infty\}$ under a Belyi map. While it is extremely difficult to try to describe all such subsets, this talk discussed a result in this direction. It was proved that given finite, disjoint subsets $S, T \subseteq C(\overline{\mathbb{Q}})$ there is always a Belyi map φ which is ramified on S and for which $\varphi(T) \cap \{0, 1, \infty\} = \emptyset$, refining a theorem of Mochizuki. This talk discussed both this theorem and a comparable theorem in positive characteristic.

Jeroen Sijsling: On descent of marked curves and maps.

Let F be a field with separable closure F^{sep}, and let X be a curve over F^{sep} that is isomorphic with all its $\text{Gal}(F^{\text{sep}} \mid F)$-conjugates. Then one can wonder whether there exists a descent of X, that is, a curve X_0 over F that is isomorphic with X over F. Surprisingly, counterexamples due to Shimura and Earle show that such a descent need not always exist. However, classical results by Dèbes and Emsalem imply that the statement does hold for smoothly marked curves. More precisely, let X be a curve as above and let $P \in X(F^{\text{sep}})$ be a smooth point on X. Then if for all $\sigma \in \text{Gal}(F^{\text{sep}} \mid F)$ there exists an isomorphism $\sigma X \to X$ taking σP to P, then there exists a curve X_0 over F and a point $P_0 \in X_0(F)$ such that (X_0, P_0) is isomorphic to (X, P) over F^{sep}. In this talk, a constructive version of this classical result was discussed that uses the branches of a morphism between algebraic curves. This allows to remove some superfluous hypotheses and to give explicit descent constructions for marked curves and Belyĭ maps. After showing these examples and their applications, some counterexamples were given for singular curves.

Michael Zieve: Monodromy groups in Galois theory.

This talk presented several types of results obtained by means of monodromy groups. These include refinements of Hilbert's irreducibility theorem, results about images of morphisms of curves over finite fields, results about reducibility of fibered products, and solutions to certain functional equations in rational or meromorphic functions.

Open Problems

In the 95-minute problem session, the following 13 open problems were discussed.

Weak/almost local Oort (David Harbater)

This focuses on the local lifting problem.

Question 1 (Weak local Oort): Which groups G have the property that some covers lift?

Question 2 (Almost local Oort): Which groups G have the property that all covers with sufficiently large conductor lift?

For a variant which seems more tractable, replace "sufficiently large conductor" by "sufficiently large first jump". Note that the groups G in question are always of the form $G \cong P \rtimes \mathbb{Z}/m\mathbb{Z}$, where P is a p-group and $p \nmid m$.

Known results:

(i) $p > 2$, $G = (\mathbb{Z}/p\mathbb{Z})^n$, $n \geq 2$: This is known to be a weak local Oort group. There is a necessary condition in order to lift, given by certain congruence conditions on the ramification.

(ii) $p = 2$, $G = Q_8$ or G a generalized quaternion group: This might be an almost local Oort group. However, it is not a full local Oort group, and moreover it is not even known whether it is weak. Note that in this case the KGB obstruction to being almost local Oort vanishes.

(iii) If G contains a subgroup of the form $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $m > 1$ and $p \nmid m$, then G is not weak local Oort.

Weyl CM curves (Frans Oort)

Let L be a field with $[L : \mathbb{Q}] = 2g$. Then L is called a Weyl CM field if it is a CM field and its Galois closure $\tilde{L}$ has the maximal possible degree for a CM field, namely $(2g)(g!)$. If so,

$$\text{Gal}(\tilde{L}/\mathbb{Q}) \cong W_g := (\mathbb{Z}/2)^g \ltimes \text{Sym}_g.$$

An algebraic curve over $\mathbb{C}$ is called a Weyl CM curve if its Jacobian is a Weyl CM abelian variety.

Question: Does there exist a Weyl CM algebraic curve for any $g \geq 4$? Can we give examples of such curves?

Remark 1. The automorphism group of a Weyl CM curve is trivial in case the curve is non-hyperelliptic; if a Weyl CM curve is hyperelliptic, the automorphism group is generated by the hyperelliptic involution. We know that we can construct CM curves by choosing curves with "many automorphisms". We see that this is of no help for the case of Weyl CM curves.

Remark 2.

(i) Coleman conjectured that for a fixed $g \geq 4$ the number of CM curves of genus g is finite [3, Conjecture 6].

(ii) For $4 \leq g \leq 7$, however, this is not correct, as was shown for $4 \leq g \leq 6$ by de Jong and Noot [6]; for a complete survey see [9].

(iii) Chai and Oort showed a modified version of Colemans conjecture: for a fixed $g \geq 4$ the number of Weyl CM curves of genus g is finite (conditional under the André-Oort condition); see [2, Proposition 3.7]. Also see [17].

Mumford curves in the Torelli locus (Frans Oort)

Let $\mathcal{A}_4$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$. Mumford constructed Shimura varieties (special subvarieties) of dimension 1 in $\mathcal{A}_4$, see [10]. Any geometric generic point corresponds to an abelian variety with endomorphism ring $\mathbb{Z}$. Hecke correspondences can be applied to obtain countably many of such curves. The (closure of the) image $\mathcal{T}_g$ of the Torelli morphism $\mathcal{M}_g \to \mathcal{A}_g$ is called the (closed) Torelli locus.

Question: Is any of these curves contained in the Torelli locus $\mathcal{T}_4 \subset \mathcal{A}_4$?

More generally, one can ask for Shimura varieties in $\mathcal{A}_g$ contained in $\mathcal{T}_g$, meeting the open Torelli locus $\mathcal{T}_g^0 \subset \mathcal{T}_g$. For a discussion, and some references see [9].

Good, ordinary reduction (Frans Oort / Folklore)

Question: Let A be an abelian variety over a number field K. Does there exist a prime p of K where A has good ordinary reduction?

In case $\dim(A) \leq 2$ the answer is affirmative, as proved on [12, pp. 370-372]; also see [18]. For a (potentially) CM abelian variety the Newton polygons of the reductions can be determined, and in this case primes of good, ordinary reduction do exist.

Irreducible polynomials with unit coefficients (Lior Bary-Soroker)

Let S_n be the set of polynomials of the form

$$f(x) = x^n \pm x^{n-1} \pm \cdots \pm 1 \tag{19.0.1}$$

that are irreducible.

Question: What proportion of polynomials of the form (19.0.1) are irreducible? In other words, what is the value of

$$\lim_{n\to\infty} \frac{\#S_n}{2^n} \ ?$$

The limit is conjectured to be 1. Known results:

(i) Poonen considered polynomials with coefficients in $\{0, 1\}$ and showed that the liminf is at least $1/n$.

(ii) Konyagin improved this bound to a (non-zero) multiple $1/\log(n)$.

We can also ask these questions modulo large p. For $p = 2$, the generalized Riemann hypothesis gives results for infinitely many n. There is also some numerical evidence. (Note that this question was also posed by “some guy on the street” [sic] on MathOverflow.)

Wild one-point covers of the projective line (David Harbater)

Fix $g \in \mathbb{Z}_{\geq 0}$, and let k be an algebraically closed field of characteristic $p > 0$.

Question 1: For which finite groups G is there a G-Galois branched cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g and with f étale outside ∞?

Questions 2 and 3 below are variants of Question 1.

Question 2: For which finite groups G is there a cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g that is étale outside ∞ and such that the monodromy group of f is isomorphic to G?

Question 3: Fix a quasi-p group G. What is the smallest positive integer g for which there exists a cover $f : Y \to \mathbb{P}^1$ over k with Y of genus g that is étale outside ∞ and such that the monodromy group of f is isomorphic to G?

Existence of connected torsors for finite group schemes (Ted Chinburg)

Question: Does there exist a finite group scheme G over $\mathbb{Q}$ such that there does not exist a connected G-torsor over $\mathbb{Q}$?

Remark. This reduces to the inverse Galois problem theory when considering constant group schemes.

For a variant, one can replace $\mathbb{Q}$ by any number field.

Lifting an automorphism to a normal domain (Frans Oort)

Question: Do there exist a field $\kappa \supset \mathbb{F}_p$, a complete, non-singular, absolutely irreducible algebraic curve C over κ, and an automorphism $b \in \mathrm{Aut}_k(C)$ such that (C, b) does not lift to any mixed characteristic normal local domain with residue field κ?

Is the question different for a perfect κ, or for a finite κ?

Remark. Let $\kappa \supset \mathbb{F}_p$ be an algebraically closed field, D a complete, non-singular, irreducible algebraic curve over κ, and let $b \in \mathrm{Aut}_\kappa(D)$. By Oort-Sekiguchi-Suwa, Green-Matignon, Obus-Wewers [11] and Pop [16] we know that (D, b) can be lifted to characteristic zero. Hence in the situation above (C, b) can be lifted to a (complete, local) mixed characteristic $R \to \kappa$ with residue class field κ. The question asks whether this can be done with a normal ring R.

Brian Conrad pointed out that if a lifting is possible to a mixed characteristic domain, this need not imply lifting is possible to a normal domain. Note that normalization of an integral domain may extend a residue class field.

Uniform boundedness of rational preimages (Michael Zieve)

Question: Fix positive integers d and D, and geometrically irreducible d-dimensional varieties X_1, X_2 defined over a degree-D number field K. Does there exist an integer N such that, for each finite morphism $\phi : X_1 \to X_2$ defined over K, the induced map on rational points $\phi : X_1(K) \to X_2(K)$ is at most N-to-1 over all points outside a proper Zariski-closed subset of $X_2(\overline{K})$? Further, can N be chosen to depend only on d and D, and not on X_1, X_2 or K?

Remark. The uniform boundedness conjecture for rational torsion on abelian varieties is equivalent to the special case of this question in which ϕ varies over all multiplication-by-n endomorphisms on abelian varieties.

Known results:

(i) Such an integer N exists (and depends only on D) when X_1 and X_2 are genus-1 curves [8].

(ii) Such an integer N exists when $X_1 = X_2 = \mathbb{A}^1$ [1]. In this case the smallest value N can take is the largest m for which K contains the real part of the m-th cyclotomic field; in particular, if $D = 1$ and $X_1 = X_2 = \mathbb{A}^1$ then the optimal value is $N = 6$.

(iii) In case $X_1 = X_2 = \mathbb{P}^1$, if $\deg(\phi)$ is sufficiently large compared to D and also ϕ admits no nontrivial factorization through an intermediate curve, then we can take N to be 2. In fact we could take N to be 1 if we exclude maps $\phi = \psi \circ x^a(x-1)^b \circ \mu$ where $\deg(\mu) = 1$ and $\gcd(a, b) = 1$, see [4].

Remark. It would be interesting to study the analogous question when K is a function field, for instance when $d = 1$. One difficulty is ruling out the possibility that the fibered product of several copies of X_1 (fibered over X_2) could have isotrivial components not contained in the fat diagonal, even though ϕ itself is not isotrivial.

Fields of definition of endomorphism rings (Kiran Kedlaya)

Question: Let A be an abelian variety of dimension g over a number field K. Let $L|K$ be the minimal extension over which all endomorphisms of A are defined. (This extension is finite Galois.)

Question: If we fix g and vary A (and possibly K), then what is the optimal bound $N(g)$ for $[L : K]$?

The current guess is that this should be $2^g g!$ for $g >> 0$.

Known results:

(i) $N(1) = 2$;

(ii) $N(2) = 48$;

(iii) $N(g) \leq 2(9n)^{2g}$ (by work of Silverberg).

Complete subvarieties of moduli spaces (Frans Oort)

Question 1: Fix $g \geq 4$. Which are the complete subvarieties $W \subset \mathcal{A}_g \otimes \mathbb{F}_p$ of codimension g?

Remark.

(i) We study complete subvarieties of $W \subset \mathcal{A}_g \otimes k$ for some algebraically closed field k. It is known that the codimension of W is at least g, as was proved by van der Geer, see [5, Theorem 3.3]. This means $\dim(W) \leq g(g-1)/2$ for a complete subvariety.

(ii) Infinitely many codimension g subvarieties exist for $g = 0$ and $g = 1$ in any characteristic.

(iii) The locus $V_0(\mathcal{A}_g \otimes \mathbb{F}_p)$ of abelian varieties of dimension g with p-rank zero is of codimension g and it is complete, see [13, Theorem 1.1].

(iv) In [14, Question 14B], we find the conjecture that $\mathcal{A}_g \otimes \mathbb{C}$ does not contain a complete subvariety of codimension g for any $g \geq 3$.

(v) This conjecture was proved to be true by Keel and Sadun, [7].

(vi) In [15, 14.2], infinitely many complete subvarieties of dimension 3 inside $\mathcal{A}_g \otimes \mathbb{F}_p$ were constructed. However we do not see how to perform an analogous construction for $g > 3$ and obtain infinitely many complete subvarieties of codimension g.

Question 2: Fix $g \geq 4$. Is it true that $V_0(\mathcal{A}_g \otimes \mathbb{F}_p)$ is the only complete subvariety $W \subset \mathcal{A}_g \otimes \mathbb{F}_p$ of codimension g?

Remark. Suppose this question has an affirmative answer for a given $g \geq 4$ and infinitely many values of p. Then for this value of g it follows that $\mathcal{A}_g \otimes \mathbb{C}$ does not contain a complete subvariety of codimension g (and in this way reproving the Keel-Sadun result).

The p-rank 0 strata of the Torelli locus (Rachel Pries)

For a prime number p and natural number g, consider the moduli space $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$ of principally polarized abelian varieties of dimension g in characteristic p and consider the Torelli locus $\mathcal{T}_g \subset \mathcal{A}_g$. Let $\mathcal{A}_g^0$ be the subscheme that parametrizes abelian varieties of dimension g with p-rank 0. For $g \geq 3$, it is known that $\mathcal{A}_g^0$ is irreducible, that its generic point has a-number 1, and that its Newton polygon has slopes $1/g$ and $(g-1)/g$. Let $\mathcal{T}_g^0 = \mathcal{T}_g \cap \mathcal{A}_g^0$.

Question 1: For p prime and $g \geq 3$, is $\mathcal{T}_g^0$ irreducible? For the generic point of each of its components, is it true that the a-number is 1 and the Newton polygon has slopes $1/g$ and $(g-1)/g$?

The answer is yes when $g = 3$ and some information is known when $g = 4$.

Characterizing lifted covers (David Harbater)

Question: Fix a prime p. Which Galois branched covers of $\mathbb{P}^1_{\mathbb{C}}$ whose ramification indices are all prime to p are lifts of smooth Galois branched covers of $\mathbb{P}^1$ in characteristic p?

For a variant, we can fix the ramification type and ask whether there exists a location of branch points that gives a positive answer above.

Known result: If the Galois group is of order prime to p, then all such covers are lifts. Beyond this case the Question is wide open.

Remark. This question essentially concerns to the study of π_1^{tame} in characteristic p. A closely related conjecture is as follows:

Conjecture (Ihara, Kyoto 2010): Let $W_p = W(\overline{\mathbb{F}_p})$ be the ring of Witt vectors over $\overline{\mathbb{F}_p}$, and let $K_p = Q(W_p)$ be the fraction field of W_p, that is, the completion of the maximal unramified extension of $\mathbb{Q}_p$. Let $f : X \to \mathbb{P}^1_{K_p}$ be a G-Galois cover with branch locus $\{0, 1, \infty\}$. If there is a K_p-point of X whose fiber is totally split, then f has potentially good reduction.

This conjecture is true for G solvable, and we can reduce it to the case where G is simple.

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Chapter 20

New Trends in Nonlinear Elliptic Equations (15w5004)

August 30 - September 4, 2015

Organizer(s): Michel Chipot (University of Zurich), Itai Shafrir (Technion-Israel Institute of Technology)

Overview The theory of elliptic equations expanded during the last decades in very different directions with various applications. Progress was made in particular in regularity theory, uniqueness techniques, fully nonlinear issues, qualitative properties of solutions, singularities, etc.. One can mention in particular several domains like Navier-Stokes equations, homogenization, calculus of variations, Ginzburg-Landau problems, singular perturbations and the Allen-Cahn equations. The workshop brought together actors of these different fields who shared their experience of the different techniques involved. Very challenging questions were concerned. A total of 25 lectures, 50-minutes long, were given by leading experts from all over the world (USA, Canada, France, Switzerland, Italy, Germany, Spain, Chile, Japan, Israel) as well as young promising researchers who are either at the postdoctoral level or in the early stages of a tenure-track position.

Scientific background The original idea beyond the workshop was to bring together mathematicians working in two different themes: Ginzburg-Landau theory and stationary Navier-Stokes equations. By the Ginzburg-Landau theory we mean both the stationary equation, known also as Allen-Cahn equation,

$$-\Delta u = (1-u^2)u\,, \tag{20.0.1}$$

and the vectorial version, that appears in Superconductivity, which in the simplest case (in the absence of magnetic field) takes the form

$$-\Delta u = (1-|u|^2)u\,, \tag{20.0.2}$$

with u taking its values in $\mathbb{R}^2$.

Regarding the scalar problem, important progress was made recently towards the resolution of the famous De Giorgi conjecture [8]. It states that if u is a solution of (20.0.1) in $\mathbb{R}^n$ which is monotone increasing in one direction, say $\frac{\partial u}{\partial x_n} > 0$, then the level sets of u must be hyperplanes, at least for $n \leq 8$. The conjecture was completely resolved in dimension two (by Ghoussoub and Gui [15]) and three (by Ambrosio and Cabre [2]). The weak form of the De Giorgi conjecture, in which the assumption that $\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$ is made, was resolved by Savin for $n \leq 8$ (see [18]). On the other hand, Del Pino, Kowalczyk and Wei [9] have recently shown that for $n \geq 9$ solutions whose level sets are not hyperplanes do exist. The De Giorgi conjecture in its strong form is still open for dimension $4 \leq n \leq 8$. Let us notice that J. Wei was one of the participants in the workshop and although his talk did not treat directly the De Giorgi conjecture, his talk on Serrin's overdetermined problem presented some analogy between the problems. In fact, Serrin's overdetermined problem deals with solutions on domains of epi-graph type,

$$\Omega = \{x \in \mathbb{R}^n \,:\, x_n > \varphi(x_1, \ldots, x_{n-1})\},$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function, namely,

$$\begin{cases} \Delta u + f(u) = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \{x_n = \varphi(x_1, \ldots, x_{n-1})\}, \\ \dfrac{\partial u}{\partial \nu} = \text{const} \text{ on } \{x_n = \varphi(x_1, \ldots, x_{n-1})\}. \end{cases} \tag{20.0.3}$$

As an example of such an analogy, Theorem 1.1 in [25] states that in dimension $n = 2$, for certain functions f, existence of a solution to (20.0.3) implies that Ω is a half-plane, and up to an isometry, $u(x) = g(x \cdot e)$ for some unit vector e.

The vectorial Ginzburg-Landau equation (20.0.2) on $\mathbb{R}^2$ arises by a "blow-up" procedure of solutions $\{u_\varepsilon\}$ to the equation

$$-\Delta u_\varepsilon = \frac{(1 - |u_\varepsilon|^2)}{\varepsilon^2} u_\varepsilon \tag{20.0.4}$$

on a bounded domain Ω in $\mathbb{R}^2$. The special case of minimizers of the Ginzburg-Landau type energy

$$E_\varepsilon(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2 \tag{20.0.5}$$

is particularly interesting. The study of the asymptotic behavior of the minimizers (and even critical points) of the energy E_ε when ε goes to zero, under Dirichlet boundary condition $g : \partial\Omega \to S^1$ was carried-up in the book [3] and many subsequent works. In the more physical Ginzburg-Landau model in Superconductivity one has to replace the energy (20.0.5) by the energy

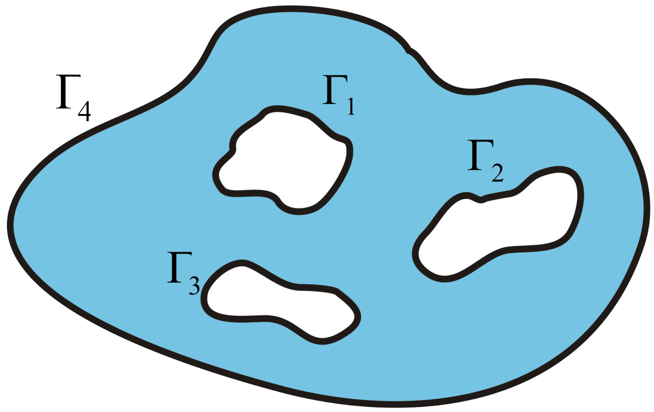
$$G_\varepsilon(u, A) = \frac{1}{2} \int_\Omega |\nabla_A u|^2 + |h - h_{ex}|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}\,, \tag{20.0.6}$$

where h_{ex} is the applied magnetic field. The resulting Euler-Lagrange equations satisfied by the minimizers (or more generally critical points) (u, A) of (20.0.6) take the form:

$$\begin{cases} -(\nabla_A u)^2 u = \dfrac{1}{\varepsilon^2}(1 - |u|^2)u \text{ in } \Omega, \\ -\nabla^\perp h = (iu, \nabla_A u) \text{ in } \Omega, \\ h = h_{ex} \text{ on } \partial\Omega, \\ \nu \cdot \nabla_A u = 0 \text{ on } \partial\Omega. \end{cases} \tag{20.0.7}$$

One of the main difficulties in the analysis of (20.0.4) and (20.0.7) is the control of the "vortices", the set on which $|u|$ is close to zero. The "vortex-balls construction" that was developed independently by Sandier and Jerrard (both participants in the workshop) played a major role in the important progress made in recent years in the mathematical analysis of solutions to (20.0.7), notably by the works of Sandier-Serfaty (see [23]).

The second main theme, stationary Navier-Stokes, was supposed to be around the Leray problem. This challenging problem was introduced in 1933 by J. Leray in a seminal paper. Suppose that a fluid is occupying a bounded domain as the one in the figure below with an outside boundary and insides multiple boundaries.



The stationary Navier–Stokes system in $\Omega \subset \mathbb{R}^n, n = 2, 3$, in to find a vector valued function $\mathbf{u}$ satisfying

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = & \mathbf{f} \quad \text{in} \quad \Omega, \\ \text{div } \mathbf{u} = & 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} = & \mathbf{a} \quad \text{on} \quad \partial\Omega, \end{cases}$$

- $\mathbf{u}(x)$ – is the velocity field, $\mathbf{u}\cdot\nabla = u_i\partial_{x_i} = \sum_i u_i\partial_{x_i}$,
- $p(x)$ – denotes the pressure of the fluid,
- ν – is the constant coefficient of viscosity,
- $\mathbf{a}(x)$ – the boundary value,
- $\mathbf{f}(x)$ – the external force.

The incompressibility of the fluid (div $\mathbf{u} = 0$) implies a necessary condition for the solvability of the problem above. One has indeed

$$0 = \int_\Omega \text{ div } \mathbf{u} dx = \int\limits_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, dS = \sum_{j=1}^{N} \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\, dS := \sum_{j=1}^{N} F_j,$$

where $\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$. For a long time the solvability of the problem above was proved only under the condition

$$F_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\, dS = 0, \;\; j = 1, 2, ..., N.$$

Recently Korobkov, Pileckas and Russo were able to solve this problem in some particular cases, but the problem is still open in its full generality (see [18], [17]). Even due to cancellations of speakers related to the Navier-Stokes theme, there were interesting discutions about these issues. Caraballo and Chipot gave two lectures regarding this theme. Quite a few lectures were also given on Ginzburg-Landau equations and related topics (by Almog, Jerrard, Mironescu, Sandier, Shafrir, Davila), but also a large variety of other topics in the elliptic theory were represented, including: Liouville-type equations, the p-laplacian, constant non-local mean curvature, extremum eigenvalues problems, Helmholtz equations with sign changing coefficients, prescribed Jacobian inequalities and many more. As a result, the topic of the workshop was less focused than expected, but instead very diversified, presenting a wide spectrum of the current state of the research in Elliptic Partial Differential Equations. A more specific description is given in the next sections.

Navier Stokes Theory

T. Caraballo [6] introduced several methods to analyze the long time behaviour of solutions to 2D Navier-Stokes models. Allowing very general delay terms, the problem becomes nonautonomous in general. For this reason he considered the theory of nonautonomous pullback attractors.

M. Chipot [7] presented existence results for the stationary non homogeneous Navier-Stokes problem in a two dimensional symmetric domain having a semi-infinite outlet. Under the symmetry assumptions on the domain, boundary value and external force he proved the existence of at least one weak symmetric solution, without any restriction on the size of the fluxes.

Ginzburg-Landau theory

R. Jerrard Starting from the Ginzburg-Landau model, he derived an effective free energy functional for nearly-parallel vortex filaments. As a consequence, he obtained the existence of solutions of the Ginzburg-Landau equations, in certain scaling regimes, possessing a collection of vortex filaments minimizing this effective energy.

P. Mironescu considered maps $u : \Omega \to \mathbb{S}^1$ having some Sobolev regularity $u \in W^{s,p}$:

a) either such maps need not have a phase φ with the same regularity as u
b) or the phase φ exists but is not controlled by the norm of u.

In case a), factorization allows to write each such u as $u = v\,w$, where v lifts and w is "smoother" than u. Case b) occurs only in dimension one.

The common theme of the proofs of the above results is the geometric detection of the energy concentration of manifold-valued maps.

E. Sandier brought together approaches of Dal Maso-Modica and Alberti-Müller to provide a framework for the analysis of multiscale problems. He applied it to a random version of a problem studied by Alberti-Müller. The approach generalizes the one in previous work of S. Serfaty and himself on the Ginzburg-Landau functional.

I. Shafrir [4, 19] Certain Sobolev spaces of S^1-valued functions can be written as a union of disjoint classes. It is interesting to study the distances between these classes. In his talk, based on a joint work with Brezis and Mironescu, he concentrated on classes in $W^{1,1}(\Omega, S^1)$, where Ω is a simply connected domain in $\mathbb{R}^N$, $N \geq 2$. He presented estimates for the minimal distance as well as the Hausdorff distance between different classes.

J. Davila was interested in singularity formation for the harmonic map flow from a two dimensional domain into the sphere. He showed that for suitable initial conditions the flow develops a type 2 singularity at some point in finite time, and that this is stable under small perturbations of the initial condition. All these results hold without any symmetry assumptions.

Y. Almog [1] investigated the time-dependent Ginzburg-Landau equations in the presence of strong currents, however weaker than the critical current where the normal state losses its stability. In the large κ limit, he proved that the superconductivity order parameter is exponentially small in a significant part of the domain, and small in the rest of it.

Scalar nonlinear elliptic equations

J. Wei [10, 25] In 1971, Serrin proved that the only bounded domain for which the overdetermined problem

$$\Delta u + f(u) = 0, u > 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega$$

$$\partial_\nu u = C \text{ on } \partial\Omega$$

admits a solution is the ball. In 1997, Berestycki, Caffarelli and Nirenberg considered the unbounded domain case, and proposed the following conjecture: If Serrin's problem admits a solution and Ω^c is connected, then Ω is either a half space, a cylinder $B \times \mathbb{R}^{N-k}$, or complement of a ball or cylinder. In the talk was discussed positive and negative answers to this conjecture. In particular, when Ω is an epigraph $\Omega = \{x_N > \varphi(x')\}$, it was shown that

(1) BCN conjecture is always true when $N = 2$,

(2) BCN conjecture is true when $3 \leq N \leq 8$ if $\frac{\partial u}{\partial x_N} > 0$,

(3) BCN conjecture is false when $N \geq 9$.

A key observation is the connection between this problem and a one-phase free boundary problem.

B. Kawohl gave a survey of solved and unsolved problems on the first nonconstant Neumann-eigenfunction for the p-Laplacian. Among other things, he studied the limit of the problem when p goes to ∞.

M. Fazly [12] provided classification and symmetry results for certain local and nonlocal elliptic PDEs with power type nonlinearities. It was the occasion to review the background on standard methods and ideas developed over the past couple of decades. Then he turned to present monotonicity formulas, Liouville theorems and one-dimensional symmetry properties.

E. Yanagida gave a survey of results of removable and non-removable singularities in elliptic and parabolic equations. Among the problems mentioned were the Laplace equation, the Lane-Emden equation, the absorption equation $u_t = \Delta u - u^p$, Fujita equation $u_t = \Delta u + u^p$.

Nonlinear parabolic equations and systems

Y. Du [11] was interested in the nonlinear parabolic problem

$$u_t - \Delta u = f(u)(x \in \mathbb{R}^N, t > 0),\ u(x,0) = u_0(x)\ \ (x \in \mathbb{R}^N),$$

where $u_0 \in L^\infty(\mathbb{R}^N)$ is nonnegative and has compact support, f is a smooth function satisfying $f(0) = 0$. He wanted to study how much of the long-time dynamics of this problem is determined by the corresponding elliptic problem

$$-\Delta u = f(u),\ u \geq 0\ \ (x \in \mathbb{R}^N).$$

M. Musso constructed global unbounded solutions for the critical nonlinear heat equation on a bounded smooth domain satisfying zero Dirichlet boundary conditions. Given an integer k, and given any set of k distinct points of the domain, which satisfy a certain condition involving Green's function of the domain, she found a positive solution for the critical heat equation blowing up at exactly those k points as time goes to infinity.

D. Kinderlehrer [16] spoke about the Poisson-Nernst-Planck system of equations used to model ionic transport is interpreted as a gradient flow for the Wasserstein distance and a free energy in an appropriate space of probability measures. The interaction term between the species arising from the Gauss law is singular which gives rise to some challenging issues. He gave a description of this situation attempting to maintain a minimal technical level including the basic format of the Wasserstein-type implicit scheme.

Liouville-type problems

G. Wolansky considered an extension of the Keller-Segel model to several cells populations. He reviewed some of the results considered before in the literature and, in particular, considered the case of conflict between two populations, that is, when population one attracts population two, while, at the same time, population two repels population one. This assumption leads to a new functional inequality which generalizes the Moser-Trudinger inequality. As an application of this inequality he derived sufficient conditions for the existence of steady states corresponding to solutions of an elliptic Liouville system.

A. Poliakovsky [22], motivated by the study of non-abelian Chern Simons vortices of non-topological type in Gauge Field Theory, analysed the solvability of some Liouville-type system in presence of singular sources. He was able to identify necessary and sufficient conditions which ensure the radial solvability of this system.

Calculus of variations

B. Dacorogna proved (Darboux theorem) that if ω_m is the standard symplectic form and f is a symplectic form, then one can find a diffeomorphism φ, with optimal regularity, satisfying

$$\varphi^{*}(\omega_m) = f \quad \text{and} \quad \delta[\varphi \lrcorner \omega_m] = 0$$

provided that f is a small perturbation of ω_m . He then apply the above result to the so-called symplectic decomposition. The connections with mass transportation and the Monge-Ampère equation were emphasized.

O. Kneuss [13] spoke about existence of solution to the prescribed Jacobian inequality coupled with a Dirichlet condition, namely

$$\begin{cases} \det \nabla\phi \geq f & \text{a.e. in } \Omega \\ \phi = \text{id} & \text{on } \partial\Omega \end{cases} \tag{20.0.8}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded smooth connected open set, $f : \Omega \to \mathbb{R}$ and where $\phi : \Omega \to \mathbb{R}^2$ is the unknown. He showed that this prescribed Jacobian inequality in the plane admits – unlike the prescribed Jacobian equation – a bi-Lipschitz solution in case of right-hand sides of class L^∞.

Other topics

X. Cabré [5] was interested in hypersurfaces of $\mathbb{R}^N$ with constant nonlocal (or fractional) mean curvature. First he proved the nonlocal analogue of the Alexandrov result characterizing spheres as the only closed embedded hypersurfaces in $\mathbb{R}^N$ with constant mean curvature. Then by the moving planes method he establishes the existence of periodic bands or "cylinders" in $\mathbb{R}^2$ with constant nonlocal mean curvature and bifurcating from a straight band.

H-M. Nguyen [21] devoted his talk to various properties and applications of the Helmholtz equations with sign changing coefficients. These equations are used to model negative index materials which are artificial structures whose refractive index are negative over some frequency range. The study of these equations faces two difficulties. First the ellipticity and the compactness are lost in general due to the changing sign coefficients. Second, the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss (the viscosity) goes to 0, might occur.

J. Fischer [14] developed a large-scale regularity theory of higher order for divergence-form elliptic equations with heterogeneous coefficient fields a in the context of stochastic homogenization. Under the assumptions of stationarity and slightly quantified ergodicity of the ensemble, he derived a $C^{k,\alpha}$-"excess decay" estimate on large scales and a $C^{k,\alpha}$-Liouville principle for any $k \geq 2$: For a given a-harmonic function u on a ball B_R, he showed that its energy distance on some ball B_r to the space of a-harmonic functions that grow at most like a polynomial of degree k has the natural decay in the radius r, at least above some minimal (random) radius r_0. His results rely on the existence of higher-order correctors for the homogenization problem, which are established by an iterative construction.

C. Wang [20] described a uniqueness result of absolute minimizers of Hamiltonian functions $H(x,p)$, provided

(i) H is lower semicontinuous, and $H(x,p)$ is convex in p;

(ii) $0 = H(x,0) \leq H(x,p)$ and $\cup_x\{p : H(x,p) = 0\}$ is contained in a hyperplane of R^n;

(iii) $H(x,p)$ is uniformly coercive in p.

S. Mardare considered a Bingham flow in a domain which is periodic in one direction. Interesting is the asymptotic behaviour of the solution to the stationary Bingham problem as the length 2ℓ of the domain (in the periodic direction) goes to infinity. The main result states that the velocity of the fluid converges strongly in the H^1-norm to the solution of a Bingham problem in the infinite periodic domain. Nevertheless, the speed of the convergence is much lower than the one obtained for the (linear) Stokes problem.

Outcome of the workshop

The workshop was a great opportunity for researchers to be updated in the last developments in nonlinear elliptic (and also parabolic) problems, to meet and interact with other researchers, either close or at least realed to their own research. It gave many opportunities to continue existing collaborations and to start new ones. The large number of open problems that were presented in the talks will certainly encourage research in the area by participants and their Ph.D. students or Postdoctoral researchers. One of the successes of the workshop was in giving the opportunity to several young promising researchers (Fazly, Fischer, Kneuss, Poliakovsky) to present their research in an important scientific meeting.

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Chapter 21

The Use of Linear Algebraic Groups in Geometry and Number Theory (15w5016)

September 13 - 18, 2015

Organizer(s): Skip Garibaldi (University of California at Los Angeles), Nicole Lemire (University of Western Ontario), Raman Parimala (Emory University), Kirill Zainoulline (University of Ottawa)

Overview of the Field

The theory of linear algebraic groups is a well established area of modern mathematics. It started as an algebraic version of the massively successful and widely applied theory of Lie groups, pushed forward most notably by Chevalley and Borel. In the hands of Serre and Tits, it developed into a powerful tool for understanding algebra, geometry and number theory (e.g. Galois cohomology). In particular, it provides a way to unify seemingly distinct statements in algebra, geometry and number theory, hence, suggesting new techniques and methods for solving problems in these areas.

For example, the Hasse-Minkowski Theorem and the Albert-Brauer-Hasse-Noether Theorem, which concern respectively quadratic forms and central simple algebras over global fields, can be viewed as special cases of the celebrated Hasse Principle in Galois cohomology of semisimple linear algebraic groups (due to Kneser, Harder, and Chernousov) which unifies these two theorems and provides many new results that would not have been even suspected nor proven before.

This philosophy has led to a creation of a vast number of new techniques and applications to different areas of mathematics: the theory of quadratic forms (Karpenko, Merkurjev, Vishik), the theory of essential and canonical dimensions (Reichstein, Merkurjev), local-global principles (Hartmann, Harbater, Krashen, Parimala, Suresh), the theory of motives (Petrov, Semenov, Zainoulline), and to the theory of torsors (Chernousov, Gille, Panin, Pianzola). For instance, Karpenko has recently shown how results on isotropy of hermitian forms (linear algebraic groups of type A) can imply corresponding results on symplectic forms and quadratic forms (groups of types B, C, D), and how the results are all connected via the theory of algebraic cycles and motives of projective homogeneous varieties.

Further afield are applications to arithmetic groups and arithmetic locally symmetric spaces (Prasad, Rapinchuk) and to number theory in the form of pseudo-reductive groups (Conrad). In the opposite direction, another trend has been using results from finite group theory to prove theorems about algebraic groups (Guralnick).

The purpose of the workshop was to exploit and to develop these new emerging links, to bring together specialists and young researchers from these areas, to stimulate new advances and developments. More precisely, it was devoted to recent applications of linear algebraic groups in algebra, geometry and number theory, especially, to techniques and results that establish new links between different areas of mathematics, such as: the proof of the Grothendieck-Serre conjecture; the breakthrough in the computation of cohomological invariants based on new Merkurjev's results concerning motivic cohomology; Conrad's proof of finiteness of fibers for fppf cohomology over global fields of prime characteristic; and applications of representations with dense orbits inspired by Bhargava's work.

Recent Developments and Open Problems

The last five years can be characterized as a boom of research activity in the area of linear algebraic groups and its applications. One should mention here recent results

On the proof of the Grothendieck-Serre conjecture. We recall that the geometric case of the Grothendieck-Serre conjecture stated in the mid 60's says that if a G-torsor (defined over a smooth algebraic variety X) is rationally trivial, then it is locally trivial (in the Zariski topology), where G is a smooth reductive group scheme over X. This conjecture has a long history: It was proven for curves and surfaces for quasi-split groups by Nisnevich in the mid-1980's. For arbitrary tori in the late 1980's by Colliot-Thelene and Sansuc. If G is defined over a field, then the conjecture is known as Serre's conjecture and was proven by Colliot-Thelene, Ojanguren and Raghunathan in the beginning of the 1990's; for most classical groups in the late 1990's by Ojanguren-Panin-Suslin-Zainoulline; for isotropic groups by Panin-Stavrova-Vavilov. However, no general proof was known up to now. Recently, Fedorov and Panin found a new original approach that proves the conjecture using the theory of affine Grassmannians coming from Langlands' program assuming the base field is infinite. Finally, Panin extended that approach to include the case of an arbitrary field.

On computations of the group of cohomological invariants. According to J.-P. Serre by a cohomological invariant one means a natural transformation from the first Galois cohomology with coefficients in an algebraic group G (the pointed set which describes all G-torsors) to a cohomology functor $h(-)$, where h is a Galois cohomology with torsion coefficients, a Witt group, a Chow group with coefficients in a Rost cyclic module M, etc. The ideal result here would be to construct enough invariants to classify all G-torsors/linear algebraic groups. The question was put on a firm foundation by Serre and Rost in the 1990's, allowing the proof of statements like "the collection of cohomological invariants of G is a free module over the following cohomology ring" for certain groups G; this theory is expounded in the 2003 book by Garibaldi-Merkurjev-Serre. Using this theory one obtains complete description of all cohomological invariants landing in degree 1 Galois cohomology for all algebraic groups, in degree 2 for connected groups, and in degree 3 for simply connected semisimple groups (Rost). In a breakthrough recent development, Merkurjev provided a complete description of degree 3 invariants for semisimple groups, solving a long standing question. This has ignited new activity, with researchers trying to understand the full power of his new methods, as well as to understand the new invariants discovered as a corollary of his results.

On length spectra of locally symmetric spaces. The answer to the question "Can you hear the shape of a drum?" is famously no, but variations of the problem such as restricting the collection of spaces under consideration or strengthening the hypotheses has led to situations where the answer is yes. In a remarkable 2009 Publ. Math. IHES paper, Prasad and Rapinchuk introduced the notion of weak commensurability of semisimple elements of algebraic groups and of arithmetic groups and used this new concept to address the question of when arithmetically defined locally symmetric spaces have the same length spectrum. In this paper they also settled many cases of the long-standing question of when algebraic groups with the same maximal tori are necessarily isomorphic. This paper and the stream of research stemming from it connects algebraic groups and their Galois cohomology – the central subject of this conference – with arithmetic groups, with geometry, and even with transcendental number theory.

On applications of the algebraic cycles and Grothendieck gamma filtration to the invariants of torsors. Let X be the variety of Borel subgroups of a simple linear algebraic group G over a field k. It was proven that the torsion part of the second quotient of Grothendieck's gamma-filtration on X is closely related to the torsion of the Chow group and hence to the group of cohomological invariants in degree 3 computed recently by Merkurjev. As a byproduct of this new striking connection one obtains an explicit geometric interpretation/description of various cohomological invariants in degree 3 as well as obtains new results concerning algebraic cycles and motives of projective homogeneous spaces (Baek-Garibaldi-Gille-Junkins-Queguiner-Semenov-Zainoulline).

On genericity theorems for the essential dimension of algebraic stacks and their applications. Techniques from the theory of algebraic stacks are used to prove genericity theorems (Brosnan-Reichstein-Vistoli) to bound their essential dimension which is then applied to finding new bounds for the more classical essential dimension problems for algebraic groups, forms and hypersurfaces. These genericity theorems have also been used in particular by Biswas, Dhillon and Lemire to find bounds on the essential dimension of stacks of (parabolic) vector bundles over curves and by Biswas, Dhillon, Hoffman to find bounds on the essential dimension of the stack of coherent sheaves over a curve.

On the classification of simple stably Cayley groups. A linear algebraic group is called a Cayley group if it is equivariantly birationally isomorphic to its Lie algebra. It is stably Cayley if the product of the group and some torus is Cayley. Cayley gave the first examples of Cayley groups with his Cayley map back in 1846. Over an algebraically closed field of characteristic 0, Cayley and stably Cayley simple groups were classified by Lemire-Popov-Reichstein in 2006. In 2012, the classification of stably Cayley simple groups was extended to arbitrary fields of characteristic 0 by Borovoi-Kunyavskii-Lemire-Reichstein. Borovoi-Kunyavskii then used this classification to classify the stably Cayley semisimple groups over arbitrary fields of characteristic 0.

On classifications of finite groups of low essential dimension. Duncan used the classification of minimal models of rational G-surfaces in his classification of the finite groups of essential dimension 2 over an algebraically closed field of characteristic 0. Beauville more recently used Prokhorov's classification of rationally connected threefolds with an action of a simple group to classify the finite simple groups of essential dimension 3.

Patching techniques. There are open questions concerning Hasse principle for homogeneous spaces under linear algebraic groups over function fields of p-adic curves. There is a surge of activities in this direction, thanks to the patching techniques and theorems due to Harbater-Hartmann-Krashen. This has led to answers to questions on Hasse principle for homogeneous spaces over several classes of linear algebraic groups. There are also answers to fundamental questions related to period-index questions for the Brauer group u-invariant of fields. New conjectures were formulated by Colliot-Thélène, Parimala and Suresh very similar to the conjectures over number fields. Reciprocity obstructions similar to the Brauer-Manin obstruction were constructed to study the obstruction to the Hasse principle. There are a host of open questions concerning the Hasse principle for rational groups which makes this area truly challenging.

Presentation Highlights

I. The first day of the workshop was devoted to general talks on the structure of linear algebraic groups and their cohomological invariants. There were morning talks by senior researcher A. Merkurjev (UCLA, USA) and 2 young researchers S. Baek (KAIST, South Korea) and R. Pirisi (Ottawa). The afternoon talks were given by senior researchers V. Chernousov (Alberta), M. Borovoi (Tel Aviv, Israel) and young researchers I. Rapinchuk (Harvard), D. Zywina (Cornell, USA).

Speaker: **Alexander Merkurjev** (University of California at Los-Angeles)
Title: *Suslin's Conjecture on the reduced Whitehead group of a simple algebra*
In the talk, the speaker reported about his proof of Suslin's Conjecture (1991) on the generic non-triviality of the reduced Whitehead group of a simple algebra. For a central simple algebra A over a field F, the reduced Whitehead group for $G = SL_1(A)$ is $\mathrm{SK}_1(A) = G(F)/[A^\times, A^\times]$. Saltman showed that G is retract rational if and only if $\mathrm{SK}_1(A_K) = 0$ for all field extensions K/F. Wang in 1950, showed that if the index of A is squarefree, then $\mathrm{SK}_1(A) = 0$. However, Platonov in 1975 gave an example of a central simple algebra A of index p^2 with non-trivial reduced Whitehead group. This and other counterexamples led to Suslin's Conjecture in 1991 that $\mathrm{SK}_1(A)$ is non-trivial generically when $\mathrm{ind}(A)$ is not squarefree. In 1993, Merkurjev proved the conjecture in the case when 4 divides the index and when $\mathrm{char}(F) \neq 2$. In 2006, Merkurjev reduced the conjecture in a characteristic free way to statements about the Chow ring of G. Namely, it is necessary to show that if $\mathrm{ind}(A) = p$, then for all field extensions K/F, $\mathrm{CH}^*(G) \to \mathrm{CH}^*(G_K)$ is surjective and if $\mathrm{ind}(A) = p^2$, then $\mathrm{CH}_i(\mathrm{SB}(A))$ are torsion free for all $i = 0, \ldots, p-2$. Recently, he proved these equivalent statements using information about the

topological filtration for Severi-Brauer varieties, the decomposition of the Chow motive for a Severi-Brauer variety and spectral sequence arguments. [13]

Speaker: **Sanghoon Baek** (Korean Advanced Institute of Science and Technology)
Title: *Semi-decomposable invariants of degree 3*
The speaker reported on semi-decomposable invariants in degree 3 for split semisimple groupss, introduced by Merkurjev-Neshitov-Zainoulline. This invariant is locally decomposable and it was shown that there is no nontrivial semi-decomposable invariant of a split simple group. In this talk, he discussed semi-decomposable invariants of a split reductive group in terms of the torsion in the codimension 2 Chow groups of a product of Severi-Brauer varieties. In particular, he presented a method to find nontrivial semi-decomposable invariants of an arbitrary split semisimple group of type A. [2]

Speaker: **Vladimir Chernousov** (University of Alberta)
Title: *Algebraic groups and their maximal tori*
The speaker surveyed recent developments dealing with characterization of absolutely almost simple algebraic groups having the same isomorphism/isogeny classes of maximal tori over the field of definition. These questions arose in the analysis of weakly commensurable Zariski-dense subgroups. In particular, for an algebraic K group, he discussed $\mathrm{gen}_K(G)$, the number of isomorphism classes of K forms of G with the same maximal tori. In the case of a finitely generated field K, he asked when $\mathrm{gen}_K(G) = 1$ or, at least finite. His talk discussed the case of absolutely almost simple K groups over number fields. Then in joint work with A. Rapinchuk and I. Rapinchuk, they showed that $\mathrm{gen}_K(G)$ was finite in that case and in fact was reduced to one element except for types A_n, D_{2n+1} or E_6. He then discussed their conjecture that $\mathrm{gen}_K(G)$ is finite over a finitely generated field of characteristic 0 or of good characteristic. He gave supporting evidence for their conjecture. [6]

Speaker: **Roberto Pirisi** (University of Ottawa)
Title: *Cohomological Invariants for stacks of algebraic curves*
The speaker discussed how to extend the classical theory to a theory of cohomological invariants for Deligne-Mumford stacks and in particular for the stacks of smooth genus g curves. He also showed how to compute the additive structure of the ring of cohomological invariants for the algebraic stacks of hyperelliptic curves of all even genera and genus three. [16]

Speaker: **Mikhail Borovoi** (Tel Aviv University)
Title: *Real Galois cohomology of semisimple groups*
In joint work with D. Timashev, the speaker explained how to compute the Galois cohomology set $H^1(\mathbb{R}, G)$ for a connected semisimple algebraic real group G using Kac diagrams, introduced by Kac to describe automorphisms of finite order of simple Lie algebras over the field of complex numbers $\mathbb{C}$. [4]

Speaker: **David Zywina** (Cornell University)
Title: *l-adic monodromy groups for abelian varieties*
For an abelian variety of dimension $g \geq 1$ over a number field K and a prime l, the l^m torsion subgroup of $A(\overline{K})$ is isomorphic to $(\mathbb{Z}/l^m\mathbb{Z})^{2g}$. This means that for each prime L, one obtains a representation of Gal_K in $\mathrm{GL}_{2g}(\mathbb{Z}/l^m\mathbb{Z})$. Combining these representations, one obtains a representation $\rho_{A,l^\infty} : \mathrm{Gal}_K \to \mathrm{GL}_{2g}(\mathbb{Z}_l)$. After increasing the size of K, $\rho_{A,l^\infty}(\mathrm{Gal}_K)$ is a finite index subgroup of $G_l(\mathbb{Q}_l)$ where G_l is the Zariski closure of the abelian variety in $\mathrm{GL}_{2g}/\mathbb{Q}_l$. The Mumford Tate conjecture states that G_l is isomorphic to $G \times_{\mathbb{Q}} \mathbb{Q}_l$ where G is the Mumford Tate group constructed from the Hodge decomposition of $A/\mathbb{C}$.

The speaker discussed the generic factorisation of the reduction of an abelian variety over a number field K with respect to a prime of good reduction of the algebraic integers of K assuming the Mumford Tate conjecture.

II. The second day of the conference featured talks about cohomological invariants and patching techniques for function fields over p-adic curves, consequences of the proof of the Grothendieck-Serre conjecture, results on essential dimension and rationality, classification of simple isotropic groups and results on quadratic forms in characteristic 2. There were 1 hour talks by E. Bayer-Fluckiger (EPFL, Switzerland), V. Suresh (Emory, USA),

J. Hartmann (Aachen, Germany) and half-hour talks by young researchers A. Auel (Yale, USA), M. Macdonald (Lancaster, UK), A. Stavrova (St Petersburg, Russia), A. Duncan (South Carolina, USA) and doctoral student N. Bhaskhar (Emory, USA).

Speaker: **Eva Bayer-Fluckiger** (École Polytechnique Fédérale de Lausanne)
Title: *Rationally isomorphic hermitian forms and torsors of some non-reductive groups*
A well known theorem about unimodular quadratic forms over a discrete valuation ring R and its quotient field F says that if two unimodular quadratic forms over R are isomorphic over F, then they become isomorphic over R. This result and many of its generalisations are consequences of the Grothendieck Serre conjecture which states that for a regular local integral domain with fraction field F. For any smooth reductive affine group scheme G over R, the induced map $H^1_{\acute{e}t}(R, G) \to H^1_{\acute{e}t}(F, G)$ is injective. Grothendieck-Serre was proved recently in the case in which R contains a field by Panin-Fedorov. One can also make this conjecture for non-connected group schemes whose connected components is reductive. The orthogonal group and its forms is a particular case of interest. Let (P, f) be a unimodular quadratic space and $\mathbf{O}(f)$ is a group scheme of isometries of f. Then the first theorem mentioned is a consequence of the analogue of Grothendieck-Serre for $\mathbf{O}(f)$ and was proved in the case of dimension of R at most 2 or when R contains a field k.

In a different direction, the first theorem was recently generalised by Auel, Parimala and Suresh to show that two quadratic forms on a semilocal Dedekin domain R containing 2 in the units, that if f, f' are quadratic forms over R with isomorphic simple coradicals, then if they are isomorphic over F, they are isomorphic over R. This result no longer follows from Grothendieck Serre since the forms are not necessarily unimodular.

The speaker discussed joint work with Uriya First dealing with generalisations of the result of Auel, Parimala and Suresh and hence generalisations of the Grothendieck-Serre conjecture in the non-reductive case. For a semilocal Dedekind domain R with $2 \in R^\times$ and its quotient field F, they proved that for two unimodular hermitian forms over (A, σ) for a hereditary R-order in a separable F-algebra and σ an R involution, then if the associated unimodular hermitian forms are isomorphic over F, then they are isomorphic over A. In this setting, this result has a cohomological analogue which states the map $H^1_{\acute{e}t}(R, \mathbf{O}(f)) \to H^1_{\acute{e}t}(F, \mathbf{O}(f))$ is injective where $\mathbf{O}(f)$ is the group scheme of isometries of f. Note here that $\mathbf{O}(f)$ has a connected component which is not reductive so that the usual Grothendieck-Serre conjecture does not apply. The result can be regarded as a first step towards a version of the Grothendieck-Serre conjecture for certain non-reductive group schemes over $\mathrm{Spec}(R)$.

She also talked about an equivariant version of that theorem about hermitian forms invariant under the action of a finite group. [3]

Speaker: **Asher Auel** (Yale University)
Title: *Algebras of composite degree split by genus one curves*
The speaker addressed the old question of whether every central simple algebra can be split by the function field of a genus one curve defined over the base field. The speaker outlined the ideas of the proofs for the known cases of algebras of degree at most 5 due to work of Artin, Swets, Clark and de Jong-Ho. He proposed a method for answering this question in the affirmative for algebras of composite degree, when the answer is known for the prime power factors. The speaker illustated his method for degree 6 and outlined the proof in that case.

Speaker: **Mark MacDonald** (University of Lancaster)
Title: *Reducing E_7 and the slice method*
The speaker discussed joint work with R. Loetscher in which they gave a definition of a (G, N) slice for an algebraic group G over a field k, X a G scheme and $N \leq G$ a subgroup, generalizing definitions by Seshadri from the sixties and Katsylo from the eighties. Their definition of a (G, N) slice is a (locally closed) N-stable subscheme $S \subset X$ such that the induced morphism $(G \times S)/N \to X$ of algebraic spaces is an open immersion. Here N acts freely on $G \times S$ via $n(g, s) = (gn^{-1}, ns)$. One of the motivations for defining (G, N) slices $S \subseteq X$ is that $k(X)^G \cong k(S)^N$ which has been used to good effect to simplify the rationality problem. They showed that the existence of a (G, N) slice of a versal G-scheme implies the surjectivity of the maps $H^1_{fppf}(L, N) \to H^1_{fppf}(L, G)$ for infinite field L containing F. Such a result implies that $\mathrm{ed}(G) \leq \mathrm{ed}(N)$ where ed refers to the essential dimension of the algebraic group. This result generalises a result of Reichstein in the characteristic zero case for linear representations of G. The speaker discussed a construction of slices of geometrically irreducible G-varieties

coming from stabilizers in general position (SGPs). He showed that for a SGP for a geometrically irreducible G-variety V, there exists a $(G, N_G(H))$ slice which is open in the fixed scheme V^H. Combining this result with their result on essential dimension and slices, they were able to get a new bound on the essential dimension of the split simply connected group of type E_7. [12]

Speaker: **Venapally Suresh** (Emory University)
Title: *Rost invariant over function fields of p-adic curves*
Let F be a field and G an absolutely almost simple simply connected algebraic group over F. For the Rost invariant $H^1(F,G) \to H^3(F, \mathbf{Q}/\mathbf{Z}(2))$, the speaker discussed the injectivity of this map when F is the function field of a p-adic curve, with special reference to $G = SL_1(A)$, where A is a central simple algebra over F of index coprime to p.

Speaker: **Nivedita Bhaskhar** (Emory University)
Title: *Reduced Whitehead groups of division algebras over function fields of p-adic curves*
The question of whether every reduced norm one element of a central simple algebra is a product of commutators was formulated in 1943 by Tannaka and Artin in terms of the reduced Whitehead group $SK_1(D)$.

The speaker addressed the question of the triviality of the reduced Whitehead group for for l torsion, degree l^2 algebras over function fields of p-adic curves where l is any prime not equal to p. The proof relies on the techniques of patching as developed by Harbater-Hartmann-Krashen and exploits the arithmetic of these fields to show triviality of the reduced Whitehead group.

Speaker: **Julia Hartmann** (RWTH Aachen)
Title: *Obstructions to Local-Global Principles for Linear Algebraic Groups*
Local-Global Principles are a very important subject in the theory of linear algebraic groups. The speaker discussed such principles for groups defined over arithmetic function fields. In that case, some obstructions come from a collection of overfields associated with patching. These obstructions are reasonably well understood for rational linear algebraic groups, but interesting examples due to Colliot-Thélène, Parimala and Suresh show that the case of nonrational groups is more difficult. Recent joint results with D. Harbater and D. Krashen provide an explanation as to why these examples occur, via the geometry of a model for the function field. [7]

Speaker: **Anastasia Stavrova** (St.Petersburg University)
Title: *Simple algebraic groups and structurable algebras*
The speaker presented a uniform proof of the well-known correspondence between isotropic simple algebraic groups and simple structurable or Jordan algebras in joint work with T. De Medts and L. Boelaert.

Speaker: **Alexander Duncan** (University of South Carolina)
Title: *Pairs of quadratic forms in characteristic 2*
In joint work with I. Dolgachev, the speaker considered smooth complete intersections of two quadrics in even-dimensional projective space. Over an algebraically closed field of characteristic not 2, it is well known that one can find a basis in which both quadratic forms are diagonal. However, this fails in characteristic 2. He presented a normal form which applies over an arbitrary field of characteristic 2. The normal form can be used to determine the automorphism groups of these varieties.

III. The third day of talks featured talks on realizing algebraic groups as automorphism groups, invariant theory for automorphism groups of simple algebras, equivariant oriented cohomology and motives of twisted flag varieties, rational orbits for groups acting on varieties, and period-index problems. One hour talks were given by M. Brion (Institut Fourier, France), V.L. Popov (Steklov Institute, Russia), G. Savin (Utah, USA). Half hour talks were given by young researchers C. Zhong (SUNY Albany,USA), O. Haution (Munich, Germany), B. Antieau (UIC, USA) and doctoral student A. Neshitov (Ottawa/Steklov).

Speaker: **Michel Brion** (Institut Fourier)
Title: *Realizing algebraic groups as automorphism groups*

The speaker addressed the question of realizing a given algebraic group as the automorphism group of some algebraic variety. He showed that every smooth connected group scheme over a perfect field is the connected automorphism group scheme of a normal projective variety. In the characteristic zero case, one could take the variety to be smooth. For a finite dimensional Lie algebra over a field of characteristic zero, he gave equivalent conditions for the Lie algebra to be derivations of the structure sheaf of some proper scheme over the field. [5]

Speaker: **Changlong Zhong** (State University of New-York at Albany)
Title: *Equivariant oriented cohomology of flag varieties*
In joint work with Calmès and Zainouilline, the speaker explained an algebraic construction of equivariant oriented cohomology of (partial and full) flag varieties and of the push-pull morphisms between these cohomology groups. In particular, he showed how for an equivariant oriented cohomology theory h over a base field k, a split reductive group G over k, a maximal torus T in G and a parabolic subgroup P containing T, the equivariant oriented cohomology ring $h_T(G/P)$ can be associated with a formal affine Demazure algebra which is the dual of a coalgebra and can be defined just using the root datum of (G,T), a set of simple roots defining P and the formal group law of h. With respect to these algebras, he showed how operators can be defined to construct push-forwards and pull-backs along geometric morphisms. A Schubert Calculus was described for these rings. [19]

Speaker: **Olivier Haution** (University of Munich)
Title: *Finite group actions on the affine space*
The speaker discussed the existence of fixed rational points for the action of a finite p-group on affine n space over a field of characteristic different than p. This question was popularised in a paper of Serre from 2009 who proved this in a number of important cases, and pointed out that the answer was unknown when the group is cyclic of order 2, the field is $\mathbb{Q}$ and $n = 3$. The list of positive known cases was extended by Esnault and Nicaise in 2011. The speaker proved the existence of a rational fixed point when k is l-special for some prime different from its characteristic and when k is perfect and fertile and $n = 3$. [8]

Speaker: **Alexander Neshitov** (University of Ottawa / Steklov Institute at St.Petersburg)
Title: *Motives of twisted flag varieties and representations of Hecke-type algebras*
In joint work in progress with N. Semenov, V. Petrov and K. Zainoulline, the speaker related the category of (cobordism-) Ω-motives of twisted flag varieties for a semisimple linear algebraic group G with the category of integer (or modular) representations of the associated Hecke-type algebra $H = H(G)$.

The algebra H was introduced and studied recently in a series of papers by Calmès, Hoffnung, Malagon-Lopez, Savage, Zainoulline, Zhao, Zhong. It has two important properties: (i) its dual over $\Omega_T(pt)$, where T is a split maximal torus of G, gives the T-equivariant cobordism ring $\Omega_T(G/B)$ of the variety of Borel subgroups of G; (ii) its complete set of generators and relations is known and resembles those of an affine Hecke algebra. [14]

Speaker: **Vladimir L. Popov** (Steklov Institute, Moscow)
Title: *Simple algebras and algebraic groups*
The speaker discussed the following questions:

(1) Given an algebraic group G, let V be a finite-dimensional algebraic G-module that admits a structure of a simple (not necessarily associative) algebra A for which $G = \mathrm{Aut}(A)$. Then V admits a close approximation to the analogue of classical invariant theory.

(2) What are the groups G for which such a V exists?

(3) Given G, what are the G-modules V for which (1) holds?

Speaker: **Gordan Savin** (Utah University)
Title: *Twisted Bhargava Cubes*
In joint work with Wee Teck Gan, the speaker discussed the problem of classifying rational orbits for pre homogeneous spaces. A classical example is $GL(n)$ acting on the space of symmetric matrices. In this case rational orbits

are parameterized by isomorphism classes of quadratic spaces. For some pre homogeneous spaces arising from exceptional groups the orbit problem has an answer in terms of (twisted) composition algebras. [18]

Speaker: **Benjamin Antieau** (University of Illinois at Chicago)
Title: *Prime decomposition in period-index problems via representation theory*
The speaker reported on joint work with B. Williams on the use of representations of projective general linear groups to extend known facts about the prime divisors of the period and index of Brauer classes that hold over fields to more general settings. The first result is that the primes dividing the period and index agree. This is proved using only exterior representations. The second result is that the index of a Brauer class is the product of the indices of each of its p-parts. This requires more complicated Young diagrams. [1]

IV. The fourth day featured morning talks about oriented motivic theories by Mark Levine (Essen, Germany), generalisations of the Grothedieck Serre conjecture by I. Panin (Steklov Institute, Russia) and the topological index of Brauer classes by young researcher B. Williams (UBC). There was a free afternoon.

Speaker: **Marc Levine** (University of Essen-Duisburg)
Title: *On the geometric part of some oriented motivic theories*
For an oriented motivic ring spectrum E in $SH(k)$, k a field of characteristic zero, there is a canonical map

$$\Omega^* \to E^{2*,*}$$

of oriented cohomology theories on Sm/k, in the sense of Levine-Morel. If E has associated formal group law (F, R), this map descends to

$$\Omega^* \otimes_L R \to E^{2*,*}$$

In joint work with S. Dai and G. Tripathi, the speaker described a criterion which implies that this second map is an isomorphism of oriented cohomology theories on Sm/k. He showed that for a wide class of examples, including MGL, Landweber exact theories and their connective covers as well as certain quotients or localizations of MGL, such as truncated Brown-Peterson theories, Morava K-theories and connective Morava K-theory, satisfy this criterion. [11]

Speaker: **Ivan Panin** (Steklov Institute at St.Petersburg)
Title: *A purity theorem*
The speaker discussed the following:

Conjecture. Let $\mathcal{O}$ be a regular local ring and K be its fraction field. Let $m\colon G \to C$ be a smooth $\mathcal{O}$-morphism of reductive $\mathcal{O}$-group schemes, with a torus C. Suppose additionally that the kernel of m is a reductive $\mathcal{O}$-group scheme. Then the following sequence

$$\{1\} \to C(\mathcal{O})/m(G(\mathcal{O})) \to C(K)/m(G(K)) \to \bigoplus_{ht(p)=1} C(K)/[C(\mathcal{O}_p) \cdot m(G(K))] \to \{1\}$$

is exact, where p runs over all height 1 primes of $\mathcal{O}$ and

$$res_p\colon C(K)/m(G(K)) \to C(K)/[C(\mathcal{O}_p) \cdot m(G(K))]$$

is the natural map (the projection to the factor group).

Theorem. The conjecture is true, if $\mathcal{O}$ is a regular local ring containing a field.

Remark. The exactness of that sequence in the middle term is used in the proof of the Grothendieck–Serre conjecture for regular local rings containing a field. [15]

Speaker: **Ben Williams** (University of British Columbia)
Title: *The topological index of period-2 Brauer classes*

The speaker outlined how one can use the homotopy theory of classifying spaces of linear groups to find obstructions to representing Brauer classes as the classes of Azumaya algebras of specific ranks, concentrating on the case of period-2 classes.

V. The last day featured talks by N. Karpenko (Alberta) about 16-dimensional quadratic forms and D. Krashen (UGA, USA) about the Clifford algebra of a morphism.

Speaker: **Nikita Karpenko** (University of Alberta)
Title: *On* 16*-dimensional quadratic forms in* I^3
The speaker discussed the mysteries related to quadratic forms with Witt class in I^3 focussing on the following questions for which 16 is the smallest dimension in which they were not understood:

- whether the forms can be can be parameterized by algebraically independent variables,
- if every form contains a proper subform with Witt class in I^2,
- how many 3-Pfister forms are needed to write the Witt class of an arbitrary 16-dimensional form in I^3 (over an arbitrary field) as their linear combination (no upper bound at all is available). [9]

Speaker: **Danny Krashen** (UGA)
Title: *The Clifford algebra of a finite morphism of schemes*
In joint work with M. Lieblich, the speaker defined a Clifford algebra associated to a finite morphism of schemes, generalizing the notion of the Clifford algebra of a homogeneous polynomial. He defined a Clifford functor from the category of finite surjective morphisms of proper k schemes to k sheaves and showed it was representable. He showed that the stack of representations of rank n of the Clifford algebra of a finite morphism of degree d is equivalent to the stack of Ulrich bundles of degree m over the finite morphism. He then described connections with relative Brauer groups and index reduction, with Ulrich bundles, and with the period-index problem for genus 1 curves. [10]

Scientific Progress Made

During the meeting, new and exciting results were reported on in the following areas:

- Merkurjev's remarkable proof of Suslin's conjecture on the generic non-triviality of the reduced Whitehead group.
- Panin's work on the Grothendieck Serre conjecture in the case of regular local rings containing a finite field.
- Work of Calmès, Neshitov, Zainoulline, Zhong to give an algebraic description of the equivariant oriented cohomology theory of flag varieties.
- Work of Neshitov, Semenov, Petrov and Zainouilline connecting motives of twisted flag varieties to representations of Hecke-type algebras.
- Krashen and Lieblich's description of the Clifford algebra of a finite morphism of schemes and applications of their methods to period index problems for curves of genus 1 and the theory of Ulrich bundles.
- Antieau and Williams' use of topological and representation theoretic methods to address period-index problems in general settings.

Outcome of the Meeting

The workshop attracted 42 leading experts and young researchers from Canada, France, South Korea, Germany, Israel, Russia, Switzerland, USA. There were 27 speakers in total: 13 talks were given by senior speakers, 12 talks by young researchers and postdocs and 2 talks by doctoral students.

The lectures given by senior speakers provided an excellent overview on the current state of research in the theory of algebraic groups in geometry and number theory. There were several new results announced, e.g. Merkurjev (on triviality of reduced Whitehead group), Panin (on Grothendieck-Serre conjecture), Krashen (on Clifford algebras and Ulrich bundles). The afternoon sessions provided a unique opportunity for young speakers to present their achievements. Numerous discussions between the participants after the talks have already led to several joint projects, e.g. by Junkins-Krashen-Lemire, Calmès-Neshitov-Zainoulline, Karpenko-Merkurjev.

The organizers consider the workshop to be a great success. The quantity and quality of the students, young researchers and the speakers was exceptional. The enthusiasm of the participants was evidenced by the frequent occurrence of a long line of participants waiting to ask questions to the speakers after each lecture. The organizers feel that the material these participants learned during their time in BIRS will prove to be very valuable in their research and will undoubtedly have a positive impact on the research activity in the area.

Participants

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Auel, Asher (Yale University)
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Bayer-Fluckiger, Eva (Ecole Polytechnique Federale de Lausanne)
Bhaskhar, Nivedita (Emory University)
Borovoi, Mikhail (Tel Aviv University)
Brion, Michel (Institut Fourier)
Calmes, Baptiste (Universit d'Artois)
Chernousov, Vladimir (University of Alberta)
Duncan, Alexander (University of Michigan, Ann Arbor)
Gille, Stefan (University of Alberta)
Gordon-Sarney, Reed (Emory University)
Grimm, David (USACH)
Hartmann, Julia (University of Pennsylvania)
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Hoffmann, Detlev (TU Dortmund)
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Krashen, Daniel (Rutgers University)
Lee, Ting-Yu (EPFL)
Lemire, Nicole (University of Western Ontario)
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Merkurjev, Alexander (University of California at Los Angeles)
Neshitov, Alexander (University of Ottawa / Steklov Institute)
Panin, Ivan (Steklov Mathematical Institute at St.Petersburg)
Parimala, Raman (Emory University)
Pianzola, Arturo (University of Alberta)
Pirisi, Roberto (University of Ottawa)
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Stavrova, Anastasia (St. Petersburg State University)
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Chapter 22

Strongly Interacting Topological Phases (15w5051)

September 20 - 25, 2015

Organizer(s): Jason Alicea (Caltech), Matthew P. A. Fisher (University of California - Santa Barbara), Marcel Franz (University of British Columbia), Yong-Baek Kim (University of Toronto)

Overview of the Field

Topology and physics comprise two ostensibly very different fields. The former represents a branch of mathematics associated with invariance of form, while the latter quantifies the behavior of nature that, fundamentally, arises from geometric laws. These disciplines have nevertheless become increasingly intertwined with the advent of 'topological phases'—wherein topological properties emerge from purely geometric rules in many-particle systems. Their behavior often defies intuition and conventional theories of matter in stunning ways, necessitating the development of new physics paradigms.

The order present in a topological phase, for instance, is characterized not by a local observable (e.g., magnetization) but rather a topological invariant that signifies global 'knotting' in a system's wavefunction. One measurable implication is that topological phases often exhibit metallic states at their boundary that are intrinsically resilient to imperfections such as impurities—in sharp contrast to standard metallic systems. Moreover, many topological phases support exotic 'fractionalized' excitations whose properties bear little resemblance to those of the constituent electrons and ions in a material. These characteristics, fascinating in their own right, endow topological phases with great technological promise in areas ranging from low-dissipation electronics to topological quantum computation.

Over the last several years the field has matured at an astonishing pace—particularly in the case of weakly correlated topological phases, whose universal properties do not rely on inter-particle interactions in an essential way. The latter are indeed now fairly well-understood due to the wealth of analytical and numerical tools available for treating weakly interacting electrons. A prominent case in point is provided by 'topological insulators', which were theoretically discovered in 2005 to arise in effectively non-interacting crystalline solids featuring strong coupling between the electrons' spin and orbital motion. Such systems possess an electrically inert interior yet necessarily conduct at their edges provided certain symmetries such as time-reversal are present. Soon after, a vast number of candidate topological insulator materials were identified via standard band structure methods, whose predictive power is outstanding in weakly correlated settings. Indeed, following such predictions many 2D and 3D topological insulators have now been observed experimentally using electron spectroscopy and transport techniques. The detailed phenomenology of topological insulators—including their unusual electromagnetic response properties, stability to symmetry-breaking perturbations, entanglement structure, etc.—can also be efficiently addressed theoretically owing to the weakly interacting nature of the problem.

Other weakly correlated topological phases enjoy a similar level of tractability. In fact an exhaustive classification of topological phases for free-electron systems in all spatial dimensions is now in place, which captures topological insulators, integer quantum Hall systems, and topological superconductors in a unified framework. (For free-boson systems the classification is trivial since without interactions Bose-Einstein condensation necessarily results.) All such weakly correlated topological phases either find intrinsic realization in specific compounds, or can be 'engineered' by judiciously combining conventional materials to essentially force electrons into nontrivial topological states. The complete experimental realization of phases in this classification remains an outstanding problem, but a concrete roadmap nonetheless exists and efforts in this direction are well underway.

Recent Developments and Open Problems

Strongly correlated topological phases—which require inter-particle interactions to exist—pose much greater challenges to both theory and experiment. Methods that have proven spectacularly successful for dealing with non-interacting systems (e.g., band structure and K-theory among others) indeed are wholly inadequate in this context. Many profound questions therefore remain only partially answered here: What is the full classification of topological phases in the interacting case, both for fermionic and bosonic systems? (Interacting bosons can exhibit topological phases.) What mathematical and numerical tools allow one to efficiently extract their essential universal properties? Which materials are likely 'intrinsic' candidates for strongly correlated topological phases? Can they be 'engineered' in a reliable way? How can their topological properties be unambiguously observed in an experiment? And how does the physics survive when realistic ingredients such as disorder and thermal fluctuations are incorporated?

Exploring such questions promises to push the boundary of our understanding of quantum mechanics and the organizing principles of nature, yet is by no means purely academic. In particular, certain strongly interacting topological phases can provide a platform for a universal quantum computer that exhibits intrinsic immunity against decoherence—the primary bottleneck that for decades has stymied efforts at fabricating a scalable quantum computer. Thus the study of highly correlated topological phases may ultimately help to solve one of the great outstanding technological problems in physics, providing immense complementary motivation for the problem.

The topic of strongly interacting topological phases has, consequently, captivated diverse segments of the physics community. Theorists are attempting a complete classification of topological phases for interacting bosons and fermions. Complementing this challenging endeavor, solvable toy models for correlated topological phases with increasing levels of realism are being constructed, and new schemes for engineering their physics in the laboratory are being developed. Another active area of theoretical research concerns the interplay between interactions and strong randomness, which may, counterintuitively, enable topological properties to survive in regimes where they are absent in an otherwise clean system. Quantum information scientists grapple with the question of how correlated topological phases can be used for applications ranging from robust quantum memory to universal quantum computation. On the experimental side, measurements suggest that solid-state systems including GaAs quantum wells, graphene, heavy fermion compounds, and magnetic insulators host correlated topological phases whose precise nature remains to be understood. Concomitantly, cold atoms researchers have devised ingenious schemes for realizing models known to support interacting topological phases in optical lattice setups.

The workshop was particularly inspired by five areas:

1) Classification of topological phases. Strong correlation is known to produce topological phases, such as fractional quantum Hall states and spin liquids, that are absent in the free-particle classification. Bosonic topological phases provide an extreme example where in fact all such states originate from interactions. Conversely, phases that are topologically distinct in the free-particle limit can, in select cases, actually lose their distinction when strong interactions are present. Recent studies have even shown that interactions can generate novel topological phases that can only exist at the boundaries of a higher dimensional system. These examples strikingly underscore the need for a more general classification of topological phases that extends into the strong correlation regime. Specific aims of the workshop for this area included (i) exploring mathematical methods necessary for this classification; (ii) applying them to characterize possible topological phases with and without symmetry requirements, as well as to boundary topological phenomena; (iii) constructing general topological invariants for these phases; and (iv) enumerating their universal physical properties.

2) Strongly correlated topological materials. Recently there has been great progress in connecting theoretical lattice models for interacting topological phases with real materials. The underlying idea is to use both strong spin-orbit coupling and Coulomb interactions to generate highly entangled many-body quantum states. Particularly fruitful are systems defined on geometrically frustrated lattices, wherein all pairwise couplings between particles cannot be satisfied simultaneously. On these lattices—which often consist of corner-sharing simplexes such as triangles and tetrahedra—the interplay between spin-orbit coupling, Coulomb repulsion, and frustration can stabilize a plethora of new interacting topological phases. Examples include correlated analogues of topological insulators that feature gapless fractionalized excitations at their boundary, 'axion insulators' exhibiting topological electromagnetic response, and Weyl semi-metals in which gapless excitations are described by two-component Dirac fermions. Several iridium-based compounds provide possible sources for such phenomena, and are therefore presently the subject of intense experimental investigation. The goal of the workshop in this area of research was to (i) understand the universal structure of lattice models that may possess topological phases; (ii) make precise connections between topological properties of the many-body wavefunction and the geometry of the underlying lattices; and (iii) obtain more precise theoretical or mathematical criteria that can be used to identify such phases in both models and real materials.

3) 'Engineered' platforms for interacting topological phases. While nature provides numerous promising host materials, many (and perhaps most) possible strongly correlated topological phases may be difficult to realize by relying solely on a system's internal dynamics. A powerful alternative method is to seek 'designer' interacting topological phases by combining well-understood ingredients that are already widely available and well-understood. Such an approach is routinely used in the context of cold atoms, where proposals exist for generating nontrivial correlated phases via artificial gauge fields, flat bands with non-trivial Chern numbers, spin models with enlarged symmetries, etc. In the solid state realm, heterostructures formed out of systems such as (Abelian) fractional quantum Hall phases and superconductors have been predicted to host exotic non-Abelian defects that can be used for universal topological quantum computation. These examples illustrate the potential power behind engineered topological phases but are by no means exhaustive. The workshop sought to address possible ways of enhancing the feasibility of existing proposals, identify means of detecting the predicted correlated topological phases and harnessing their properties for applications, and work towards more general schemes for designing arbitrary interacting topological phases.

4) Many-body localization. Most topological phases are characterized by an energy gap to bulk excitations. This energy gap is critical in protecting topological degeneracies that can serve as decoherence-free quantum memory or as the basis for fault-tolerant quantum computation. When temperatures are well below this energy gap, unwanted 'stray' excitations can in principle be exponentially suppressed leaving the protected manifold essentially untouched. But at higher temperatures thermally induced excitations can be disastrous, destroying quantum memory or dephasing a quantum computation. In many potential topological quantum systems, avoiding such thermal dephasing effects will, in practice, be exceedingly challenging. Will such 'thermalizing' effects always be a showstopper? Perhaps not. Remarkably, when an isolated quantum system is subject to a quenched random potential (disorder), a Many-Body-Localized (MBL) phase can exist within which thermalization is simply not operative. In an MBL phase energy can become localized by the disorder and the system cannot self-thermalize, even when the system possesses a finite energy density. The implications of many-body localization for topological phases is in its infancy, and many questions need addressing. Is it possible to stabilize quantum memory in an MBL phase? Can one characterize topological order in an MBL phase which has no energy gap? What are the implications for braiding and quantum computation in the MBL setting?

5) Quantum information applications. One of the holy grails of the field is to utilize correlated topological phases to construct a universal, decoherence-free quantum computer. (The key idea is that certain correlated topological phases allow storage and processing of quantum information in a non-local manner that is intrinsically immune to local environmental perturbations that ordinarily cause decoherence.) The workshop will provide a venue for discussing many fundamental questions related to this ultimate goal. For instance, what implications does many-body localization have for the stability of qubits? What is the 'optimal' correlated topological phase for this application? More precisely, what phase of matter allows one to run a given quantum algorithm with the fewest number of qubits, and using the smallest number of elementary operations? On a more practical level, how can such an 'ideal' correlated topological phase be constructed experimentally and manipulated for quantum computation? As the technology develops, what initial applications can be realized when the number of qubits is relatively low? Addressing such questions can provide valuable long-term direction for the field.

Presentation Highlights

Our workshop gathered a broad spectrum of researchers studying many facets of the problem. The following broad themes were explored: Classification of topological phases; frontiers in Majorana-zero-mode platforms, with an emphasis on interaction-driven physics; non-Abelian defects in topological phases; the interplay between symmetry and topology; new developments in the half-filled Landau level and connections to topological insulators; and topological materials including Weyl semimetals, magnetic insulators, and a new type of 'cohomological insulator'. Five extended overview talks highlighted progress and open challenges in these areas:

1. Lukasz Fidkowski (Stony Brook), 'Symmetry and Topological Phases: an overview'
In this talk I will review the recent progress in topological phases with symmetries. A central focus will be Symmetry Protected Phases (SPTs), and I will discuss various approaches to studying these, including microscopic models, group cohomology, effective Chern-Simons field theories, and non-linear sigma models. I will emphasize open issues, including SPTs in higher dimensions and those beyond the cohomology classification, and SPTs built out of fundamental fermion degrees of freedom.

2. Alexey Gorshkov (JQI) 'Topological Phases in Atomic, Molecular, and Optical Systems'
We will first review schemes for taking advantage of the tremendous degree of control recently achieved in AMO (atomic, molecular, and optical) systems to realize topological phenomena. In particular, we will emphasize unique features of AMO systems such the abundance of bosonic platforms, accessibility of far-out-of-equilibrium dynamics, and natural occurrence of interactions decaying as tunable power laws. We will then focus on a few examples such as SPT phases with ion crystals, various fractional quantum Hall states with dipoles, and parafermionic zero modes with ultracold neutral bosons.

3. Charlie Marcus (University of Copenhagen), 'Majorana update'
This talk surveyed experimental progress in synthesizing topological superconductors fashioned from semiconductor wire/superconductor hybrid architectures. Breakthroughs on both the fabrication and measurement end were discussed, including new signatures of Majorana zero modes in ultra-clean, hard-gap devices.

4. Todadri Senthil (MIT), 'Half-filled Landau level, topological insulator surfaces, and three dimensional quantum spin liquids'
We synthesize and partly review recent developments relating the physics of the half-filled Landau level in two dimensions to correlated surface states of topological insulators in three dimensions. The latter are in turn related to the physics of certain three dimensional quantum spin liquid states. The resulting insights provide an interesting answer to the old question of how particle-hole symmetry is realized in composite fermion liquids. Specifically the metallic state at filling $\nu = \frac{1}{2}$ - described originally in pioneering work by Halperin , Lee, and Read as a liquid of composite fermions - was proposed recently by Son to be described by a particle-hole symmetric effective field theory distinct from that in the prior literature. We show how the relation to topological insulator surface states leads to a physical understanding of the correctness of this proposal. We develop a simple picture of the particle-hole symmetric composite fermion through a modification of older pictures as electrically neutral "dipolar" particles. We revisit the phenomenology of composite fermi liquids (with or without particle-hole symmetry), and show that their heat/electrical transport dramatically violates the conventional WiedemannFranz law but satisfies a modified one. We also discuss the implications of these insights for finding physical realizations of correlated topological insulator surfaces.

5. Xiao-Gang Wen (MIT) '2+1D Bosonic/Fermionic topological orders with/without symmetry'
2+1D bosonic/fermionic topological orders with/without symmetry can be described in a unified way by the so called "non-degenerate braided fusion categories (UBFC) over a symmetric category"; where the symmetric category describes a fermionic product state without symmetry or a fermionic/bosonic product state with symmetry G. I will describe those mathematical notion in simple terms and discuss the table of bosonic/fermionic topological orders. For example, we find that, up to invertible $p+ip$ fermionic topological orders, there are only four fermionic topological orders with one nontrivial topological excitation: (1) the $K = [-10; 02]$ fractional quantum Hall state, (2) a Fibonacci bosonic topological order $2^B_{14/5}$ stacking with a fermionic product state, (3) the time-reversal conjugate of the previous one, (4) a primitive fermionic topological order that has a chiral central charge $c = \frac{1}{4}$, whose only topological excitation has a non-abelian statistics with a spin $s = \frac{1}{4}$ and a quantum dimension $d = 1 + \sqrt{2}$.

Numerous shorter, more specialized talks were also presented by experts in the field:

1. Maissam Barkeshli (Station Q), 'Superconductivity Induced Topological Phase Transition at the Gapless Edge of Even Denominator Fractional Quantum Hall States'
We show that every even-denominator fractional quantum Hall (FQH) state possesses at least two robust, topologically distinct gapless edge phases if charge conservation is broken at the boundary by coupling to a superconductor. The new edge phase allows for the possibility of a direct coupling between electrons and emergent neutral fermions

of the FQH state. This can potentially be experimentally probed through geometric resonances in the tunneling density of states at the edge, providing a probe of fractionalized, yet electrically neutral, bulk quasiparticles. Other measurable consequences include a charge e fractional Josephson effect, a charge e/4q quasiparticle blocking effect in filling fraction $p/2q$ FQH states, and modified edge electron tunneling exponents. We also discuss similar phenomena in quantum spin liquids, which suggests new probes of fractionalization in such states.

2. Erez Berg (Weizmann), 'Topological phenomena in periodically driven systems: the role of disorder and interactions'
Periodically driven quantum systems, such as semiconductors subject to light and cold atoms in optical lattices, provide a novel and versatile platform for realizing topological phenomena. Some of these are analogs of topological insulators and superconductors, attainable also in static systems; others are unique to the periodically driven case. I will describe how periodic driving, disorder, and interactions can conspire to give rise to new robust steady states, with no analogues in static systems. In disordered two-dimensional driven systems, a phase with chiral edge states and fully localized bulk states is possible; this phase can realize a non-adiabatic quantized charge pump. In interacting one dimensional driven systems, current carrying states with excessively long life times can arise.

3. Andrei Bernevig (Princeton), 'Cohomological insulators'
This talk presented theory and experiment for a new type of topological insulator exhibiting novel topological features in the band structure over extended regions of momentum space.

4. Parsa Bonderson (Station Q), 'Symmetry, Defects, and Gauging of Topological Phases'
We examine the interplay of symmetry and topological order in 2+1 dimensional topological phases of matter. We describe how the global symmetries of the microscopic system act on the emergent topological degrees of freedom. A general framework is provided to classify symmetry fractionalization in topological phases, including phases that are nonAbelian and symmetries that permute the quasiparticle types and/or are anti-unitary. We develop a theory of extrinsic defects (fluxes) associated with elements of the symmetry group. This provides a general classification of symmetry-enriched topological phases derived from a topological phase of matter with unitary on-site symmetry. The algebraic theory of the defects, known as a G-crossed braided tensor category, allows one to compute many properties, such as the number of topologically distinct types of defects associated with each group element, their fusion rules, quantum dimensions, zero modes, braiding exchange transformations, a generalized Verlinde formula for the defects, and modular transformations of the G-crossed extensions of topological phases. We also examine the promotion of the global symmetry to a local gauge invariance, wherein the extrinsic G-defects are turned into deconfined quasiparticle excitations, which results in a different topological phase.

5. Fiona Burnell (University of Minnesota), 'Correlated topological and symmetry-breaking order: geometrical frustration and anyon condensation on the lattice'
A number of solvable topological orders admit phase transitions in which quasiparticles condense to bring the system to a new phase with reduced, but non-trivial, topological order. I will describe a transition of this type carried out on a lattice, in which the condensed phase is geometrically frustrated and spontaneously breaks translation symmetry. This symmetry breaking coexists with a residual Z_2 (abelian) topological order, leading to an interesting interplay between quasiparticles and fluctuations in the local order parameter.

6. Jennifer Cano (Princeton), 'Chirality-Protected Majorana Zero Modes at the Gapless Edge of Abelian Quantum Hall States'
We show that the $\nu = 8$ integer quantum Hall state can support Majorana zero modes at domain walls between its two different stable chiral edge phases without superconductivity. This is due to the existence of an edge phase that does not support gapless fermionic excitations; all gapless excitations are bosonic in this edge phase. Majorana fermion zero modes occur at a domain wall between this edge phase and the more conventional one that does support gapless fermions. Remarkably, due to the chirality of the system, the topological degeneracy of these zero modes has exponential protection, as a function of the relevant length scales, in spite of the presence of gapless excitations, including gapless fermions. These results are compatible with charge conservation, but do not require it. We discuss generalizations to other integer and fractional quantum Hall states, and classify possible mechanisms for appearance of Majorana zero modes at domain walls.

7. Yong Chen (Purdue), 'Transport experiments in topological insulators'
I will discuss our recent transport experiments on high-quality topological insulators thin films and nanowires with insulating bulk and surface-dominated conduction. We reveal a number of unique transport properties of spin-helical Dirac fermion topological surface states, such as the half-integer quantum Hall effect, helical spin-polarized

current, and half-integer Aharonov-Bohm effect. If time allows, I may also discuss measurements probing proximity induced superconductivity and interaction effects.

8. Paul Fendley (Oxford), 'Strong Zero Modes and Eigenstate Phase Transitions'
Gapless edge or zero modes surviving the presence of disorder are common in a topological phase of matter. "Weak" zero modes, guaranteeing ground-state degeneracy, necessarily survive throughout a topological phase. A more dramatic effect occurs in the Ising chain/Majorana wire: "strong" edge zero modes result in identical spectra in even and odd fermion-number sectors, up to exponentially small finite-size corrections. There is a presumption that disorder is necessary to stabilize strong zero modes in the presence of interactions, but I show that their presence in a clean system is not a free-fermionic fluke. In this talk I construct an explicit strong zero mode in the XYZ chain/coupled Majorana wires; this operator possesses some remarkable structure apparently unknown in the integrability literature. I also present evidence for strong zero modes in the parafermionic case, implying the existence of an unconventional eigenstate phase transition where the strong zero mode disappears, leaving only the weak one.

9. Josh Folk (University of British Columbia), 'Trivial edges in a topological material: investigations of InAs/GaSb quantum wells'
Double quantum wells in InAs/GaSb are believed to host a highly tunable quantum spin Hall state when front and backgate voltages align the bottom of the conduction band (InAs) below the top of the valence band (GaSb). To date, the most direct evidence of the quantum spin Hall state in this material has been the observation of conducting edges when bulk is insulating. The conductance properties of those edges are close to those expected for topologically-protected edges, but no direct measurement of edge helicity has been reported and the weak magnetic field dependence of the edge conductance remains a puzzle. Here, we report measurements of edge conductance in InAs/GaSb quantum wells that are clearly tuned to a trivial (non-topological) regime with gate voltages. This observation is robust, in the sense that it is seen in every sample, independent of processing conductions or wafer details, though at at quantitative level edge conductance varies by almost an order of magnitude when considering many samples over a wide range of temperatures. In some samples, these trivial edges are found to become highly resistive below 100mK with a temperature dependence suggestive of variable range hopping, raising hopes that new processing schemes may be developed to eliminate unwanted non-topological edge conduction from this otherwise-promising material.

10. Marcel Franz (University of British Columbia), 'New phases from interacting Majorana fermions'
Vortices in the Fu-Kane model (describing a superconducting surface of a 3D topological insulator) are known to host Majorana zero modes. By adjusting a single system parameter – the global chemical potential – the zero modes can be tuned to the regime of strong interactions. In this talk I will describe the simplest interacting system that can be built from these ingredients: a 1D Majorana chain with nearest neighbor hopping and the most local 4-fermion interaction. The system exhibits a complex phase diagram with interesting phases and phase transitions between them. These include a gapless Ising phase for attractive interactions separated from a doubly degenerate gapped phase at strong coupling by a quantum critical point in the tricritical Ising universality class. For weak repulsive interactions we find an interesting gapless phase with coexisting Luttinger liquid and Ising degrees of freedom. The latter is separated from a 4-fold degenerate gapped phase at strong coupling by a novel generalization of the commensurate-incommensurate transition.

11. Liang Fu (MIT), 'Bit from it: building a robust quantum computer from Majorana fermions'
This talk described a new way of implementing surface codes for quantum computation based on Majorana networks.

12. Roman Lutchyn (Station Q), 'Interplay between Kondo and Majorana Interactions in Quantum Dots'
We study the properties of a quantum dot coupled to a topological superconductor and normal leads and discuss the interplay between Kondo- and Majorana-induced couplings in quantum dots. The latter appears due to the presence of Majorana zero-energy modes localized, for example, at the ends of the one-dimensional topological superconductor. We investigate the phase diagram of the system as a function of Kondo and Majorana interactions using a renormalization-group analysis, a slave-boson mean-field theory, and numerical simulations using the density-matrix renormalization-group method. We show that, in addition to the well-known Kondo fixed point, the system may flow to a new fixed point controlled by the Majorana-induced coupling, which is characterized by nontrivial correlations between a localized spin on the dot and the fermion parity of the topological superconductor and the normal lead. We compute full counting statistics of charge fluctuations, which highlights some peculiar

features characteristic to this Majorana fixed point.

13. Mike Hermele (CU Boulder), 'The flux-fusion anomaly test and bosonic topological crystalline insulators'
I will describe a method, the flux-fusion anomaly test, to test for anomalous symmetry fractionalization in some two-dimensional (d=2) symmetry-enriched topological (SET) phases, where the symmetry may include spatial symmetries and/or time reversal. The anomalous fractionalization patterns thus identified cannot occur in strictly d=2 systems, but are realized as surface theories of d=3 symmetry-protected topological phases. This leads to several new examples of d=3 bosonic topological crystalline insulators, and some understanding of physical properties at their surfaces. Time permitting, I will briefly mention applications of the anomaly test to d=3 SET phases, and to Z_2 spin liquids in d=2 Heisenberg antiferromagnets.

14. Taylor Hughes (UIUC), 'Abelian Topological Phases: Symmetries, Defects, and Entanglement'
In this talk we will review the quasi-particle lattice of 2+1-d Abelian topological phases described by a K-matrix. From the lattice structure we will discuss the classification of global anyonic symmetries and their related twist defects, and briefly mention the consequences of gauging the anyonic symmetries of a parent topological phase. Finally, we will discuss a new development for the entanglement of Abelian phases which allows for modifications to the sub-leading correction to the area law. These modifications stem from a non-uniqueness in the generation of the topological phase, which we explicitly show using a coupled-wire construction.

15. Michael Levin (University of Chicago), 'Bulk-boundary correspondence for 3D symmetry-protected topological phases'
Symmetry-protected topological (SPT) phases can be thought of as generalizations of topological insulators. Just as topological insulators have robust boundary modes protected by time reversal and charge conservation symmetry, SPT phases have boundary modes protected by more general symmetries. In this talk, I will discuss the relationship between bulk and boundary properties of 3D SPT phases with unitary symmetries.

16. Nate Lindner (Technion), 'The Ising bagel: Non-Abelian statistics enriched by defects and their zero modes'
Non-Abelian topological phases of matter can be utilized to encode and manipulate quantum information in a non-local manner, such that it is protected from imperfections in the implemented protocols and from interactions with the environment. The condition that the non-Abelian statistics of the anyons supports a computationally universal set of gates sets a very stringent requirement which is not met by many topological phases. We consider the possibility to enrich the possible topological operations supported by a non-Abelian topological phase by introducing defects into the system. We show that such defects bind zero modes which form a unique algebra that goes beyond the parafermionic algebra describing defects in Abelian phases. Furthermore, we show that by coupling zero modes, one can obtain a set of topological operations that implements a universal set of gates. We also discuss lattice models of interacting defects and their implications to edge phases of non-Abelian topological phases.

17. Yuan-Ming Lu (Ohio State University), 'Measuring symmetry fractionalization in quantum spin liquids'
Motivated by mounting numerical evidence for spin liquid ground states in the many two-dimensional frustrated spin models, here we develop systematic methods to measure the global and crystal symmetry quantum numbers of fractional excitations. We show that the symmetry fractionalization patterns in a quantum spin liquid can be measured by a dimensional reduction regime, which relates the two-dimensional symmetric topological orders to one-dimensional symmetry protected topological phases. This general framework is directly applicable to numeric results obtained in 2d DMRG studies, and can be generalized to other gapped topological orders in two dimensions.

18. Max Metlitski (KITP), 'Particle-vortex duality of 2D Dirac fermion from electric-magnetic duality of 3D topological insulators'
Particle-vortex duality is a powerful theoretical tool that has been used to study bosonic systems. Here we propose an analogous duality for Dirac fermions in 2+1 dimensions. The physics of a single Dirac cone is proposed to be described by a dual theory, QED3 with a dual Dirac fermion coupled to a gauge field. This duality is established by considering two alternate descriptions of the 3d topological insulator (TI) surface. The first description is the usual Dirac cone surface state. The second description is accessed via an electric-magnetic duality of the bulk TI coupled to a gauge field, which maps it to a gauged topological superconductor. This alternate description ultimately leads to a new surface theory - dual QED3. The dual theory provides an explicit derivation of the T-Pfaffian state, a proposed surface topological order of the TI, which is simply the paired superfluid state of the dual fermions. The roles of time reversal and particle-hole symmetry are exchanged by the duality, which connects some of our results

to a recent conjecture by Son on particle-hole symmetric quantum Hall states.

19. Julia Meyer (Grenoble), 'Topological Josephson φ_0-junctions'
We study the effect of a Zeeman field on the current-phase relation of a topological Josephson junction, based on the edge states of a two-dimensional quantum spin Hall insulator or a nanowire with strong spin-orbit coupling. We show that, in the helical regime, the Zeeman field along the spin quantization axis of the states at the Fermi level results in a large anomalous Josephson effect that allows for a supercurrent to flow in the absence of superconducting phase bias. We relate the associated field-tunable phase shift in the Josephson relation of such a φ_0-junction to the existence of a so-called helical superconductivity, which may result from the interplay of the Zeeman field and spin-orbit coupling.

20. Roger Mong (Pittsburgh), 'Dirac composite fermions in the half-filled Landau level'
One of the most spectacular experimental findings in the fractional quantum Hall effect is evidence for an emergent Fermi surface when the electron density is nearly half the density of magnetic flux quanta ($\nu = 1/2$). The seminal work of Halperin, Lee, and Read (HLR) attributed this to the formation of composite fermions, bound states of an electron and a pair of vortices. We use infinite cylinder DMRG to provide compelling numerical evidence for the existence of a Fermi sea of composite fermions for realistic interactions between electrons at $\nu = 1/2$. Moreover, we show that the phase is particle-hole symmetric, in contrast to the theory of HLR. Instead, our findings are consistent if the composite fermions are massless Dirac particles, at finite density, similar to the surface state of a 3D topological insulator.

21. Stevan Nadj-Perge (Delft), 'Majorana fermions in atomic chains on a superconductor'
Majorana fermions are zero-energy excitations predicted to localize at the edge of a topological superconductor, a state of matter that can form when a ferromagnetic system is placed in proximity to a conventional superconductor with strong spin-orbit interaction. With the goal of realizing a one-dimensional topological superconductor, we have fabricated ferromagnetic iron atomic chains on the surface of superconducting lead [1]. Using highresolution spectroscopic imaging techniques, we show that the onset of superconductivity, which gaps the electronic density of states in the bulk of the chains, is accompanied by the appearance of zero-energy end-states. This spatially resolved signature, corroborated by other observations and theoretical modeling [2], provides evidence for the formation of a topological phase and edge-bound Majorana states in this system. Our results demonstrate that atomic chains are viable platform for future experiments to manipulate Majorana bound states [3] and to realize other 1D and 2D topological superconducting phases. [1] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science 346, 602 (2014). [2] Jian Li, Hua Chen, Ilya K. Drozdov, A. Yazdani, B. Andrei Bernevig, A.H. MacDonald, Phys. Rev. B 90, 235433 (2014). [3] Jian Li, Titus Neupert, B. Andrei Bernevig, Ali Yazdani, ArXiv:1404.4058 (2014).

22. Masaki Oshikawa (University of Tokyo), 'Symmetry protection of critical phases and global anomaly in 1+1 dimensions'
Classification of gapless quantum phases remains very much open. Symmetries are naturally expected to play an important role here, as in the case of gapped quantum phases. In this talk, we argue that there is a protection of bulk gapless critical phases by discrete symmetry. We demonstrate this for the SU(2)-symmetric quantum antiferromagnetic chains and their effective field theory, SU(2) Wess-Zumino-Witten (WZW) theory as an example. The SU(2) WZW theory is characterized by a natural number k, which is called level. In the presence of the SU(2) and a certain discrete Z_2 symmetry, they are classified into the two "symmetry-protected" categories: one corresponds to even levels and the other to odd levels.

23. Dima Pikulin (University of British Columbia), 'Strongly Interacting Majorana Fermions on the Topological Insulator Surface'
We discuss how to engineer strongly interacting phases of Majorana fermions on the surface of a topological insulator in contact with an s-wave superconductor. We suggest that tuning single parameter, surface chemical potential, we can go between weakly- and strongly-interacting regimes. For a special value of the chemical potential the symmetry class BDI is realised. This allows us to suggest an experimental test for the Fidkowski-Kitaev Z_8 periodicity, a realisation of the interacting-enabled topological crystalline phase first suggested by Lapa, Teo, and Hughes, as well as its generalisations to higher dimensions.

24. Shinsei Ryu (UIUC), 'Bulk/boundary correspondence in SPT phases'
Many of (but not all of) interesting physical (in particular topological) properties of topological phases and symmetry protected topological phases can be "inferred" from their boundary (end, edge, surface, ..) field theories. In

particular, the presence of quantum anomalies in boundary field theories (or lack thereof) gives a way to diagnose bulk topological properties. I will discuss such bulk/boundary correspondence in various examples in 2d and 3d.

25. Kirill Shtengel (UC Riverside), 'Quantum infidelity'
This talk issues related to quantum information in Majorana systems.

26. Dam Son (University of Chicago), 'Particle-hole symmetry and the nature of the composite fermion'
I will describe the new picture of the half-filled Landau level, according to which the composite fermion is a massless Dirac fermion in the limit of vanishing Landau-level mixing. Such a fermion is characterized by a Berry phase of pi around the Fermi disk. Physical consequences of the new picture are outlined.

27. Ady Stern (Weizmann), 'Current at a distance and resonant transparency in Weyl semi-metals'
This talk discussed a new way of detecting Weyl semi-metals using remarkably accessible transport experiments.

28. Hidenori Takagi (Max Planck), 'Exotic electronic states produced by strong spin-orbit coupling in complex Ir oxides'
In 5d Iridium oxides, a large spin-orbit coupling of ~ 0.5 eV, inherent to heavy 5d elements, is not small as compared with other relevant electronic parameters, including Coulomb U, transfer t and crystal field splitting D, which gives rise to a variety of exotic magnetic ground states. In the layered perovskite Sr2IrO4, spin-orbital Mott state with Jeff=1/2 is realized due to the novel interplay of those energy scales [1-3]. Despite the strong entanglement of spin and orbital degrees of freedom, Jeff=1/2 iso-spins in Sr2IrO4 was found to be surprisingly isotropic, very likely due to a super-exchange coupling through almost 180 degree Ir-O-Ir bonds [4]. The temperature dependence of in-plane magnetic correlation length of Jeff=1/2 iso-spins, obtained from inelastic x-ray resonant magnetic scattering, was indeed well described by that expected for two-dimensional S=1/2 Heisenberg antiferromagnet [5]. The three-dimensional analog of Sr2IrO4, SrIrO3 perovskite is very close a band insulator due to lattice distortion but a Dirac semimetal protected by crystalline symmetry [6]. Upon increasing effective Coulomb U, magnetism emerges and creates a gap at Dirac nodes, giving rise to a semimetal to magnetic insulator transition. This can be realized by controlling the dimensionality and hence the effective U in (SrIrO3)m/SrTiO3 (m: number of SrIrO3 layer) super-lattice structure [7]. With reducing m, a transition to an insulator, accompanied with magnetism was clearly observed. At m=1, single layer, the transport remains insulating even above the magnetic ordering temperature, indicative of the increased Mott character. When Jeff=1/2 iso-spins interact with each other through 90 degree Ir-O-Ir bonds, very anisotropic bond dependent ferromagnetic coupling is expected, unique to strong SOC system. Complex Ir oxides with honeycomb and more recently identified hyper-honeycomb lattices [8], where x-, y- and z- 90 degree Ir-O-Ir bonds are realized, may be candidates for quantum spin liquid expected for the Kitaev model. Very likely due to the superposition of additional magnetic couplings not included in the Kitaev model [9], in reality, a long range magnetic ordering emerges at low temperatures in those compounds. Hyper-honeycomb b-Li2IrO3, though eventually show a marginal ordering, appears to be located at the critical vicinity to the Kitaev spin liquid. With application of pressure of $>$ 2GPa, in fact, the long range ordering fades out and replaced by an inhomogeneous spin liquid-like state. 1) B. J. Kim et al., Phys. Rev. Lett. 101, 076402 (2008). 2) B. J. Kim et al., Science 323, 1329 (2009). 3) S. Fujiyama et al., Phys. Rev. Lett. 112, 016405 (2014). 4) G. Jackeli and G. Khaliullin, Phys. Rev. Lett. 102, 017205 (2009). 5) S. Fujiyama et al., Phys. Rev. Lett. 108, 247212 (2012). 6) Y. Chen et al. ,Nat. Commun. 6:6593 doi: 10.1038/ncomms7593 (2015). 7) J. Matsuno et al., Phys. Rev. Lett. 114 247209(2015). 8) T.Takayama, et al., s Phys. Rev. Lett.114, 077202 (2015) 9) A.Kitaev, Annals of Physics 312 2 (2006).

29. Ashvin Vishwanath (Berkeley), 'Particle-vortex duality of Dirac fermions: Linking topological insulators and superconductors to the half filled Landau level'
We derive a dual description of a 2+1 dimensional Dirac fermion, analogous to the duality of bosons to vortices. The dual theory is found to be QED_3, which is also composed of a single Dirac fermion, coupled to a gauge field. We connect the 3+1d topological insulator, which displays a Dirac surface state, and connect it to a topological superconductor, whose surface states supplies the dual description. All known surface phases of topological insulators are shown to emerge from the dual description. Remarkably, this has consequences for particle hole symmetric states in the half-filled Landau level. In particular, the composite Fermi liquid is shown to be closely related to the surface state of a topological insulator. An alternate to the Moore-Read Pfaffian, which respects particle hole symmetry, is also identified.

30. Amir Yacoby (Harvard), 'Controlled Finite Momentum Pairing and Spatially Varying Order Parameter in Proximitized HgCdTe Quantum Wells'

This talk reported on measurements in long HgTe Josephson junctions subjected to an in-plane magnetic field.

31. Norman Yao (Berkeley), ‘Quantum control in the many-body localized phase’

In thermal phases, the quantum coherence of individual degrees of freedom is rapidly lost to the environment. A number of recent works have postulated that the manybody localize phase may be promising for quantum information applications owing to the slow decay of local coherences. I will try to sharpen this intuition and describe what it means to locally ”manipulate” a many-body system. Working with a specific one dimensional model of interacting spins at infinite temperature, I will describe a protocol that allows one to encode quantum information, perform a universal set of gates, and readout the resulting state. Our protocol utilizes protected qubits that emerge at the boundary between a symmetry-protected topological and trivial phase.

Outcome of the Meeting

On the broadest level, one of our chief objectives was to provide a venue whereby diverse topics in strongly interacting topological phases could be seamlessly unified. For instance, we hoped to relate questions on the formal mathematical physics end to experiment and even applications. A related goal we hoped to achieve was to foster new interdisciplinary collaborations, which would undoubtedly produce exciting breakthroughs in the field.

We believe that our workshop succeeded in fulfilling these objectives. Participants included both experimentalists and theorists with expertise spanning a wide range of complementary areas. Ample informal discussion time allowed participants to engage in extensive dialogue concerning material presented during the program. Particularly spirited interactions revolved around issues concerning the half-filled Landau level, dualities for the topological insulator surface, Majorana platforms including semiconducting wires and quantum spin Hall materials, Weyl semimetals, classification of phases and defects, and spin liquids. We expect that the conference will both influence ongoing work (many talks at the conference presented unpublished results) and produce new collaborations.

Participants

Alicea, Jason (California Institute of Technology)
Barkeshli, Maissam (Microsoft Station Q)
Berg, Erez (Weizmann Institute)
Bernevig, Andrei (Princeton University)
Bonderson, Parsa (Station Q, Microsoft Research)
Burnell, Fiona (University of Minnesota)
Cano, Jennifer (Princeton)
Chen, Yong (Purdue University)
Dam Thanh , Son (University of Chicago)
Fendley, Paul (Oxford)
Fidkowski, Lukasz (University of Washington)
Fisher, Matthew (University of California - Santa Barbara)
Folk, Joshua (University of British Columbia)
Franz, Marcel (University of British Columbia)
Fu, Liang (Massachusetts Institute of Technology)
Gorshkov, Alexey (Joint Quantum Institute)
Hermele, Michael (University of Colorado Boulder)
Hughes, Taylor (University of Illinois)
Kim, Yong Baek (University of Toronto)
Levin, Michael (University of Chicago)
Lindner, Nate (Technion)
Lu, Yuan-Ming (Ohio State University)
Lutchyn, Roman (Microsoft Station Q)
Marcus, Charles (University of Copenhagen)
Metlitski, Max (Kavli Institute for Theoretical Physics)
Meyer, Julia (CEA Grenoble)
Mong, Roger (University of Pittsburgh)

Nadj-perge, Stevan (Caltech)
Oshikawa, Masaki (University of Tokyo)
Pikulin, Dmitry (University of British Columbia)
Ryu, Shinsei (University of Illinois at Urbana)
Shtengel, Kirill (University of California - Riverside)
Stern, Ady (Weizmann Institute)
Takagi, Hidenori (Max Planck Institute for Solid State Research)
Todadri, Senthil (Massachusetts Institute of Technology)
Vishwanath, Ashvin (Harvard University)
Wen, Xiao-Gang (Massachusetts Institute of Technology/Perimeter Institute)
Yacoby, Amir (Harvard)
Yao, Norman (UC Berkeley)

Chapter 23

Approximation of High-Dimensional Numerical Problems - Algorithms, Analysis and Applications (15w5047)

September 27 - October 2, 2015

Organizer(s): Christiane Lemieux (University of Waterloo), Ian Sloan (University of New South Wales), Henryk Woźniakowski (University of Warsaw and Columbia University)

Overview of the Field

High-dimensional problems occur in many scientific disciplines. They arise for various reasons, including the analysis of data in which each sample is defined by the measurement of hundreds of different quantities or the use of models that are based on a large number of variables. This can occur, for example, because of the nature of the system under study, or because the process of interest is observed over a period of time that is discretised over a large number of steps. Questions that need to be answered about such problems often amount to integrating a function or finding a representation for it that is amenable to further investigation. For example, determining the expectation of a complicated function over the system under study can be formulated as a multivariate integration problem, while determining a probability distribution for such a function or solving a system of PDEs would fall under the umbrella of function approximation.

When dealing with high-dimensional problems, it is rarely possible to obtain exact solutions. One must then resort to numerical methods. For high dimensional problems traditional numerical methods are often not feasible. In fact, for many such problems we know that ALL methods suffer from the curse of dimensionality. This can be proved by establishing lower bounds on the number of function evaluations, or more generally on the number of information operations that are necessary to obtain a small error. These bounds are often exponentially large as the dimension increases. This is the subject of information-based complexity [TWW88]. Complexity of high-dimensional continuous problems is nowadays a very popular research subject, and many people have contributed to this area. This sub-area of information-based complexity is called tractability of multivariate problems [NW08-12]. Typically, if we consider a high-dimensional problem defined over a space of functions for which each variable and all groups of variables are equally important then the curse of dimensionally occurs. Luckily, for most high-dimensional problems, not all variables and groups of variables are equally important.

Another interesting area giving rise to high-dimensional problems is the study of PDEs with random fields in their coefficients (typified by the flow of oil or water through a model material whose permeability is a random field) that are parametrised in a space of moderate to high dimension (under names such as polynomial chaos, stochastic Galerkin and stochastic collocation). Such applications present great challenges to the existing high-dimensional theories and also include infinite-dimensional integration problems. The study of integration and approximation problems where there is no a priori bound on the dimension has been a field of active research over the last few years [HW01, Gne10, KSWW10, PW11].

The main goal of tractability study is to understand which high-dimensional problems do not suffer from the curse of dimensionality, or more generally, what properties of high-dimensional problems make them tractable. For tractable problems we then want to find implementable methods whose cost is at most polynomial in the dimension and in the error threshold. Monte Carlo (MC) and quasi-Monte Carlo (QMC) based methods are often used for that purpose, and discussing the most recent advances on these methods for high-dimensional problems was a main focus of this workshop.

Main Topics Covered in the Workshop

The workshop was organized into four main themes arranged by day in terms of the actual scheduling of the workshop. Below, we discuss these themes and summarize the main results that were presented.

Adaptive and Bayesian-related methods

Multilevel Monte Carlo methods have become increasingly popular as a tool to improve the efficiency of MC methods [1]. It was enlightening to start the workshop with Mike Giles' talk discussing ideas (something that has not been tested yet), tricks (something that worked once) and techniques (something that worked more than once) to apply these methods successfully on a range of problems, and find ways of combining them with other well-known approaches for variance reduction.

While QMC methods are known to provide approximations with error that converges to zero faster than MC methods, there has not been so much work (until recently) on developing methods providing problem-specific information on how many sample points should be used to achieve a certain level of error. The work presented by Lluís Antoni Jiménez Rugama discussed recent methods and algorithms designed by himself and his co-authors Fred Hickernell and Da Li to do exactly that. An important advantage of their approach is that it doesn't require the user to have a lot of information on the function to integrate, e.g., on the decay rate of its Fourier coefficients or on the weights defining the underlying function spaces.

Using QMC methods to search for an optimal solution has not received a huge amount of attention in the last few years compared to other type of problems, so it was very interesting to see that topic discussed in Mathieu Gerber's talk who presented both theoretical and numerical results on a new approach for this problem. Interesting discussions followed his presentation, in particular on the choice of which coordinate of the low-discrepancy point set should be used for the acceptance/rejection step required by the method.

The talk by Christoph Schwab discussed certain problems arising in uncertainty quantification. To perform estimation related to the parametric Bayesian posterior densities involved in such problems, it was first demonstrated that such densities belong to a class of weighted Bochner spaces of functions of countably many variables, with a particular structure of the QMC quadrature weights. Based on that, error bounds for higher order QMC quadratures were obtained. Numerical results showing the performance of hybridized versions of the fast component-by-component construction for QMC point sets were shown on problems with a dimension up to 10^4.

Markus Weimar presented some new results on the use of adaptive wavelet Galerkin schemes to solve operator equations and discussed how they relate to their non-adaptive counterpart. This presentation led to interesting discussions among participants on the nature of tractability and how it relates to actual algorithms to address the problem under study.

Tractability and complexity of integration and approximation

As mentioned above, the research areas of tractability and complexity are tackling the important problem of determining which type of (high-dimensional) problems can be tackled with d-dimensional integration algorithms (quadrature rules) without suffering from the curse of dimensionality. In his talk, Erich Novak discussed a so-called universal algorithm for integration making use of a well-chosen $d \times d$ matrix satisfying the so-called Frolov property, a well-chosen transformation on the d-dimensional space, and then also explained how to randomize this construction. Error bounds for this algorithm were also provided.

The presentation by Greg Wasilkowski also discussed algorithms for integration in high-dimensional spaces but using a very different approach based on the idea of setting the "unimportant" variables to 0 instead of evaluating them at a value prescribed by the cubature rule. In this way, we can work with only the k most important variables, which reduce the computational cost. Of course, one needs to know when it is reasonable to proceed in this way, which is what the talk was about, making use of γ-weighted anchored space to do so and providing error bounds to motivate the approach. The form of the error bound also enables the determination of appropriate values for k leading to a desired error-threshold, assuming the weights defining the function space are known. Further results were given about the equivalence between this type of space and ANOVA-type spaces, with also an indication of whether or not the proposed approach can also apply to the latter spaces.

Moving away from integration, the talk by Michael Griebel was instead on the topic of tensor-product approximations of analytic functions, which appear in diverse problems arising from uncertainty quantification, econometrics and physics. More precisely, the approximations studied in this talk were obtained by truncating infinite series representation based on tensor product of univariate basis terms. Working with weighted analytic functions, error bounds can be obtained for these approximations, which in turn can be represented using certain exponential sums. The infinite-dimensional case was also discussed, in addition to the use of Leja points to handle interpolation and quadrature in such spaces.

The problem of multivariate L^p approximation of Hölder classes in the presence of Gaussian noise was discussed in Leszek Plaskota's talk. Here, the idea is to construct approximations for a d-dimensional function f based on information given by a vector of the form $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, where $y_j = f(\mathbf{x}_j) + e_j$ for $1 \leq j \leq n$, and where the e_j's are independent Gaussian random variables with mean 0 and variance $\sigma_j^2 > 0$. The error of approximation φ is defined as the supremum of the L_p norm of the expected error $f - \varphi(\mathbf{y})$ over all functions in an Hölder class $H^d_{r,\varrho}$. Two main results were discussed for this problem: first, if arbitrarily many observations N with arbitrary variances σ_j^2 satisfying $\sum_j \sigma_j^{-2} \leq N$ are allowed, then the minimal error is of order $N^{-(r+\varrho)/(2(r+\varrho)+d)}$. Second, if only n observations with fixed variances $\sigma^2 > 0$ are allowed then the minimal error is of order $\max\{(\sigma^2/n)^{(r+\rho)/(2(r+\rho)+d)},\ n^{-(r+\rho)/d}\}$. These results thus give insight on the observation-gathering model used and its effect on the convergence of the error of the corresponding approximation.

In Josef Dick's talk, the topic of integration for functionals of the form $\int f(yA)dy$ was discussed, where y is a d-dimensional vector and A is a $d \times d$ matrix. The problem of interest here is to find good quadrature points for which the product YA can be computed efficiently and also yields a small integration error, where Y is an $n \times d$ matrix containing on each row a d-dimensional quadrature point. He then established that lattice point sets (polynomial or not) of a certain form can provide quadrature points with those properties. Examples drawn from calculations related to the multivariate normal were given, in addition to problems arising from PDE with random coefficients. An interesting open question mentioned at the end of the talk was to ask whether or not it is possible to obtain the same efficiency gain with randomized constructions.

The final talk in this session was discussing the curse of dimensionality for smooth functions. After going over the precise mathematical definition of this concept (curse of dimensionality), Aicke Hinrichs reviewed known results providing necessary and sufficient conditions for this problem to happen when integrating smooth functions, which were expressed in terms of the bounds involved in the Lipschitz properties of the class of functions considered. A generalization of this result was then presented by making use of recent deviation inequalities obtained by O. Guedon and E. Milman, thus closing some gaps in previous work.

High-dimensional problems and methods for SDEs and PDEs

Stefan Heinrich discussed the problem of finding a strong solution to scalar stochastic differential equations depending on a parameter, where the goal is to find numerical approximations for all parameter values simultaneously. To do so, suitable convergence results for the Banach space valued Euler-Maruyama scheme in spaces of martingale type 2 were first given. Then a multilevel scheme involving two embedded Banach spaces was presented to construct an approximation. Finally, he explained that the parametric problem can be cast into this embedded Banach space setup, from which a multilevel method for the strong solution of parametric stochastic differential equations results. The talk included the presentation of convergence rates for classes of input functions with different levels of smoothness. Furthermore, the rates were shown to be optimal by proving matching lower bounds.

The topic of parameterized PDE models was discussed in Clayton Webster's presentation. After reviewing some possible numerical strategies to find solutions for such problems, results for best s-term and quasi-optimal approximations were given, where the idea is to build polynomial approximations for these solutions by appropriately choosing an index set determining the polynomials that will enter the approximation. The talk then went over two main topics: the first one being to present a general methodology for analyzing the convergence of best s-term and quasi-optimal Taylor and Legendre approximations, and the second one being to develop alternative approximation strategies to recover these results, in particular using compressed sensing and weighted ℓ_1-minimization.

The last talk of the day by Paweł Przybyłowicz was on the problem of finding global approximations for stochastic differential equations of the form $dX(t) = a(t, X(t))dt + c(t)dN(t), t \in [0, T]$ where $N = \{N(t)\}_{t\in[0,T]}$

is a non-homogeneous Poisson process with an intensity function $\lambda = \lambda(t) > 0$. The idea is to consider arbitrary algorithms based on a finite number of observations of the Poisson process, and some of the results presented in this talk were describing the exact convergence rate of the minimal errors that can be achieved by this type of method. Two different classes of methods were studied, using equidistant and nonequidistant sampling for the process N. It was shown that methods based on nonequidistant meshes are more efficient than those based on the equidistant sampling. Methods based on a regular sequences of discretizations and algorithms that use adaptive step-size control were also discussed.

Construction of high-dimensional point sets and sequences

The last day started with Art Owen's talk on the construction of low-discrepancy point sets and sequences over spaces other than the usual unit hypercube. He described a generalization of the van der Corput sequence from the unit interval to general volumes, using the triangle as an example. From there, he showed how higher-dimensional constructions can be obtained by taking Cartesian products, and provided discrepancy bounds for the corresponding sequences. The talk also discussed the generalization of scrambled nets to such spaces and provided results for the variance of the correponding estimators, assuming a certain smoothness for the integrand.

A construction that shares properties with scrambled nets but is not formally within the realm of low-discrepancy point sets is Latin Hypercube Sampling. In Natalie Packham's presentation we heard about a modification to this sampling mechanism whereby vectors of random variables that have a certain dependence structure can be sampled. The properties of the corresponding estimator were derived, among other things allowing for the construction of confidence intervals for the quantities of interest. Numerical comparisons between this approach and QMC-based ones were provided on an example from finance.

The van der Corput sequence and its multidimensional extension to Halton sequences were the main topic discussed in Roswitha Hofer's presentation. Different ways of extending the definition of these sequences were reviewed. The first reviewed extensions were those based on the use of a different mapping to be applied within the reverse-digital expansion; the second ones were those based on varying the numeration system, for example via the use of Cantor expansions, as has been done by other authors. A new idea in that direction was presented, making use of formal u/v-adic expansions, and for this extension a discrepancy bound was provided. Another new class of extensions was discussed, using the idea of replacing the ring of integers by a ring of polynomials over a finite field. In this new representation, an interesting question is that of choosing appropriate polynomials $u_j(X), v_j(X)$ for each dimension $j = 1, \ldots, d$ that are required to define this type of sequence. Discrepancy bounds were given in that context as well.

We had another talk on Halton sequences, although going in a quite different direction where the goal is to study a randomized version of these sequences based on a $p-$adic shift, and then studying the worst mean-square error over a weighted anchored Sobolev space of functions. Different bounds on this error were then established, using different assumptions on the behaviour of the weights in the function space. These results imply, among other things, that there exist shifts for which the corresponding shifted Halton sequence has an optimal convergence rate for its error. Based on this, a constructive method to find these shifts based on a component-by-component approach was presented.

The presentation by Daniel Rudolf was not directly connected to the construction of low-discrepancy sequences, but rather to the concept of discrepancy itself (and the related concept of dispersion) and more specifically to the problem of finding the largest empty box amidst a point set. An important issue is to determine how this quantity depends on the dimension of the space and the number of points in the point set. After reviewing known results on this question, a new result improving the known lower bound was presented, with an insightful presentation of the main steps of the proof for this result.

The talk by Friedrich Pillichshammer was focusing on point sets and sequences whose L_p-discrepancy achieve the optimal rate of convergence. After reviewing known results on lower bounds for this type of discrepancy, the case of symmetrized van der Corput sequences in one dimension was examined. Results showing the existence (and the fact that a construction can be found) for sequences in d dimensions achieving the optimal rate of $(\log N)^{d/2}$ were presented as well. He then discussed the L_p discrepancy of order 2 digital (t, d)-sequences over $\mathbb{F}_2$, showing they could achieve the optimal rate as well (up to a constant depending on t). Finally, results extending the ones presented for L_p-norms to other norms such as the L_∞ norm were presented .

Outcome of the Meeting

The meeting was very much seen as a success by everyone who attended, with many participants expressing the desire that another BIRS workshop be organized in the future to enable the type of information-sharing and enlightening discussions that took place at this workshop, also commenting on the beautiful and inspiring venue. The PhD students and postdoctoral researchers who have attended felt quite privileged to have had the chance to attend the meeting, present their work (in some cases) and have a chance to discuss informally their work with experts in their research area. Several small groups of researchers who regularly work together took the opportunity enabled by the not-so-packed schedule to advance their common projects, while new collaborations were formed among participants. The latter were enabled by the fact that as organizers, we made sure to invite a large enough number of people who are outside the more common circles attending other workshops and conferences in our research area. In conclusion, we certainly feel that the workshop produced the results that we wanted and are very happy with the outcome.

Participants

Dick, Josef (University of New South Wales)
Faure, Henri (Institut de Mathmatiques de Marseille)
Gerber, Mathieu (Harvard University)
Gilbert, Alexander (University of New South Wales)
Giles, Mike (University of Oxford)
Griebel, Michael (Universitaet Bonn)
Heinrich, Stefan (University of Kaiserslautern)
Hickernell, Fred J. (Illinois Institute of Technology)
Hinrichs, Aicke (Johannes Kepler UniversitŁt Linz)
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Chapter 24

The Geometry, Algebra and Analysis of Algebraic Numbers (15w5054)

Oct 4 - 9, 2015

Organizer(s): Francesco Amoroso (Caen), Igor Pritsker (Oklahoma State), Christopher Smyth (Edinburgh), Jeffrey Vaaler (UT Austin)

The participants gathered in Banff on 4 October 2015, smiled upon mild, sunny weather, with clear skies giving almost uninterrupted views of the spectacular surrounding mountains.

They came to the workshop from a variety of mathematical backgrounds: most were number-theorists of various kinds: algebraic, transcendental, analytic, computational, ... The common thread among them was, of course, an interest in algebraic numbers.

Overview of the Field and Recent Developments

Algebraic numbers are fundamentally arithmetic objects, a property coming from the integer coefficients of their minimal polynomials. However, their study can involve the deployment of a surprisingly diverse arsenal of mathematical techniques. As well as classical algebra (fields, Galois theory, ...), these include analytic methods [2, 9], algebraic number theory [17, 19], combinatorial methods [29], methods from algebraic and toric geometry [5], potential theory [33, 35], dynamical systems [21, 41]. Classical geometric methods are used, for instance, for studying the position of algebraic numbers in the complex plane [9]. Connections with von Neumann algebras have been revealed, leading to formulations of natural non-commutative analogues of some classical questions about algebraic numbers. Algebraic numbers coming from restricted classes arise naturally in the study of hyperbolic manifolds.

Mahler measure and heights. The workshop capitalized on recent progress and stimulated further development in this area, where the central problem is the 80-year old Lehmer conjecture [15, 42], on the smallest Mahler measure of a nonzero noncyclotomic algebraic integer. Thus, while this fundamental problem is old and difficult, there has been steady progress towards its positive solution in recent years. Some important advances include its proofs for polynomials having a bounded number of monomials [16], with odd coefficients [11] and for algebraic numbers that generate Galois extensions of the rationals [5], generalizations of Dobrowolski-type bounds for multiplicatively independent algebraic numbers [5] and for multivariate polynomials [4], and an absolute lower bound for the height in abelian extensions [6]. The polynomials of smallest Mahler measure coming from integer symmetric matrices have also been found [31]. There are many rapidly developing related directions such as computing explicit values of the Mahler measure, its connection with algebraic geometry and dynamics [41], counting results [16], infinite fields with the Bogomolov [10] and the Northcott property [20], etc.

The legacy of Schur and Siegel. In is of great interest to highlight the inheritance of Schur and Siegel in this area. Their achievements in other areas are so substantial that these have perhaps not received enough attention up to now. One important topic is the Schur-Siegel-Smyth trace problem on the smallest limit point for arithmetic means for totally real and positive algebraic integers [39, 40, 43, 1]. Another is the application of the Schur theory of Hardy functions to the study of Pisot numbers [14], and, more generally, to the theory of nonreciprocal polynomials. The distribution of sets of conjugates algebraic numbers in the complex plane, whose study began with Kronecker [8], and was continued by Schur [39] and Fekete [23], is still far from being well understood. A notable exception is the area of asymptotic equidistribution for algebraic numbers of small height (or small Mahler measure), which received substantial attention in recent years (see, e.g., [8, 44, 22, 45]). The ideas of equidistribution were successfully applied in many problems, including the Schur-Siegel-Smyth trace problem, cf. [35].

Relations between conjugate algebraic numbers. The problem of how conjugate algebraic numbers may be connected algebraically is poorly understood – the difficulty of Lehmer's problem being a consequence of this. A related problem is to find strong lower bounds for discriminants of algebraic numbers, current known exponential lower bounds all coming from consideration of field discriminants. A breakthrough on this would have consequences for Lehmer's problem. Other connections between conjugates have been studied in [7, 17].

Integer Chebyshev problem and other extremal problems for integer polynomials. The problems of minimizing norms by polynomials with integer coefficients date back to at least the work of Hilbert of 1894 [24]. They were developed by Fekete and many others [3, 12, 34], but the integer Chebyshev problem remains open even in its classical setting. It has intimate connections with the distribution of conjugate algebraic integers, and with the Schur-Siegel-Smyth trace problem.

Applications of functional analysis. Recent work has lead to new theorems about the Weil height of algebraic numbers using techniques from functional analysis. In some cases the results can be stated in the classical language of algebraic number theory, but the proofs use a special Schauder basis for the Banach spaces identified in [2]. At present proofs using classical methods of algebraic number theory are not known, but are highly desirable to develop.

Open Problems

The workshop incorporated three problem sessions where many participants submitted new open problems and exchanged ideas on the well known ones. A complete collection of discussed open problems is attached to this report due to its size. We state only several central problems below.

1. **Lehmer's problem.** Is there an absolute constant $C > 1$ such that, for any nonzero algebraic integer α, the product of the moduli of those conjugates of α (including α itself) that are of modulus at least 1 is either 1 or at least C.

   The example of α being a zero of "Lehmer's polynomial"

   $$z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$$

   shows that such a C, if it exists, is at most $1.176\ldots$. An optimistic form of this problem, called 'Lehmer's conjecture (strong form)', says that this constant C does exist, and equals $1.176\ldots$.

   For references to Lehmer's problem see for instance [42].

2. **The Schur-Siegel-Smyth trace problem.** For which real numbers r are there only a finite number of totally real and positive algebraic integers β such that $\mathrm{Trace}(\beta)/\deg(\beta)$ is less than r?

   It is known that this is true for all $r < 1.79193$ – see Liang and Wu [27] – but false for $r \geq 2$. An obvious conjecture is that it is true for all $r < 2$, but doubt has been thrown on this by the result of Smyth, improved by Serre (see Appendix to Aguirre and Peral [1]), which shows that the 'auxiliary function' method used to obtain results of this type cannot produce bounds greater than 1.8983021.

   For a survey of the trace problem see [1].

3. **Algebraic integers contained in an interval of length** 4. Pólya proved that any interval $[a,b] \subset \mathbb{R}$ of length $b-a<4$ contains only finitely many complete sets of conjugate algebraic integers, see [39, p. 391]. If $b-a>4$ then each $[a,b]$ contains infinitely many complete sets of conjugate algebraic integers by the result of Robinson [37]. It is well known that intervals of the form $[a,a+4],\ a \in \mathbb{Z}$, also contain infinitely many such sets. For example, one can consider the roots of polynomials $2T_n(x/2)$ contained in $[-2,2]$ for all $n \in \mathbb{N}$, where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial.

   Problem. Characterize all $a \in \mathbb{R}$ such that each interval $[a,a+4]$ contains infinitely many complete sets of conjugate algebraic integers.

   A survey of related problems with further references can be found in McKee [30].

4. **The conjecture of Schinzel and Zassenhaus.** For a nonzero algebraic integer α, of degree d say, $\mathrm{house}(\alpha)$ is the maximum modulus of $|\alpha|$ and any of its conjugates. The Schinzel-Zassenhaus conjecture [38] asserts that there is a constant $c>1$ such that $(\mathrm{house}(\alpha))^d$ is either 1 or at least c.

   If the conjecture is true, there would, for fixed d, be a constant $c_d \geq c$ such that for all α of degree d then $(\mathrm{house}(\alpha))^d$ is either 1 or at least c_d. Boyd [15] has conjectured that for d divisible by 3 the largest value for c_d is $c_d = \theta_0$, the real zero of $z^3 - z - 1$, attained for $d = 3k$ and α a zero of $z^{3k} + z^{2k} - 1$. Computation by Boyd [15] and also by Rhin and Wu [36] provide evidence for this strong conjecture, and also suggest possible values for c_d for other values of d.

   The strongest result to date in the direction of the conjecture is due to Dubickas [18], who proved that

$$\mathrm{house}(\alpha)\ > 1 + (\tfrac{64}{\pi^2} - \varepsilon)(\log\log d/\log d)^3/d \quad \text{for } d > d_0(\varepsilon).$$

   The Schinzel-Zassenhaus conjecture is readily seen to be a consequence of 'Lehmer's conjecture' – see [42].

5. **The integer Chebyshev problem.** For a real interval I, its Chebyshev constant given by

$$t(I) = \inf_P \max_{x \in I} |P(x)|^{1/\deg P},$$

   where the infimum is taken over all non-constant monic polynomials in $\mathbb{C}[x]$. The *integer Chebyshev constant* $t_{\mathbb{Z}}(I)$ has a similar definition, except that now the infimum is taken over all not identically zero (not necessarily monic) polynomials in $\mathbb{Z}[x]$. It is known from the results of Fekete [23] that

$$\frac{|I|}{4} \leq t_{\mathbb{Z}}(I) \leq \frac{\sqrt{|I|}}{2},$$

   where $|I|$ is the length of I. It is also known that $t_{\mathbb{Z}}(I) = 1$ when $|I| \geq 4$. However, no exact value for $t_{\mathbb{Z}}(I)$ is known for any interval of length between 0 and 4. Can you find $t_{\mathbb{Z}}(I)$ for any interval I with $|I| < 4$?

   For more information and references see for instance Amoroso [3], P. Borwein and Erdélyi [12] and Pritsker [34]. There is also a version of the problem where the polynomials over which the infimum is taken *are* restricted to be monic, as well as having integer coefficients – see P. Borwein, Pinner and Pritsker [13].

Presentation Highlights

Shabnam Akhtari, University of Oregon

Akhtari is one of a small group of number theorists working on the difficult problems related to the Thue Diophantine equation. Let $F(x,y)$ be an absolutely irreducible, homogeneous polynomial with integer coefficients, and degree greater than or equal to three. A basic problem in number theory is to describe the set of integer solutions to the Diophantine equation $F(x,y) = m$ for nonzero integers m. It is know from the work of Thue that for each m such an equation has only finitely many solutions. Current research is directed at estimating the number of solutions and the height of solutions. Akhtari gave a general overview of such questions, and reported on her recent results that bound the number of solutions to the special Thue equations $ax^n - by^n = c$. This work is closely related to the problem of finding effective measures of irrationality for the algebraic numbers that are roots of the equation $F(x,1) = 0$.

Fabrizio Barroero, Scuola Normale Superiore di Pisa

The topic of counting algebraic integers as well as integer polynomials of bounded height is steadily gaining popularity. One of the pioneering results in this direction was obtained by Chern and Vaaler, who proved an asymptotic formula for the number of polynomials with bounded Mahler measure. It is possible to recast their formula as an asymptotic for the number of algebraic integers of bounded degree and height. A generalization of this asymptotic result was proved by Widmer in a multiterm form. Barroero used similar techniques to count monic polynomials by counting lattice points in a more general setting of polynomials over arbitrary number field of fixed degree. This requires to use a recent result of Barroero and Widmer on counting points of an arbitrary lattice in definable sets in an o-minimal structure. The main term of the asymptotic result comes from the volume of a certain bounded domain defined by a generalization of the Mahler measure.

Yann Bugeaud, Université de Strasbourg

In his talk "On the approximation of transcendental numbers by algebraic numbers of bounded degre", Yann Bugeaud has surveyed recent and less recent results closely related to a long-standing conjecture formulated by Wirsing in 1961 on approximation of real numbers by algebraic numbers of bounded degree. This subject is somehow related to Dujella's talk on root separation of integer polynomials.

Emanuel Carneiro, Instituto nacional de Matematica Pure e Aplicada

Let $\xi_1, \xi_2, \ldots, \xi_N$ be distinct real numbers, let $\delta_m \le \min\{|\xi_m - \xi_n| : n = 1, 2, \ldots, N$, and $n \neq m\}$ for each $m = 1, 2, \ldots, N$, and let $\Delta = \min\{\delta_1, \delta_2, \ldots, \delta_N\} \le \min\{|\xi_m - x_n| : m \neq n\}$. Then a generalization of Hilbert's inequality proved by H. L. Montgomery and R. C. Vaughan, asserts that

$$\left|\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{a_m \overline{a_n}}{\xi_m - \xi_n}\right| \le \pi \Delta^{-1} \sum_{n=1}^{N} |a_n|^2,$$

for all complex numbers $a_1, a_2, \ldots a_N$. Montgomery and Vaughan also proved a weighted version of the form

$$\left|\sum_{m=1}^{N} \sum_{\substack{n=1 \\ n\neq m}}^{N} \frac{a_m \overline{a_n}}{\xi_m - \xi_n}\right| \le \frac{3\pi}{2} \sum_{n=1}^{N} \delta_n^{-1} |a_n|^2.$$

Inequalities of this sort originated in attempts to prove sharp forms of the large sieve inequality. Since their work it has long been a problem to prove the weighted version with the constant $3\pi/2$ replaced by the constant π, which is known to be best possible. Carneiro described new joint work with F. Littmann which provides a new and very original proof of the second inequality, but with the slightly inferior constant 2π. This is of interest because the Carneiro/Littmann method uses a new type of extremal function related–but not seen before–to the theory of Beurling-Selberg extremal functions. Clearly the Carneiro/Littmann extremal function is not quite the right one; hence the inferior constant. But it is significant progress that the weighted Hilbert inequality is now seen to be connected to the theory Beurling-Selberg extremal functions. It is to be hoped that this will lead to a new extremal function providing a new proof with the conjectured constant π.

Stephen Choi, Simon Fraser University

Several problems of analytic number theory arise from the study of Littlewood polynomials. These are polynomials with all coefficients equal to 1 or -1. There name is attached to questions raised by Littlewood concerning the size of such polynomials on the unit circle. By Parseval's identity a Littlewood polynomial $F(x)$ of degree N satisfies

$$\left(\int_0^1 \left|F\left(e^{2\pi i x}\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} = \sqrt{N+1}.$$

Littlewood asked if there exist such a polynomial with supremum norm on the circle not much larger than $\sqrt{N+1}$. Stephen Choi is an expert on such questions, and gave a general overview of recent research on bounds for the L^1 norm of Littlewood polynomials.

Paulius Drungilas, Vilnius University

Drungilas discussed a kind of abc problem for algebraic numbers: for which triples (a, b, c) of positive integers do there exist algebraic numbers α, β and γ of degrees a, b, c respectively with $\alpha + \beta + \gamma = 0$? Such triples are called *sum-feasible*. Building on earlier work, he finds, in joint work with Dubickas and Luca, all such triples with $a \leq b \leq c \leq 7$. He also discussed the multiplicative analogue ('product-feasible' triples) where, in the definition, $\alpha + \beta + \gamma = 0$ is replaced by $\alpha\beta\gamma = 1$. He showed with Dubickas that every sum-feasible triple is also product-feasible, but that the two notions are not equivalent since, for instance, the triple $(2, 3, 3)$ is product- but not sum-feasible.

Arturas Dubickas, Vilnius University

In his talk "Counting dominant and degenerate polynomials," Arturas Dubickas presented a wide range of estimates on the number of polynomials with integer coefficients of fixed degree and bounded height that have certain special properties. In particular, he considered dominant polynomials (those that have one root whose modulus is strictly greater than the moduli of the remaining roots) and degenerate polynomials (that have a pair of distinct roots whose quotient is a root of unity). One of the main results shows that asymptotically all monic integer polynomials are dominant, meaning that the proportion of dominant polynomials among all integer polynomials tends to 1 as their heights increase to infinity. If integer polynomials are not monic, then dominant polynomials represent a positive fraction that generally depends on their degree. The asymptotic value of this fraction is known only for some degrees, and in general represents an open problem. The motivation for this study arose from linear recurrence sequences, and all the results are joint with Min Sha (Sydney).

Finally, the speaker stated a series of asymptotic formulas for the number of reducible polynomials and irreducible polynomials in terms of their heights. The latter problem is an old one: in the monic case the corresponding formula was established by Chela in 1963.

Andrej Dujella, University of Zagreb

On the BIRS workshop, Andrej Dujella gave a talk titled "Root separation for reducible integer polynomials", reporting on a recent joint work with Bugeaud. He consider the question how close to each other can be two distinct roots of an integer polynomial $P(X)$ of degree d. He compare the distance between two distinct roots of $P(X)$ with its height $H(P)$, defined as the maximum of the absolute values of its coefficients. The first result in this direction in due to Mahler, who proved that the distance is $> c(d)H(P)^{-d+1}$, for an explicit constant $c(d)$, depending only on d. In the talk, he present some recent results in the opposite direction, obtained by constructing explicit parametric families of (monic) reducible polynomials having two roots very close to each other.

Tamas Erdelyi, Texas A&M University

The order of vanishing for integer polynomials at 1 is an old and well known problem. It was considered for many special classes such as Littlewood, Newman and other kinds of integer polynomials. For example, the difficult problem of Prouhet, Tarry, and Escott in Diophantine equations has an equivalent reformulation as a question of this kind.

Tamas Erdelyi used the Coppersmith-Rivlin type inequalities as his main analytic tool to give estimates for the order of vanishing at 1 for polynomials with restricted coefficients. In fact, he presented various generalizations of the original Coppersmith-Rivlin inequality (obtained in the case of ℓ_∞ norm) to the full range of ℓ_p norms.

Michael Filaseta, University of South Carolina

It is well known that any monic irreducible non-cyclotomic polynomial with integer coefficients must have a root outside the closed unit disk. Many papers are devoted to the estimates on how far from the unit disk such roots must lie, which is directly related to the famous Lehmer conjecture. In this talk, Michael Filaseta discussed analogous regions different from disks for which similar estimates hold. These regions are constructed in a very special way from cyclotomic polynomials.

Michael Filaseta gave a surprising application of such regions to the problem on irreducibility of polynomials with non-negative coefficients. A result of A. Cohn implies that if the coefficients of a polynomial $f(x)$ are non-negative and bounded by 9, and if $f(10)$ is a prime number, then f is irreducible. The new approach using constructed regions allows to replace 9 with the sharp bound 49598666989151226098104244512918 for the coefficients. Moreover, its gives sharp bounds on coefficients under the modified assumption that $f(b)$ is prime for other positive integers b.

Several questions were stated concerning roots in these regions that would lead to further developments in the applications to irreduciblity.

Paul Fili, Oklahoma State University

Zannier asked a question about finiteness of the set of parameters $c \in \mathbb{C}$ such that $z = 0, 1$ are both preperiodic under iteration of $f_c(z) = z^2 + c$. It is now customary to refer to such questions as problems on unlikely intersections in arithmetic dynamics. Baker and DeMarco were able to answer this question in the affirmative using equidistribution. Using the metric of mutual energy and some discrete approximation techniques, Paul Fili proved an effective degree bound for the algebraic integers c arising in the question of Zannier. The main idea behind the proof is similar to that of Baker and DeMarco. Namely, the probability measure equally supported on the Galois conjugates of such a number c would tend to the equilibrium measures of both Mandelbroit sets M_0 and M_1 as the degree of c grows, which is impossible as these equilibrium measures are distinct. Hence the degree of c must be bounded. The notion of distance between two (adelic) measures defined via the mutual energy metric, and the triangle inequality for this metric, allow to effectively estimate the distance between the mentioned measures. Together with discrete energy approximation techniques, these ideas give an effective degree bound for the set of parameters c in question.

Robert Grizzard, University of Wisconsin

Robert Grizzard gave a talk entitled "Remarks on diophantine approximation in the multiplicative group and generalized Lehmer problems," discussing the following ideas. The absolute logarithmic Weil height $h : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty)$ induces a norm on the $\mathbb{Q}$-vector space $\mathbb{G}_m(\overline{\mathbb{Q}})/\text{tors} = \overline{\mathbb{Q}}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ (which is written multiplicatively!), where the torsion subgroup in this case simply consists of roots of unity. If Γ is any subgroup of $\mathbb{G}_m(\overline{\mathbb{Q}})$, we define a height function relative to Γ, written $h_\Gamma : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty)$, as follows: $h_\Gamma(\alpha) = \inf_{\gamma\in\Gamma} h(\alpha/\gamma) = \inf_{\gamma\in\Gamma} h(\alpha\gamma)$. If we interpret $h(\alpha/\beta)$ as the distance between α and β, then $h_\Gamma(\alpha)$ is the distance from α to the subgroup Γ. We discuss the result, joint with J. D. Vaaler, that for any field $k \subseteq \overline{\mathbb{Q}}$, we have

$$w_k(\alpha) \geq h_{k^{\text{div}}}(\alpha) \geq \frac{1}{2} w_k(\alpha),$$

where $w_k(\alpha) = \max\left\{h(\sigma\alpha/\alpha) \mid \sigma \in \text{Gal}(\overline{\mathbb{Q}}/k)\right\}$, and $k^{\text{div}} = k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.

By combining the lower bound on $h_{k^{\text{div}}}$ with a Dobrowolski-type estimate of Amoroso and Zannier, one achieves a bound of the form

$$h_{k^{\text{div}}}(\alpha) \geq \frac{c}{[k^{ab}(\alpha) : k^{ab}]^{2+\varepsilon}},$$

when k is a number field. Using this, Grizzard explained a connection to the "degree one form" of the generalized Lehmer problems recently proposed by G. Rémond.

Adam Hughes, University of Texas at Austin

Let $\overline{\mathbb{Q}}^\times$ be the multiplicative group of nonzero algebraic numbers, let $\text{Tor}\big(\overline{\mathbb{Q}}^\times\big)$ denote its torsion subgroup, and write $\mathcal{G} = \overline{\mathbb{Q}}^\times/\text{Tor}\big(\overline{\mathbb{Q}}^\times\big)$ for the quotient group. It is known that $\mathcal{G}$ has the structure of a vector space over $\mathbb{Q}$ written multiplicatively. The map $\alpha \mapsto h(\alpha)$ is well defined on $\mathcal{G}$, where $h(\alpha)$ is the absolute logarithmic Weil height of a coset representative, because distinct representatives of each coset differ only by a root of unity. Moreover, elementary properties of the Weil height imply that the map $\alpha \mapsto h(\alpha)$ is a norm on the vector space $\mathcal{G}$ written multiplicatively. Hence the completion of this vector space is a real Banach space. In a recent paper Allcock and Vaaler [2] showed that this Banach space $\mathcal{X}$ is isomorphic to a co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$, where Y is the set of all places of $\overline{\mathbb{Q}}$ with an inverse limit topology, $\mathcal{B}$ is the σ-algebra of Borel subsets of Y, and λ is an invariant measure defined on $\mathcal{B}$ that is induced by the action of the absolute Galois group $\text{Aut}\big(\overline{\mathbb{Q}}/\mathbb{Q}\big)$. The isomorphism is given explicitly by a map $\alpha \mapsto f_\alpha(y)$, where f_α is a continuous function on Y with compact support. If $\mathcal{F} \subseteq \mathcal{X}$ is the collection of all continuous functions on Y with compact support that have the form $y \mapsto f_\alpha(y)$, then $\mathcal{F}$ is dense in $\mathcal{X}$. Hughes gave a report on several recently developed directions for future research in this setting. A basic discovery of Hughes is a Banach algebra structure in the space $L^1(Y, \mathcal{B}, \lambda)$. Possible implications of this discovery for classical problems about the Weil height of algebraic numbers were also described.

Jonas Jankauskas, Vilnius University Jankauskas described joint work with Dubickas where they completely solve the equations $\alpha_1 = \alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$ in conjugate algebraic numbers α_i of degree up to 8. The techniques used are diverse: they involve methods from linear algebra, Galois theory and some combinatorial arguments. They also solve these equations when the α_i are Pisot numbers and their conjugates.

Matilde Lalín, Université de Montréal

Let $F(x_1, x_2, \ldots, x_N)$ be a nonzero Laurent polynomial in N variables with complex coefficients. The Mahler measure of F is given by

$$M(F) = \exp\left\{ \int_{(\mathbb{R}/\mathbb{Z})^N} \log\big|F\big(e^{2\pi i x_1}, e^{2\pi i x_2}, \ldots, e^{2\pi i x_N}\big)\big| \,\mathrm{d}\boldsymbol{x} \right\}.$$

If $N = 1$ then Jensen's identity leads to an alternative expression for the value of Mahler's measure as a product over the roots of $F(z) = 0$ which occur outside the unit circle. If $N \geq 2$ no such simple formula is known, and the problem of evaluating $M(F)$ for Laurent polynomials with integer coefficients is much more difficult. It is known, from relatively few examples, to be connected to special values of certain L-functions. More explicit conjectures of Beilinson have guided much recent research in this field. Lalin described recent joint work with D. Samart and W. Zudilin which establishes an identity between the Mahler measure of the Laurent polynomial $x + x^{-1}y + y^{-1} + 3$ and a special value of an associated elliptic curve. This confirms a conjecture of David Boyd based on computer calculations of the two values to very high precision. Several further examples were described.

James McKee and Pavlo Yatsyna, University of London

McKee and Yatsyna discussed a 22-year-old conjecture of Estes and Guralnick to the effect that any monic integer polynomial of degree d without multiple roots will occur as the minimal polynomial of some integer symmetric matrix. This was known to be true only for $d \leq 4$. They show that the conjecture is false for all $d \geq 6$, leaving only $d = 5$ the only unsolved degree. Their idea was to find a lower d-dependent bound for the span of the roots of a minimal polynomial of degree d of an integer symmetric matrix, and then exhibit polynomials that did not satify these bounds, for all sufficiently large d. This disproved the conjecture for those d, and ad hoc methods further eliminated all lower degrees down to $d = 6$. Thus $d = 5$ is the only unsolved case – see problem list.

Michael Mossinghoff, Davidson College

Let $a_1, a_2, \ldots, a_N$, be a finite sequence of integers such that $a_n = \pm 1$ for each $n = 1, 2, \ldots N$. It will be convenient to set $a_n = 0$ if $N + 1 \leq n$. Such a finite sequence is called a *Barker sequence* if it satisfies the condition

$$\sum_{n=1}^{\infty} a_n a_{n+k} \in \{-1, 0, 1\}$$

for every positive integer k. The associated *Barker polynomial* is $P(z) = a_1 + a_2 z + \cdots + a_N z^{N-1}$. From the definition of a Barker sequence it follows that the absolute value of a Barker polynomial on the unit circle is relatively flat. Mossinghoff spoke about the relationship between Barker polynomials and several questions in combinatorics, number theory and analysis. The primary conjecture in this subject asserts that there are only finitely many distinct Barker sequences. Indeed, the longest known Barker sequences has length $N = 13$, and it may be true that no Barker sequence exists with $N > 13$. Mossinghoff described an extensive numerical search for long Barker sequences that produced the following result: if $N > 13$ is the length of a Barker sequence, then either $N = 3979201339721749133016171583224100$, or $N > 4(10)^{33}$. Several additional arithmetic restrictions were described that limit in various ways the possible values of $N > 13$, if N is the length of a Barker sequence.

Lukas Pottmeyer, Universitat Basel

In his very interesting (and beautifully presented) talk "On Narkiewicz's property (P)", Lukas Pottmeyer presented his recent proof of the Narkiewicz conjecture. We say that a field F has property (P) if and only if there is no infinite subset $X \subseteq F$, such that $f(X) = X$ for any polynomial $f \in F[x]$ with $\deg(f) \geq 2$. Let $\mathbb{Q}^{(d)}$ be the compositum of all number fields of degree $\leq d$. In 1963 Narkiewicz conjectured that $\mathbb{Q}^{(d)}$ has property (P) for all positive integer d. He proved his conjecture for $d = 1$. Later on, building on more general results, Bombieri and Zannier (2001) proved the case $d = 2$. Building on a equidistribution theorem for points of small height on the

v-adic Berkovich line $\mathbb{P}_v^{\mathrm{Berk}}$ (due to several authors) Pottmeyer proves that a Galois extension F/K with uniformly bounded local degree satisfies a Uniform Bogomolov Property (USB): *any* canonical height associated to a rational map in $F(x)$ of degree > 1 is $> c > 0$ outside the set of zero-height points. This is a deep generalization of one of the main result of Bombieri and Zannier (2001). He then remark that (USB) implies (P).

Igor Pritsker, Oklahoma State University

Pritsker's talk was devoted to the problems of Schur on the limit points for the arithmetic means of conjugate algebraic numbers contained in the closed unit disk and in the real line. There are many recent results on the equidistribution of algebraic numbers of small height according to certain equilibrium measures arising in potential theory. This equidistribution is expressed in terms of weak convergence for the counting measures of algebraic numbers to the corresponding equilibrium measure, so that means of algebraic numbers converge to the first moment (center of mass) of the limiting measure. It is well known that the limiting equilibrium measure for the unit disk is the normalized arclength on the unit circle, and its center of mass is at the origin, which answers a question of Schur about algebraic numbers in the unit disk. Moreover, such methods allow to make more precise quantitative statements about means by using certain discrepancy estimates.

Another application of this approach is the Schur-Siegel-Smyth problem on the smallest limit point of the mean trace of totally positive algebraic integers. It was shown that this smallest limit point is 2 for many classes of algebraic numbers equidistributed in subsets of the real line. This includes algebraic numbers whose minimal polynomials satisfy various extremal properties, as well as those arising in polynomial dynamics. Various generalizations of this problem to the means of powers of algebraic numbers were also presented.

Georges Rhin, Université de Lorraine

Rhin, in collaboration with El Otmani and Sac-Épée, developed an innovative method for finding totally positive algebraic integers of small mean trace. They applied it to find 41 totally positive algebraic integers of degree 17 and trace 31. One of these immediately gives a Salem number of degree 34 and trace -3, the smallest known degree for which a Salem of trace -3 has been found. The idea (e.g., for $d = 17$) is as follows: first find an upper bound (6.69 for $d = 17$) for the largest conjugate. This is done using a particularly effective version of the auxiliary function method due to Flammang. Then choose 16 numbers uniformly at random from $(0, 6.69)$, and search for monic integer polynomials $x^{17} - 31x^{16} + \dots$ that alternate in sign at these values. This is an integer programing problem which, if it is successful in finding such a polyomial that is also irreducible, produces the required algebraic integers. The method is repeated again and again with new random numbers until no new polynomials have been found for some time, making it likely that all polynomials sought have been found.

Robert Rumely, University of Georgia

This talk contained a rather comprehensive survey on the current state of knowledge regarding arithmetic applications of potential theory. This includes the Fekete theorem stating that any compact set of capacity less than one, stable under complex conjugation, contains only finitely many complete sets of conjugate algebraic integers. Another important result is the Fekete-Szegő theorem on existence of infinitely many complete sets of conjugate algebraic integers in every open neighbourhood of a stable compact set of capacity at least one. Robert Rumely also covered the Pólya-Carlson rationality criterion, Ferguson's theorem on functions taking algebraic integer values on a set, and results on approximation of functions by polynomials with integer coefficients. The second part of the talk was devoted to the far reaching generalizations of all of the above classical results, predominantly obtained by the speaker.

Charles Samuels, Christopher Newport University

Samuels described a surprisingly explicit method of computing the Metric Mahler measure of rational numbers p^a/q^b using the continued fraction expansion of $\log q/\log p$. The Metric Mahler measure, being an infimum evaluated over a potentially infinite set, is not easy to compute. Further, in joint work with Jankauskas, Samuels showed that the potentially infinite set can be replaced by a finite one. This set is, however, not easy to find explicitly. For these particular numbers it could in fact be found, and so the Metric Mahler Measure evaluated.

Andrzej Schinzel, Polish Academy of Sciences

By ternary linear recurrence over a number field K we mean a sequence u_n in K satisfying $u_n = a_1u_{n-1} + a_2u_{n-2} + a_3u_{n-3}$, where $a_1, a_2, a_3 \in K$. In his talk, Andrzej Schinzel explained a local-global principle connecting solubility of congruence $u_n \equiv 0 \pmod{P}$ for almost all prime ideals P of K and solubility of the equation $u_n = 0$.

More precisely, Schinzel showed that for every algebraic number field K and every ternary simple linear recurrence u_n in K with the companion polynomial $(z-1)(z-a_1)(z-a_2)$ where $a_1^2 = a_2^x$ for $x = 0, 1$ or 2, if the congruence $u_n \equiv 0 \pmod{P}$ is soluble for almost all, in the sense of density, prime ideals P of K, then the equation $u_n = 0$ is soluble in integers n. A similar result holds if $K = \mathbb{Q}$ and the companion polynomial satisfies $a_1^3 = a_2^x$ for $x = 0, 1$ or 2. In the opposite direction, Schinzel showed that there exist a real quadratic field K and a ternary simple linear recurrence u_n in K with the companion polynomial $(z-1)(z-a_1)(z-a_1^3)$ such that the congruence $u_n \equiv 0 \pmod{P}$ is soluble for all prime ideals P of K, but the equation $u_n = 0$ is not soluble in integers n. The open question is, whether every ternary simple linear recurrence in $\mathbb{Q}$ satisfies the local-global principle.

Christopher Sinclair, University of Oregon

The statistics of roots of a degree N polynomial chosen uniformly from the set of polynomials with Mahler measure bounded by 1 can be analyzed using machinery developed to study the eigenvalues of random matrices. The underlying reason for this is that the joint density of roots contains a Vandermonde term which arises from the Jacobian of the change of variables from roots to coefficients.

The line of research discussed in this talk started with the work of Chern and Vaaler, who computed the volume of the set of (coefficient vectors) degree N polynomials with Mahler measure less than or equal to 1 for both real and complex coefficients. This volume is the main term for an asymptotic estimate on the number of integer polynomials with bounded height. Sinclair later realized that these volume could be interpreted as either the Pfaffian (in the real case) or a determinant (in the complex case) of a Gram-like matrix. The similarity of the partition function calculation in Gaussian ensembles with the Pfaffian volume calculation of Chern and Vaaler, lead Sinclair and Yattselev to introduce the Mahler ensembles: random (real or complex) degree N polynomials chosen uniformly from the set with Mahler measure bounded by 1. Using random matrix machinery, Sinclair and Yattselev recently derived the scaled kernels that give detailed information about the root statistics of random polynomials from the bounded Mahler measure ensemble. They also produced delicate results on the limiting distribution densities of these roots in various sets.

Martin Widmer, University of London

A subset of the algebraic numbers is said to have the Nortcott Property (N) if all of its subsets of bounded height are finite. It is known since 1949 that all number fields have Property (N) but the first examples of infinite degree were given by Bombieri and Zannier in 2001. In particular, they showed that $\mathbb{Q}^{(2)}$, the composite field of all quadratic extensions, has Property (N). However, for $\mathbb{Q}^{(3)}$ there is not much evidence in neither direction. In his talk "Around the Property (N)" Widmer proposed a strategy to make some progress on this problem, by introducing a notion of size for composite fields of cubic extensions. The size is a real number between 0 and 1, and 1 is attained, e.g., for $\mathbb{Q}^{(3)}$. The work of Bombieri and Zannier shows the existence of a subfield of $\mathbb{Q}^{(3)}$ with Property (N) and size 1/2. Widmer uses a different approach (relying on his previous work) that establishes such a subfield with the slightly larger size 3/5.

Qiang Wu, Southwest University of China

Wu made an new contribution to the Schur-Siegel-Smyth trace problem by applying a variant of the auxiliary function method to find the four totally positive monic irreducible reciprocal integer polynomials of smallest mean trace. He used another variant of the method to find the reciprocal algebraic integer of degree d having smallest maximum modulus of its conjugates for every even d up to $d = 42$.

Scientific Progress Made

This meeting used an excellent opportunity to bring experts in many different areas together, and enabled them to learn from and build on each others' specialized knowledge. Thus, on the one hand the topic of the workshop was clearly focused, while on the other hand the diversity of participants' interests gave the workshop great potential for cross-fertilisation between different areas of mathematics.

Our workshop was the first event that Pottmeyer participated in, and was focused exactly on his mathematical interests. As a postdoc it is extremely important to make ones research public and to get in contact to many experts in ones field. He enjoyed every talk and could gather new input for further research projects. For instance, he benefited very much from a discussion with Fili and Hughes on the use of equidistribution theorems on the topic of unlikely intersections, a discussion with Widmer on fields with the so called Northcott property, and from a conversation with Schinzel. Some very interesting questions raised by the latter will surely occupy Pottmeyer thoughts for some time.

During the workshop Ranieri and Widmer had several discussion on the connection between Property (N) (a topic that Widmer have studied for a long time) and decidability problems in mathematical logic. This opens a new door and led to interesting new applications of some of Widmer's results and to interesting new questions that might be tackled by number theoretic techniques.

Grizzard was able to work on joint research projects with Vaaler and with Mossinghoff during the workshop. He also found the problem sessions to be very useful for thinking of new research ideas.

Amoroso had a discussion with Smyth after a problem session and was able to make a significant breakthrough towards accurately describing the set $\{h(\alpha)\}$ of Weil heights for $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois.

Carniero and Vaaler had several discussions on the interesting new extremal function that appeared in Carniero's lecture (describing joint work of Carniero and F. Littmann) on the weighted Hilbert inequality. They hope to continue their effort to establish the best constant in the weighted Hilbert inequality by using more refined functions from the general theory of Beurling-Selberg extremal functions.

Dubickas discussed with Pritsker several results related to the problems of Schur on algebraic numbers in various sets. In particular, Dubickas mentioned his earlier results on trace and heights of algebraic integers that are closely related to the questions presented by Pritsker at a problem session. Their communication continued via e-mail after the conference.

Other participants enjoyed very much the inspirational atmosphere of the workshop and interesting discussions with colleagues. Some of them let us explicitly know that they came to the most of the talks, listening with attention and interest to all the speakers.

Outcome of the Meeting

The aim of our conference was to preserve and enhance the positive momentum, and to create a strong foundation for further progress in this rich area. We believe that active discussions and emerging collaborations should lead to solutions of important problems. Another significant outcome of the meeting is the list of problems that will be circulated among the participants. The list of problems includes approximately thirty important directions for future research. These were discussed by participants during the problem sessions, and the problems have been somewhat expanded in the written versions. Besides the obviously useful conversations among participants, the extensive list of problems is tangible evidence of a successful meeting.

Participants

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Carneiro, Emanuel (Instituto Nacional de Matematica Pura e Aplicada)
Choi, Stephen (Simon Fraser University)
Coons, Michael (University of Newcastle)
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Dubickas, Arturas (Vilnius University)
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Fili, Paul (Oklahoma State University)
Greaves, Gary (Tohoku University)
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Pinner, Christopher (Kansas State University)
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Chapter 25

Multiscale Modeling of Cell Wall Mechanics and Growth in Walled Cells (15w5050)

October 18 - 23, 2015

Organizer(s): Anja Geitmann (Universit de Montréal), Kerwyn Huang (Stanford University)

The workshop was the first of its kind since it represented a venue that brought together people from very different backgrounds that typically know each other's publications but do not meet at scientific conferences as the respective fields are extremely diverse.

- Different scientific disciplines: mathematics, physics, engineering, biology

- Focus on biological organisms across kingdoms: bacteria, fungi, plants

Especially for the scientists working on bacteria, who essentially never attend meetings frequented by fungal or plant biologists, this meeting represented an opportunity to interact with a new set of colleagues with synergistic interests and expertise. Similarly, the representatives with mathematical and physical backgrounds rarely attend biological conferences and vice-versa. Collaborations typically take place at the individual level between single labs, whereas cross-pollination between highly different projects is rare. The most important comment by participants was therefore the appreciation for this opportunity to finally meet people that approach the topic 'cell wall' with similar or complementary strategies, but in different biological systems. By consequence, many of the workshop participants met for the first time at BIRS, and rapidly identified common interests and opportunities for interaction and collaboration. The following examples illustrate the variety of contributions and different structural scales at which the cell wall was discussed at the workshop:

Molecular structure of the cell wall

The cell wall is an envelope that surrounds the cells of most organisms except for those of animals. The molecular structure of cell walls differs significantly between different biological kingdoms. In bacteria the wall is built from a single type of proteoglycan, whereas in plants and fungi the wall is composed of multiple polysaccharidic polymers with a great variety of structure and biochemistry. Despite these differences the fundamental laws governing shape formation in cells is similar since independent of cell type this process relies on yielding of the wall under the force provided by the internal turgor pressure. The following presenters gave detailed overviews of the properties of the different types of cell wall:

Olivier Ali (INRA Lyon) reported on force-driven polymerization of a hydrogel as a fundamental mechanism behind cell wall matrix expansion. While many molecular players involved in growth control have been identified in the past decades, it is often unknown how they mechanistically act to induce specific shape changes during development. Plant morphogenesis results from the turgor-induced yielding of the extracellular, load-bearing cell wall. Its mechano-chemical equilibrium appears as a fundamental link between molecular growth regulation and the effective shape evolution of the tissue. For animal cells, it has been well established that the dynamics of load-bearing structures, such as the cortical actin network, relies on their force-driven polymerization. The authors tried to determine if turgor-induced morphogenesis in plants originates from the same kind of force-driven polymerization of cell wall components.

Molecular interactions are modeled by **Michael Crowley** (National Bioenergy Center). The authors have studied the specific interactions in cellulose in its various forms and found evidence of how many of the structures find their stability and how conversions between structures occur. Further they are looking in detail at the molecular interactions and dynamical properties between cellulose and many of the other plant cell wall components. Using several approaches to atomic and molecular modeling, they found many interesting properties of cell wall components that are consequences of different kinds of stress on the polymers and polymer complexes.

Experimental tools for quantitative measurement of cell wall architecture and mechanics

Modeling the cell wall behavior requires detailed information on the structure, architecture and mechanical behavior of the material composing the wall. Novel experimental methods have recently contributed to enhancing the researchers' ability to visualize the cell wall and to measure its mechanical properties.

Bara Altartouri (University of Montreal) presented experimental tools used to visualize cell wall components. Insight into cellulose architecture can be obtained by field emission scanning electron microscopy and atomic force microscopy. While these techniques provide exquisite detail in terms of spatial resolution and mechanical properties of the cell wall, they only have limitated capability to follow the dynamic changes over time. The authors have begun exploring the feasibility of using optical microscopy to examine the orientation and anisotropy of cellulose at high spatial resolution.

Mohammad Shafayet Zamil (University of Montreal) used micromechanical testing combined with finite element modeling to assess the mechanical behavior of the substance responsible for the adherence between the walls of neighboring cells. The authors have taken an onion outer epidermal cell wall profile as representative of a multicellular material system and developed a framework of multiscale finite element method (FEM) computational model to scale-up mechanical properties from subcellular to tissue scale. A 3D repetitive volume element (RVE) was developed in a way so that if arrayed in X and Y directions will produce an idealized onion epidermal wall tissue patch. In a RVE, wall fragments from four adjacent cells are attached by a distinct continuous layer also known as middle lamella (ML), and therefore the RVE contains both subcellular and extracellular scale parameters. By changing the material properties of the ML and the anisotropy of cell shape (width to length or WL ratio of a cell), their respective contributions were investigated from subcellular to RVE scale. The RVE mechanical responses were one more time scaled up to a 7.2X2.7 mm tissue scale. It was observed that from subcellular to tissue scale the ML mechanical properties have little to no impact on overall mechanical responses. The shape factor was found to have contribution on mechanical responses both from subcellular to RVE and from RVE to tissue scales. Form subcellular to RVE scale, the anisotropy in modulus values decreases as the WL ratio approaches 1, i.e., become isotropic in shape. However, from RVE to tissue scale the shape factor does not change the anisotropy in modulus value. An interesting finding was that the anisotropy in modulus value trend was reversed from subcellular-RVE to RVE-tissue scale. The tissue scale modulus values and their anisotropy were validated by experimental results at similar length scale, which were in good agreement with the computational estimations.

Mechanics of cell growth

Cellular growth in walled cells is governed by completely different physical principles compared to growth and movement in animal cells. Modeling of the growth process necessitates a mechanical representation of the cell wall and its deformation under the effect of load application by turgor.

Adelin Barbacci (INRA Nantes) reported on his model of growth that integrates mechanical properties and biosynthesis of the wall. Expansive growth of plant cell is conditioned by the cell wall ability to extend irreversibly. Growth is then possible thanks to two processes. The first process consists in the development of a tensile stress in the cell wall caused by turgor pressure and the modulation of its mechanical properties through chemorheological processes. The second process is the biosynthesis and the assemblage to the existing wall of new cell wall elements. Despite its central aspect for expansive growth, evolution of mechanical properties of cell wall remains poorly understood. Recent results emphasizing the complexity of the relations between cell wall polysaccharides and emerging properties justify the development of models. In a previous work, the autho proposed a model, based on irreversible nonequilibrium thermodynamics, considering cells as dissipative structures. In this first proposition, mechanical properties of the cell wall and stress tensor were coarsely described to focus on coupling between temperature, turgor pressure and polysaccharides biosynthesis. The present work aims at modeling kinetics of mechanical properties of the cell wall in function of chemorheological process. The development of the model is based on Gibbs-Duhem relations and kinetic relations acting on generalized non-equilibrium forces relative the reshuffling of the cell wall.

Plant cell wall growth remains a complex problem as documented by **Daniel Cosgrove** (Penn State University). In the context of plant cell growth, wall extensibility refers to the ability of the cell wall to extend irreversibly. Plant cell walls are more extensible at low pH, a phenomenon called acid growth and known to be mediated by expansins. The action of expansins on wall mechanics will be contrasted with that of family-12 endoglucanases such as Cel12A. Mutagenesis of Cel12A gives novel insights into the nature of acid growth. The results suggest a new paradigm for how plant cells regulate the activities of cell wall enzymes and also imply that meso-scale features of cell wall architecture may play a crucial role in such regulation.

Fission yeast (Schizosaccharomyces pombe) displays multiple cellular shapes as discussed by **Fred Chang** (Columbia University). How do these cells form rods of a certain shape and size, and how do they grow and divide? Chang has embarked on studying the role of the cell wall and cell mechanics. He finds that after cell division, a mechanical process is responsible for producing the rounded shape of the new end, in which turgor pressure simply pushes out the cell wall of the septum in the absence of cell growth. He is also interested in developing models for how turgor pressure and cell wall insertion produces the rounded shape of cell ends while they are growing. New results suggest that softening of the cell wall by itself is able to initiate cell growth, suggesting the presence of mechano-sensory pathways involved in regulation of cell polarity.

Bacterial cell wall was also discussed by **Simon Foster** (Sheffield University). Bacterial cell wall peptidoglycan is essential for the life of most bacteria. It determines cell shape, and its biosynthesis is the target for many important antibiotics. The fundamental chemical building blocks of peptidoglycan are conserved: repeating disaccharides cross-linked by peptides. However, despite this relatively simple chemistry, how this is manifested into the myriad bacterial shapes and how this single macromolecule remains dynamic permitting cell growth and division has largely remained elusive. The advent of new microscopy approaches is beginning to revolutionize our understanding of the architecture of this polymer and to reveal novel insights into its biosynthesis and hydrolysis. Atomic force microscopy has demonstrated a complex, nanoscale peptidoglycan architecture in diverse species, which meets the challenges of maintaining viability and growth within their environmental niches by exploiting the bioengineering versatility of the polymer. The application of super-resolution fluorescence microscopy, coupled with new chemical probes has begun to reveal how this essential polymer is synthesized during growth and division.

Sun Sean (Johns Hopkins University) described how combination of physical and chemical processes are involved in determining the bacterial cell shape. In standard medium, gram-negative Escherichia coli cells are rod-shaped, and maintain a constant diameter during exponential growth. The authors demonstrate that by applying precisely controlled compressive forces to growing rod-shaped E. coli, cells no longer retain their rod-like shapes but grow and divide with a flat pancake-like geometry. The observed deformation is reversible: deformed cells can recover back to rod-like shapes in several generations after compressive forces are removed. During compression, the cell elongation rate, proliferation rate, DNA replication rate, and protein synthesis are not significantly different from those of the normal rod-shaped cells. They find that during the observed shape transition, peptidoglycan synthesis takes place along the entire envelope of the cell rather than constrained to the lateral cell wall. Quantifying the rate of cell wall growth under compression reveals that the cell wall growth rate depends on the local cell curvature. MreB not only influences the rate of cell wall growth, but also how the growth rate scales with cell geometry. They discuss a mechanochemical model of E. coli cell wall growth and morphogenesis, and suggests an active mechanical role for MreB during cell wall growth.

There is a feedback between cell wall and cellular growth, as reported by **Ethan Garner** (Harvard University). Rod shaped bacteria elongate by the action of cell-wall synthetic complexes that move circumferentially around the cell width. These motions that are thought to reflect the insertion of new cell wall material. These synthetic complexes, located on the outside of the cell, are linked to MreB filaments bound to the cytoplasmic membrane surface. Each filament/enzyme complex moves around the rod independently, with adjacent complexes moving in opposing directions. To understand how the independent, disconnected motions of these filament/enzyme complexes are able to orient their motions 90 degrees to the long axis of the bacteria, the authors observed their dynamics as they deformed and reformed bacteria from rods into spheres. To cause this transition, the authors modulate the levels of teichoic acids and PBP2, each titrating the width and the rigidity (straightness) of the rod shaped Bacillus subtilis. They find that as they decrease cell rigidity, the cellular width increases, up to a point at which rod shape fails. They find that the motions of MreB are circumferentially organized in rods of all widths, yet become anisotropic and disorganized in spheres. By confining these disorganized cells in chambers near the width of normal cells, they authors see that MreB aligns its motion along this externally imposed axis, indicating that the elongation system can sense the aspect ratio of the cell. This orientation also occurs within confined spheroplasts, and MreB assembled inside liposomes, indicating this orientation is dependent upon filaments alone. The authors can watch how this system reorganizes as they convert spheres back into rods by suddenly increasing the magnesium or teichoic acid levels. In isolated cells, they observe the orientation and shape transitions occur by spheres emitting rods from one point, suggesting a local, not global, reorganization of growth, a result corroborated with FDAA staining. The initially emitted rods are near the normal bacterial width, suggesting that this machinery is tuned to both sense and propagate a given cellular radius. These results suggest that the elongation machinery encodes an intrinsic sensor of cell width, one that creates rod shape by orienting their motion of synthesis along given curvatures. The authors suggest that feedback between curvature, orientation of synthesis, and cell wall stability provides a robust mechanism to initiate and maintain rod shaped growth, independent of the pre-existing shape or cell wall organization. This mechanism would allow cells to form and grow as rods based on local rules rather than long-range organization.

Fungi have a very different molecular composition of the cell wall compared to plants and bacteria. **Amir Jafari Bidhendi** (University of Montreal) reported on fungal spores, which are spherical structures with extremely thick and resistant walls. Nevertheless, the spores of mycorhyzal fungi that are produced commercially as biological fertilizers can get damage during the production process. The authors therefore wanted to characterize the biomechanical properties of the spore walls. Glomus intraradices spores are formed both inside and outside the root, representing two very different environments. Using microindentation the authors examined the spore stiffness and its correlation with size, wall thickness and origin of the spores. A finite element model of the microindentation procedure was developed to analyze the influence of different geometrical and material constants on deformation behavior of the spores. To relate the mechanics to the biochemical composition of the wall, they determined the subcellular localization of the structural cell wall components chitin, β(1,3) glucan and glomalin using confocal laser scanning microscopy.

Cell growth and supply of cell wall material

Sustained growth of walled cells requires the supply and assembly of new cell wall material. The regulatory role of this process is rarely incorporated in mechanical models of growth, but various approaches to do so point at future avenues in this field.

Salomon Bartnicki-Garcia (Centro de Investigación Científica y de Educación Superior de Ensenada) discussed fungal cell wall morphogenesis powered by vesicle delivery as predicted by computer modeling. Hyphal wall synthesis is highly polarized. It originates from a sharp gradient of synthesis centered on the dome of the hyphal tip. It has been recently found that the enzymes that make the two principal polymers of most fungal walls, chitin and beta-1,3-glucan, are transported in different vesicles. In most fungi, the vesicles congregate in a structure called the Spitzenkörper. An exercise in computer simulation of hyphal morphogenesis led to the realization that the Spitzenkörper functions as a vesicle distribution center whose linear advance generates the gradient of cell-making vesicles necessary to produce a continues tubular cell. The present challenge is to elucidate the mechanisms by which vesicles move in out of the Spitzenkörper.

Rishi Bhalerao (Umea Plant Science Center) investigates vesicular trafficking in ethylene and auxin mediated differential growth. Differential growth across tissue layers is mediated by asymmetric auxin distribution. The authors are using apical hook development, bending of hypocotyl immediately after seed germination, as a model to investigate how differential growth is achieved. Asymmetric auxin distribution essential for apical hook development is mediated by the concerted action of auxin influx- and efflux carriers many of which localize to the plasma membrane (PM). The abundance of auxin carriers at the PM is dynamically regulated through vesicle trafficking processes such as endocytosis, recycling and secretion.

Mechanics of multicellular tissues

The behavior of plant tissues is complex since plant cells adhere to each other and influence each other's growth behaviour, necessitating study of how these interactions affect tissue mechanics.

Arezki Boudaoud (Ecole Normale Superieure de Lyon) discussed how the mechanical properties of plant tissues determine how they grow into well-defined shapes. Little is known on the mechanics of internal tissues in developing organisms, making it difficult to assess their contribution to morphogenesis. Here the authors investigated the mechanics of internal cell walls in plants, more specifically in Arabidopsis. They combine confocal imaging of tissues in 3D, nano-indentation, and mechanical models. Their results provide support to the pressurized shell model of the shoot apex and reveal the differences between the shoot apex, the hypocotyl, and the root, shedding light on the contrast in organogenesis between the root and the shoot apex.

The presentation by **Siobhan Braybrook** (Sainsbury Lab) emphasized that pectin is a mechanically important component of the plant cell wall. Recent literature highlights a key role for an overlooked component of the cell wall, the pectin homogalacturonan (HG), in plant development. The authors have been exploring the role of pectin in plant cell and organ growth using Arabidopsis thaliana cotyledons and hypocotyls. They combine immunohistochemistry, growth and shape analysis, and mechanical measurements to understand how these organs and their cell shapes are generated. They have also begun exploring how pectin gels themselves behave mechanically as a result of chemical alteration, and furthermore what effect these changes have in cell wall material mimics. Their results indicate that HG pectin modification controls the magnitude of plant growth through precise chemical mechanisms.

Tissue mechanics determines tissue failure under stress, as studied by **Douglas Cook** (NYU Abu Dhabi). Cook's group studies stalk failure, focusing primarily on crop stalk biomechanics. In maize and other crops, failure often initiates in or near the meristematic tissue above each node, and almost always occurs due to compressive stresses. The localized buckling of individual cell walls is one hypothesized mode of failure propagation: (i.e. failure in the first cell leads to increased stresses in neighboring cells and so forth). The authors have conducted mechanical tests and imaging studies to try to determine where failure initiates, but this approach is problematic. If failure initiates on the surface of the stalk it may be visible using one of several optical techniques, but failure that initiates at any internal location is not visible optically until it is manifest on the surface. Is there a way to identify the location of initial failure? Could cell wall modeling provide an avenue for studying failure initialization and perhaps lead to insights that are not possible through purely experimental approaches? This presentation represented an excellent case study for the modelers in the room who immediately brainstormed on the problem.

Rosemary Dyson (University of Birmingham) presented that the arrangement, and dynamic rearrangement, of cell wall components lead to emergent anisotropic mechanical properties which govern the directional growth of individual plant cells and hence the tissues these cells combine to form. Growth is therefore inherently a multiscale process, with the microstructure determining the mechanical properties of a cell wall segment, which in turn governs the expansion of individual cells and hence the whole tissue. However, this resultant tissue-level growth can also produce active rearrangement of the cell wall microstructure, which will again alter the macroscale mechanical properties. Understanding this interplay is essential to determining the mechanical origins of three-dimensional macroscale plant root behaviour, for example twisting and tropic responses. The author discussed her current work incorporating these macroscale effects on the microscale structure into mathematical models of plant root growth. She focused in particular on the effects of the current macroscale state of the tissue on the deposition, dispersion and reorientation of cellulose microfibrils, and how this manifests as changes to the evolving cell wall mechanical properties. This in turn determines the directional growth of cells and tissues to produce observable changes in plant phenotype.

Henrik Jönsson (Lund University) emphasized that physical forces have been suggested to contribute in generating form in plant tissue, and more recently act as main input to patterning processes themselves. Plants can guide their pressure-driven growth into anisotropic shapes by strengthening the walls by laying out cellulose fibers in specific directions. He discussed different efforts modeling morphodynamics in plants. Focus was on how stress and strain can provide input to elastic properties and plastic growth, and when these signals fail. He showed how plants' orientation of cellulose fibers represent an energy-minimizing principle similar to topology optimization.

Tissue and organ development are intimately linked as discussed by **Przemyslaw Prusinkiewicz** (University of Calgary). Early leaves of the model plant Physcomitrella patens consist of a single layer of cells. Although cellular patterns are thus easily observable, live imaging of developing Physcomitrella leaves has remained technically challenging. To describe and understand Physomitrella leaf development from the initial apical cell to the mature leaf form, they constructed a computational model integrating the information gathered from microphotographs of leaves in different stages of development and from short live-imaging sequences. A particularly puzzling observation was the transformation of diagonally-oriented cells near the leaf apex into staggered files of longitudinally-elongated rectilinear cells closer to the leaf base. To explain it, the authors considered two hypotheses: (1) diagonally-oriented cells are rotated to their final rectilinear position by a medio-lateral gradient of growth rates, and (2) diagonal cells undergo shape changes caused by inhomogeneous expansion of different wall segments. A biomechanical model implementing the second hypothesis produced cellular patterns consistent with the observations of real leaves. In addition to characterizing Physcomitrella leaf development at the cellular level, the model points to a possibly broad morphogenetic role of the growth-tensor discontinuities in symplastic development.

Richard Smith (Max Planck Institute for Plant Breeding Research) explained various modeling tools specifically developed for plant tissue modeling. This is especially true for mechanical models, where non-intuitive behavior often arises even from very simple assumptions. Although the field of mechanical modeling with the finite element method (FEM) is well developed, growth is typically not included, and many methods are not well suited for modeling plant cells. He presented FEM models he developed specifically for 3D plant cells. He uses the models for quantitative work to interpret force measurements from experiments, as well as qualitative models for growing 3D cellular plant tissues.

Joseph Turner (University of Nebraska-Lincoln) employs similar modeling strategies to analyze time-dependent material response of plant cell walls. One very importance aspect for growth is the time-dependent (viscoelastic) nature of the cell wall that allows the wall to relax and expand. In this presentation, the role of viscoelasticity and its impact on measurement interpretation and modeling are discussed. In particular, the computational modeling of cell wall behavior must include an appropriate time-dependent component in order for force-displacement responses to have physical relevance. In addition, model inputs are often derived from scanning probe microscope experiments (e.g., nanoindentor, atomic force microscope) that are now being applied to plants. However, data analysis is often made under the assumption of a quasi-static material response for which the loading rate is presumed unimportant. Here, the impact of such assumptions was discussed and quantified using several examples.

Participants

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Chapter 26

Women in Geometry (15w5135)

November 1 - 6, 2015

Organizer(s): Ailana Fraser (University of British Columbia), Catherine Searle (Wichita State University), Elizabeth Stanhope (Lewis and Clark College)

Overview

There is currently an underrepresentation of women in the mathematics faculty of Ph.D. granting universities, a relative underrepresentation of women in Geometry/Topology and low visibility of women at Geometry conferences. For example, in the last fifteen years roughly 30% of all mathematics Ph.D.s in the United States have been awarded to women, with only 20% of these degrees awarded in Geometry/Topology to women. Despite this, the percentage of tenure-track and tenured female math faculty members at Ph.D. granting institutions in the U.S. is only 11%. (Statistics from Proceedings of the National Academy of Sciences publications as well as the Annual Survey of the Mathematical Sciences.)

The overarching goal of the Women in Geometry (WIG) workshop was to increase the strength and visibility of the community of women geometers. The workshop hosted seven different research teams with anywhere from four to seven women each in the following areas: Calibrated geometry, Geometric aspects of soliton equations, Metric geometry, Minimal submanifolds, Ricci flow, Spectral geometry and Symplectic geometry. Led by women established in these areas of geometric research, each team worked together on an open problem in their respective area. Before the program began team leaders provided participants with a synopsis of the research problem and necessary background reading materials. Once at BIRS, the majority of the time was spent in working groups pursuing collaborative research. Further background was developed through team seminars held during the first day, as needed. Seven plenary talks, one for each group, were scheduled during the first four mornings to provide attendees with new insights on current trends in geometry, to build a feeling of community, to promote discussion and collaboration within and between groups, as well as to inspire participants to consider expanding and broadening their own research programs. On the last day of the program each team reported back to the full workshop on progress made and future goals.

Other highlights of the program included "Open Problem Sessions" scheduled on Monday evening and a "Group Discussion" on Tuesday evening. For the Open Problem Sessions, each team created a problem list, much as is done at AIM conferences (see the following link: http://aimpl.org). Participants were encouraged to think about open problems in their area, and try to arrive at the worksop with at least one problem to add to the team's problem list, or a place in their work where they would welcome ideas from others at WIG. Part of the motivation for doing this was to encourage teams to think about possible future collaborations, both with their own team members and as well as with members of other teams. In addition to the talks from each area, the exercise helped all participants gain a better understanding of what problems are open in each area as well as a better idea of what questions are of interest. The Group Discussion on Tuesday evening was an informal, but guided, conversation on the current status of women in mathematics with a view to finding ways to promote change. Topics included funding and job opportunities, improving the representation of women as speakers at conferences, how to lobby for funding for childcare to attend conferences, how to combat bias on award panels, and the mentoring of female graduate students as well as women in the early stages of their careers.

Presentation Highlights

There were seven plenary talks scheduled on the first four mornings of the workshop. A representative or representatives from each team gave a colloquium style lecture related to their team's research problem in one of these seven hour-long talks. The talks were accessible to all WIG participants for at least the first 30-40 minutes. A goal of WIG was for participants to broaden their horizons a bit and get to know what are important problems in each other's areas. WIG aimed to establish strong working relationships within teams. These lectures reached beyond this to lay the groundwork for possible cross-team collaboration.

Christine Guenther from Pacific University, Casey Kelleher from U.C. Irvine, and Xuan Hien Nguyen from Iowa State University, representing the Ricci Flow group, spoke on <u>An overview of geometric flows, Renormalization group flow, Self-similar surfaces under mean curvature flow, and Higher order Yang-Mills flow</u>.

Carolyn Gordon from Dartmouth College, representing the Spectral Geometry group, spoke on <u>The Steklov problem on orbifolds</u>.

Dusa McDuff from Barnard College and Columbia Unversity, representing the Symplectic Geometry group, spoke on <u>Symplectic and contact geometry</u>.

Gloria Mari-Beffa from the University of Wisconsin, representing the Geometric Aspects of Soliton Equations group, spoke on <u>Geometric realizations of completely integrable PDEs</u>

Chikako Mese from Johns Hopkins University, representing the Minimal Submanifolds group, spoke on <u>Harmonic maps and its generalization in singular geometry</u>.

Sema Salur from the University of Rochester, representing the Calibrated Geometries group, spoke on <u>Manifolds with special holonomy</u>.

Christina Sormani from CUNY Graduate Center, representing the Metric Geometry group, spoke on <u>Metric Geometry and Rectifiability</u>.

Scientific Progress Made

The majority of the time during the workshop was spent in working groups pursuing collaborative research. Here we summarize scientific progress made by each team. Each entry in the list contains: the team research area, the team members (with team leaders underlined), a description of the area, and a summary of the project and progress made during the workshop.

Calibrated geometries

Members: <u>Jeanne Clelland</u> (University of Colorado, Boulder), Rebecca Glover (University of Rochester), Eleonora Di Nezza (Imperial College), Kimberly Moore (University of Cambridge), <u>Colleen Robles</u> (Duke University), and <u>Sema Salur</u> (University of Rochester)

Area: A calibrated manifold is Riemannina manifold (M, g) of dimension n equiped with a p-form ϕ whihc is closed and such that $\phi|_\xi = \lambda \mathrm{vol}_\xi$ with $\lambda \leq 1$ for any oriented p-dimensional subspace ξ of the tangent space to M, where vol_ξ is the volume form of ξ with respect to the metric g. The theory of calibrations is due to Harvey and Lawson and others.

Summary: The team made progress on two problems. Let (M, ϕ) be a G_2–manifold.

(1) Sema Salur has conjectured the existence of a natural contact structure on M which is compatible with ϕ. There are at least two different possible constructions/definitions of the desired contact structure. The team made partial progress toward demonstrating the viability of the first construction in the flat case $M = \mathbb{R}^7$.

(2) The associative 3–folds $X \subset M$ are characterized by an exterior differential system (EDS) $\mathcal{I} \subset \Omega(M)$.

Associated to that system is a characteristic cohomology (CC). The CC gives obstructions to the following problem. Given a closed surface $S \subset M$, when can the surface be realized as the boundary of an associative submanifold X? The CC yields "moment conditions" on S which obstruct the existence of X. The team studied the flat case $\mathbb{R}^7$ and showed that there exist both cohomology obstructed and unobstructed surfaces.

Geometric aspects of soliton equations

Members: Annalisa Callini (College of Charleston), Katrin Leschke (University of Leicester), Gloria Mari-Beffa (University of Wisconsin), and Jing Ping Wang (University of Kent)

Area: Soliton equations are non-linear evolution PDEs that admit many remarkable properties. Many soliton equations can be constructed from Lie algebra splittings. Terng and Uhlenbeck showed that the properties of soliton equations can be obtained in a unified way from Lie algebra splittings. Soliton equations occur naturally in submanifold geometry and in geometric curve flows.

Summary: There were two problems the group discussed, with some progress. The first was the geometric interpretation of the Miura transformation at the curve evolution level. The KdV equation has a geometric realization as a projective flow of curves, meaning that as the curves flow following the geometric realization, their projective invariant flows following KdV. This can be generalized to Adler-Gel'fand-Dikii flows in $\mathbb{RP}^n$. The so-called Miura transformation $u \to v' - v^2$ takes solutions of KdV to solutions of mKdV. On the other hand, mKdV does not have geometric realizations in projective space, only in Euclidean spaces. The question was: what is the interpretation of the Miura transformation at the curve level? Early on, the group noticed that if u is a projective invariant of a curve, then v given as $u = v' - v^2$ is invariant only under an affine subgroup of the projective group PSL(2). Since then we have generalized it to $\mathbb{RP}^n$ and we have identified the Miura map at the level of curve evolutions. We are now trying to find a measure that will tie the new moving frame to a parallel frame usually associated to mKdV realizations.

Since Chuu-Lian Terng did not attend, Katrin Leshke was left a little isolated (she works in problems closer to Chuu-Lian's), so instead of working on the other problems we had proposed, we looked for one closer to Katrin's area of specialization. She has been working on Darboux transforms of surfaces, while JP Wang has worked on Darboux transforms of integrable PDEs. We had long discussions about how to bridge the two approaches, and we saw some similarities that we plan to pursue. We did not have any breakthroughs, but we learned a lot from Katrin.

Metric geometry

Members: Maree Jaramillo (University of Cennecticut), Raquel Perales (SUNY at Stonybrook), Rajan Priyanka (U.C. Riverside), Catherine Searle (Wichita State University), Anna Siffert (Max-Planck-Institut), and Christina Sormani (Lehman College and CUNY Graduate Center)

Area: Metric geometry has blossomed from its origins when Gauss first defined curvature, to the study of Alexandrov spaces of curvature bounded from below, and most recently to the application of Gromov-Hausdorff convergence in Perelman's proof of the Poincare Conjecture. There are many open questions in this active field and new synthetic notions of convergence are being developed.

Summary: The Metric Geometry group considered questions of intrinsic flat convergence and Alexandrov spaces during the workshop. The intrinsic flat convergence, first defined by Sormani and Wenger for oriented countably H^m rectifiable metric spaces has more flexibility than Gromov-Hausdorff convergence.

All Alexandrov spaces are countably H^m rectifiable, but not necessarily orientable, however using the existence of ramified orientable double covers, which themselves are Alexandrov spaces with the same lower curvature bound defined in Harvey and Searle, one can pass to such spaces to consider the intrinsic flat convergence.

In particular, they studied the convergence of Alexandrov spaces with curvature bounded below with respect to intrinsic flat convergence, to determine whether the Gromov-Hausdorff limits and intrinsic flat limits agree when they exist, and to determine what properties are conserved under this limit.

The problem breaks up naturally into two problems.

Problem 1. Prove that every oriented Alexandrov space is an integral current space.

The notion of integral current space as in Sormani and Wenger extends the notion of a Riemannian manifold with boundary: it involves finding a bi-Lipschitz collection of charts almost everywhere which preserve the orientation and then proving the boundary has finite mass, where the mass is defined to be the mass of the current structure.

Problem 2. Prove that the Gromov-Hausdorff and Intrinsic flat limits of oriented Alexandrov spaces without boundary that are non-collapsing and have uniformly non-negative curvature agree.

This has been proven for Riemannian manifolds with non-negative Ricci curvature in Sormani and Wenger. Major progress on Problem 2 is found in Li and Perales, where they consider integral current spaces with non-negative Alexandrov curvature. Problem 2 follows once they have proven Problem 1.

The time spent at the workshop was used to consider the case where the Alexandrov space X has no boundary. The group made significant progress on this question and hopes to complete this case soon, which they will then publish in a peer-reviewed journal. A large number of the participants plan to then collaborate to investigate the case where the Alexandrov space has boundary. We note that this will also require generalizing the result in Li and Perales to include the case where X may have boundary.

Minimal submanifolds

Members: Christine Breiner (Fordham Unviersity), Ailana Fraser (University of British Columbia), Lan-Hsuan Huang (University of Connecticut), Chikako Mese (Johns Hopkins University), Pam Sargent (University of British Columbia), Karen Uhlenbeck (University of Texas Austin and Institute for Advanced Study), and Yingying Zhang (Johns Hopkins University)

Area: Solutions to variational problems are critical points of an energy functional and as such are deeply connected with physical phenomenon. Understanding these solutions is a foundational aspect of geometric analysis and has classical roots dating back to the ancient Greeks. As a particular example, minimal surfaces are critical points for area and on sufficiently small scales are area minimizers.

Summary: Our goal is to prove the following conjecture:

Conjecture: Let Σ be a compact Riemannian surface, X a compact CAT(1) space and $\varphi : \Sigma \to X \in C^0 \cap W^{1,2}$. Then there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a conformal harmonic map $v : \mathbb{S}^2 \to X$.

CAT(1) spaces are metric spaces with an upper curvature bound determined via comparison triangles on the unit sphere.

In the smooth setting, the conjecture was first shown by Sacks-Uhlenbeck in 1981. They used a perturbed energy functional that implied convenient Sobolev embedding theorems. They then had to establish that the solutions to the perturbed energy problems converged to a harmonic map as the perturbed energy converged to the Dirichlet energy. Among other things, their work relied on the Palais-Smale theory for the perturbed functional. Jost demonstrated that one may instead use the idea of harmonic replacement. Jost's argument also used the Palais-Smale condition but we noticed that this is unnecessary. Our argument demonstrates that the main tools needed for harmonic replacement are: existence of a minimizer for the Dirichlet problem on sufficiently small balls, convexity of the energy, and a compactness theory for uniform limits of harmonic maps on small balls.

During this week, we worked out the necessary steps to prove each of the previously mentioned elements. We also noted that we should carefully prove the regularity theory for minimizers. We expect to produce at least two papers based on this project. The first paper will address the existence and regularity of Dirichlet solutions. The second paper will address the previously mentioned conjecture. We might also attempt to answer questions of energy quantization in a third paper. Our time together was incredibly productive and fruitful and we are grateful to have had the opportunity to spend a week together at BIRS.

Ricci flow

Members: Christine Guenther (Pacific Unviersity), Casey Kelleher (U.C. Irvine), Xuan Hien Nguyen (Iowa State University), and Guofang Wei (U.C. Santa Barbara)

Area: The Ricci flow is informally the process of stretching the metric in directions of negative Ricci curvature, and contracting the metric in directions of positive Ricci curvature, as a way of smoothing out irregulairties in the metric. It is the primary tool used in Hamilton-Perelman's solution of the Poincar conjecture, as well as in the proof of the differentiable sphere theorem by Brendle-Schoen.

Summary: In the field of geometric flows, one studies the evolution of geometric objects in time. The flows have extensive applications to physical problems such as crystal growth and eroding stones, as well as geometric problems such as the Thurston Geometrization conjecture. Important examples include the curve shortening flow, the mean curvature flow, and the Ricci flow. A prototype for many analytical techniques is the heat equation.

Our Geometric Flow group was unique in that, due to cancellations before the BIRS workshop, each of the four members of the group brought ideas for problems on which to work (instead of working on a problem that was posed beforehand). Throughout the week we discussed the feasibility of these projects, and decided to work on four of them. To this end we have set up a Dropbox in order to easily share information at a distance. The problems are as follows:

Problem A: Given an entire graph u over $\mathbb{R}^2$ that satisfies the equation for self-translating surfaces under mean curvature flow:

$$\sqrt{1+|Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=1,$$

must the graph be the rotationally symmetric "bowl soliton"?

This Bernstein-type problem for self-translating hypersurfaces in $\mathbb{R}^{n+1}$ is solved in dimension $n \geq 3$ (the answer is no). In dimension $n=2$, the conclusion holds under the additional assumption that the initial surface $u(\mathbb{R}^2)$ is non-collapsed (Haslhofer) or convex (Wang).

Progress: We wrote the equation in polar coordinates and decided to study possible energy functionals and monotonicity formulas in order to control the area growth.

Problem B: Ricci Flow for Smooth Metric Measure Space

One may consider the flow

$$\begin{aligned}\frac{\partial g}{\partial t} &= -2Rc - 2\nabla\nabla f\\ \frac{\partial f}{\partial t} &= \Delta f - |\nabla f|^2.\end{aligned}$$

Is positivity of the Bakry-Emery Ricci tensor $Rc + \nabla\nabla f$ preserved in dim $n=3$?

In Hamiltons original paper in 1982, he showed that the positivity of the Ricci tensor is preserved under the Ricci flow, by applying the tensor maximum principle to the evolution equation of the Ricci tensor.

Progress: During the BIRS workshop we considered the evolution equations of the curvatures, and decided to work with weighted differential operators and try to apply the maximum principle.

Problem C: The second order Renormalization Group flow (RG-2) equation is a nonlinear perturbation of the Ricci flow that arises in quantum field theory, and is given by

$$\frac{\partial}{\partial t}g = -2Rc - \frac{\alpha}{2}Rm^2.$$

Here $\alpha > 0$, Rc is the Ricci tensor, and $Rm^2_{ij} := R_{iklm}R_j^{klm}$. The fixed points of the RG-2 flow in dimension $n=3$ include the space forms, as well as the product spaces $\mathbb{H}^2\times\mathbb{R}$ and $\mathbb{S}^2\times\mathbb{R}$ (allowing $\alpha<0$).

In the cases of Nil, Sol, and $\mathrm{SL}(2,\mathbb{R})$, the flow exhibits dichotomous behavior depending on the initial conditions, with a curve separating the "Ricci flow" type behavior from asymptotic solutions where all directions collapse to zero. Are the dividing curves for Nil, Sol, and $\mathrm{SL}(2,\mathbb{R})$ self-similar solutions? Can one use this flow to construct a flow whose fixed points are locally homogeneous spaces?

Progress: During the Workshop we decided to write out the soliton equation for the geometries in question, and see if the dividing curves could be shown to satisfy them, as well as writing out the 3-loop flow in dimensions $n=4$.

Problem D: We decided to investigate the functional

$$\int|\nabla Rm|^2,$$

both varying the connection and the metric, to see if the resulting flow has interesting properties; in particular, it may be useful for finding geometric structures such as symmetric spaces.

Progress: We will proceed with calculations guided by previous work on the variation of $\int |Rm|^2$ and the Yang Mills functional.

Spectral geometry

Members: Teresa Arias-Marco (Universidad de Extremadura), Emily Dryden (Bucknell University), Carolyn Gordon (Dartmouth College), Asma Hassannezhad (Max-Planck Institute for Mathematics), Allie Ray (Trinity College), Liz Stanhope (Lewis and Clark College)

Area: A central question in Spectral Geometry is: "What information about a geometric object is encoded in the eigenvalue spectrum of the Laplace operator acting on differentiable functions on that object?" This compelling question, often phrased as "Can you hear the shape of a manifold?", has yielded a rich area of research.

Summary: Our project was to consider the Steklov problem on orbifolds.

Our first goal was to obtain an upper bound on Steklov eigenvalues in the orbifold setting. We considered previous results by Hassannezhad in the Riemannian setting, and showed that the main arguments in the proof could be adapted to the orbifold setting for dimension two. One tool in this proof is the Ricci flow. Consultation with the Women in Geometry Geometric Flow team greatly expedited our solution of an important case. This gives us the main strategy for generalizing this result, although we still need to consider more minor details. In the future, we intend to publish this result as well as considering the higher dimensional case.

Our second goal was to consider whether the Steklov spectrum on 2-orbifolds determines the number of boundary components as well as their lengths. We looked at a similar result by Girouard-Parnovski-Polterovich-Sher in the Riemannian setting, and are currently working through their proof, adapting steps as needed to the orbifold setting. Once obtained, this result will distinguish 2-orbifolds with an odd number of singular boundary components from Riemannian surfaces.

The group has applied for support to the Max Planck Institute in Bonn, Germany, and received an informal acceptance, in order to meet together again in the summer of 2016 for future collaboration.

Symplectic geometry

Members: Maia Fraser (University of Ottawa), Ailsa Keating (Columbia Unversity), Joan Licata (Australian National University), Dusa McDuff (Barnard College, Columbia University), Sheila Sandon (University of Strasbourg), and Lisa Traynor (Bryn Mawr College)

Area: Symplectic and contact structures are closely related geometric structures that can be put on even (resp. odd) dimensional smooth manifolds. By Darboux's theorem these structures are locally unique so that all invariants are global in nature. Moreover in each case the group of structure-preserving transformations has a rich structure.

Summary: The symplectic geometry group discussed a variety of topics that eventually converged to three questions. Overlapping teams from our group are interested in pursuing each of these.

1. Symplectic Embeddings

   One of the classical results in symplectic geometry is Gromov's Non-Squeezing Theorem for symplectic balls. In dimension 4, interesting embedding results due to McDuff exist for ellipses, and we began by reviewing a recent result addressing the "stabilized" problem: for what A can $E(1, S, T)$, embed symplectically into $B^4(A) \times \mathbb{R}^2$? Hind gives an explicit construction which is close to optimal, and we hope to generalize his technique to high dimensions in which essentially nothing is currently known.

2. Legendrian norms

   Two of the members of the group (Fraser & Sandon) have studied metrics on the group of contactomorphisms of a fixed contact manifold. In-progress work of Traynor and a collaborator associates to a pair of cobordant Legendrian submanifolds the minimal length of a Lagrangian cobordism between them. We have developed a potential connection between these two topics: to a contactomorphism, we may associate a pair of cobordant Legendrian submanifolds of a higher-dimensional contact manifold. We conjecture that the length associated to these submanifolds is bounded from below by a function of the norm of the original contactomorphism.

3. (Non-)squeezing and metric properties of the group of contactomorphisms

   Prior work of McDuff and others in the symplectic setting can be interpreted as addressing this relationship, and in the contact setting, fundamental work of Eliashberg-Kim-Polterovich establishes a connection between null homotopies of certain loops of contactomorphisms and squeezing of domains in a related contact manifold. We are looking at shorter homotopies that reduce one of the existing norms (or related real-valued length) of loops and whether this corresponds to new squeezing phenomena.

Outcome of the Meeting

Each of the research teams made significant progress on their research problems during the workshop. The teams plan to continue their collaborations, and resulting publications are expected.

Participants benefited on the individual level by building background knowledge on a new problem, by strengthening and broadening their research programs, and, in some cases, by being provided with a re-entry point after being sidetracked by any or all of family duties, high service loads or high teaching loads. By building teams that included women at all career stages, from advanced graduate students and recent Ph.D.s to associate professors seeking to invigorate their research programs to senior researchers, the workshop formed mentoring and collaborative networks that will strengthen the careers of all participants. All attending gained an overview of seven exciting areas of current research in geometry, and all contributed to progress in their own area.

The community of women geometers was strengthened by WIG's supportive research community, mentorship of women just beginning or at the middle of their research careers, and the new collaborative links forged between women geometers working within and between their respective areas of specialization. It is important to note that the areas of geometry featured in the WIG program are strongly interrelated, so the potential for cross-area collaboration is high and at least one group of women from three different teams have told us of their intention of forging a future collaboration. The visibility of the community of women geometers was increased by highlighting the work of established female leaders in geometry, by bringing attention to the work of outstanding new women geometers, and, very simply, by having this many women together to do geometry research.

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Chapter 27

Homogeneous Structures (15w5100)

November 8 - 13, 2015

Organizer(s): Claude Laflamme (University of Calgary), Lionel Nguyen Van The (University of Aix-Marseille), Stevo Todorcevic (University of Toronto), Robert Woodrow (University of Calgary)

Overview of the Field

A relational first order structure is homogeneous if every isomorphism between finite substructures extends to an automorphism. Familiar examples of such structures include the rational numbers with the usual order relation, the countable random and so called Rado graph, and many others. Countable homogeneous structures arise as Fraïssé limits of amalgamation classes of finite structures, and have connections to model theory, permutation group theory, combinatorics (for example through combinatorial enumeration, and through Ramsey theory), and descriptive set theory. Our principal objective at this workshop was to promote close interactions between different fields of mathematics affected by recent developments related to homogeneous structures, including researchers in the areas of combinatorics, descriptive set theory, dynamical systems, group theory, metric spaces, and model theory.

Some of the mainstream recent themes that have emerged include the following.

- Universal objects: Examples include the Fraïssé theory in logic and generalizations in model theory, universal graphs in combinatorics, the universal Urysohn space in topology.
- Homogeneous structures: Automorphism groups of homogeneous structures, Set Homogeneity, Polish groups and topological dynamics, structural Ramsey theory, constraint satisfaction, omega-categoricity and amalgamation constructions, metric homogeneous structures, and classification results.

Universal objects are central to Mathematics in a sense that they may reflect properties and non-properties of a given class of structures. They are typically very homogeneous, and hence the deep connection between these areas. Recent applications across the above themes and disciplines provide a unique opportunity to gather experts with knowledge from various mathematical angles, from model theorists who provide techniques for constructing such objects, to permutation group theorists who can provide insight into the automorphism groups of these rich structures.

Recent Developments and Open Problems

While the initial focus of the area was on the classification of various homogeneous structures and their Ramsey properties, the subject has since widened extensively to include connections to infinite permutation groups, dynamical systems through questions about continuous actions of Polish groups on compact spaces, making extensive use of other tools such as Ramsey classes. In other directions, versions of constraint satisfaction problems were established through infinite (usually homogeneous) structures, and Fraïssé limits have been extended to metric structures.

These various developments are well exemplified by the variety of open problems which arose at the workshop, a few of them listed here.

Itaï Ben Yaacov, Université Claude Bernard

Problem: Let G be a Polish Roelcke precompact group with compatible left invariant metric d_L. Define a new metric as $d_u(g,h) = \sup_{k \in G} d_L(gk, hk)$. Is there any relationship between discreteness of d_u and the fact that G cannot act transitively by isometries on a complete metric space?

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Gabriel Conant, University of Notre-Dame

Problem: Suppose that S is an infinite subset of positive reals that has the 4-value condition, as defined in [1]. Fix $A \subset S$ finite. Is there a finite S_0 with the 4-value condition such that $A \subset S_0 \subset S$?

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Cameron Freer, Gamalon Labs

Call a relational structure highly homogeneous when for every k, its automorphism group acts transitively on the k-element sets. Peter Cameron classified the countable highly homogeneous relational structures in 76, and all of them turn out to satisfy the strong amalgamation property.

Problem: Is there an elementary proof of this latter fact, without referring to the classification?

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Robert Gray, University of East Anglia

A relational structure M is set-homogeneous if whenever two finite substructures U and V are isomorphic, there is an automorphism $g \in \mathrm{Aut}(M)$ such that $g(U) = V$. As an example, consider the countably infinite graph $R(3)$ defined as follows. Its vertex set is a countable dense subset of the unit circle with no two points make an angle of $2\pi/3$ at the centre of the circle. Its edge set is given by $x \sim y$ iff $0 < \arg(x/y) < 2\pi/3$.

Problem 1: [from [DGMS94]] Are $R(3)$ and its complement $\overline{R(3)}$ the only countable set-homogeneous graphs which are not homogeneous?

Problem 2: [from [GMPR12]] Is there a countable set-homogeneous tournament that is not homogeneous?

Related to this, there is also the problem:

Problem 3: Classify the countably infinite set-homogeneous digraphs.

A structure (with an age which has finitely many n-element structures up to isomorphism) is homogenizable if it can be made homogeneous by adding finitely many relations to the language (without changing the automorphism group).

Problem 4: [from [DGMS94]] Is there a set-homogeneous structure which is not homogenizable?

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Jordi Lopez-Abad, Instituto de Ciencias Matemáticas

Problem 1: [from [1]] Does the dual Ramsey theorem hold for equidistributed partitions?

Problem 2: Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Is $\mathrm{iso}(\mathbb{G})$ compactly approximable, ie does it admit an increasing chain of compact subgroups whose union is dense? Is $\mathrm{iso}(\mathbb{G})$ Lévy?

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Martino Lupini, California Institute of Technology

Problem 1: [from [4]] Let $\mathbb{G}$ be the Gurarij space, and $\mathrm{iso}(\mathbb{G})$ be its linear isometry group. Does it have the automatic continuity property, ie is every algebraic homomorphism from $\mathrm{iso}(\mathbb{G})$ to a separable group continuous?

Problem 2: Given a Fraisse class (or a metric Fraisse class) $\mathcal{C}$ with limit M, is there a natural assumption on $\mathcal{C}$ that guarantees that for every countable substructure X of M, $\mathrm{Aut}(X)$ embeds in $\mathrm{Aut}(M)$ via some embedding j of X in M so that every automorphism of $j(X)$ extends to an automorphism of M?

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Micheal Pawliuk, University of Toronto

Problem: Let $\mathbb{S}$ denote the semi-generic digraph. Is $\mathrm{Aut}(\mathbb{S})$ uniquely ergodic, ie is there a unique invariant Borel probability measure whenever $\mathrm{Aut}(\mathbb{S})$ acts continuously and minimally on a compact Hausdorff space? Equivalently, is there a unique invariant Borel probability measure on the universal minimal flow of $\mathrm{Aut}(\mathbb{S})$?

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Michael Pinsker, Charles University

Problem 1: Let $F \subset \omega^{\omega^3}$ be topologically closed, closed under composition, and containing the set $\{\pi_1, \pi_2, \pi_3\}$, where π_i denotes the i-th projection. Assume that there exists a map $\xi : F \to \{\pi_1, \pi_2, \pi_3\}$ which preserves composition. Is there a continuous such map?

Problem 2: Same question as above, assuming that the set of functions $\{x \mapsto f(x, x, x) : f \in F\} \subset \omega^\omega$ contains an oligomorphic group.

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Maurice Pouzet, Université Claude-Bernard / University of Calgary

First, here are two old problems about the notions of indivisibility and minimality introduced by Roland Fraïssé. A relational structure R is indivisible (resp. age-indivisible) if for every partition of its domain into two parts, say A, B, one of the induced structures $R_{\restriction A}$ and $R_{\restriction B}$ embeds R (resp. has the same age as R). See [2]. An indivisible structure is obviously age-indivisible.

Problem 1: If R is age-indivisible, does there is an indivisible relation with the same age as R? Note that the answer is yes if the maximum of the arities is 3. See [5].

A relational structure R is minimal for its age if every induced substructure with the same age as R embeds R (see [2]). Not every age has a structure which is minimal for this age (see [4]), but, trivially, if R is age-indivisible and minimal for its age, then R is indivisible.

Problem 2: If R is age-indivisible, is there R' with the same age that is also minimal for its age?

Next, here are some problems on the collection of orbits of an oligomorphic group. The profile of a relational structure M is the function φ_M which counts for every non-negative integer n the number $\varphi_M(n)$ of n-element substructures of M counted up to isomorphy, see [6]. When M is homogeneous, this counts the orbits of the action of $Aut(M)$ on the n-element subsets of the domain of M. The age of a group G of permutations on a set V is the set $\mathrm{Age}(G)$ of orbits of finite subsets of V. This set can be ordered as follows: for two orbits O', O'' we set $O' \leq O''$ if there are subsets F' and F'' of V such that $F' \subseteq F''$, $O' = Orb(F')$ and $O'' = Orb(F'')$. As an ordered set, $\mathrm{Age}(G)$ is ranked; elements of rank n being the orbits of n-element subsets of V. The function φ_G which counts for each integer n the (cardinal) number $\varphi_G(n)$ of orbits of n-element subsets is the the orbital profile of G. If the number of these orbits is finite for each integer n then G is oligomorphic (see [1]).

Problem 3: If φ_G is not bounded above by some exponential function of n then $\mathrm{Age}(G)$ contains an infinite antichain.

The following result yields groups whose ages have no infinite antichain (use the test given in [7]).

Theorem Let M be a relational structure. If for every non-negative integer n the class $\mathrm{Age}_{n^-}(M)$ of finite substructures N of M with n-labelled elements, say $(N, \{a_1, \ldots a_n\})$, has no infinite antichain, then there is some M' with the same age as M whose theory is $\aleph_0$-categorical and inductive. Moreover, if the set of initial segment of $\mathrm{Age}_{\omega^-}(M) := \bigcup_{n<\omega} \mathrm{Age}_{n^-}(M)$ has no infinite antichain then $\mathrm{Age}(G')$, where $G' := Aut(M')$, has no infinite antichain.

Finally, here are some problems about polynomially bounded profiles. Cameron conjectured that if G is a permutation group, φ_G is bounded above by some polynomial function of n then $\varphi_G(n) \simeq a.n^k$ for some $a > 0$ and $k \in \mathbb{N}$ (see [1]). Macpherson [3] asked if the fact that φ_G is bounded above by some polynomial function implies that the Cameron algebra of G is finitely generated (this algebra is made of linear combinations of members of $\mathrm{Age}(G)$ (see [1]). A positive answer implies that the generating series associated to the profile is a rational fraction; Cameron's conjecture would follow.

We may note that there are relational structures whose profile is bounded above by some polynomial and in fact the generating series is a rational fraction but for which the age algebra of Cameron is not finitely generated [8].

A description of those groups whose orbital profile is bounded above by some polynomial has yet to come. I propose the following approach: According to Schmerl [9] a relational structure M is cellular if its domain V is the disjoint union of a finite set F and a set which can be identified to the cardinal product $K \times L$ such that (1) for every permutation f of L the map $(1_K, f) \cup 1_F$ is an automorphism of M. I propose a variation of this notion, replacing (1) by the following condition:

(2) The substructures induced on two finite sets A and A' with the same cardinality are isomorphic provided that $(2a)$ $A \cap F = A' \cap F$ and $(2b)$ the frequency vectors $\chi_{A \setminus F}$ and $\chi_{A' \setminus F}$ are equal (the frequency vector $\chi_{A \setminus F}$ associates to every non-empty subset K' of K the number $\chi_{A \setminus F}(K') := |\{\ell \in L : A \cap (K \times \{\ell\}) = K' \times \{\ell\}\}|$).

I will say that $M$is set-cellular if it has such a decomposition.

Problem 4:

1. If $\varphi(M)$ is bounded above by some polynomial then M is set-cellular (the converse holds trivially).

2. If M is set-cellular the generating series of the profile is a rational fraction.

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Norbert Sauer, University of Calgary

Problem: Let R be a relational structure on $\mathbb{N}$. Consider the poset $\mathbb{P}$ of all copies of R in itself, ordered by inclusion. How does it look like? For example, does it have an infinite antichain? Note that it is known that the cardinality of $\mathbb{P}$ is either 1 or continuum. Furthermore, if R is ω-categorical, the second alternative always holds and $\mathcal{P}(\mathbb{N})$ embeds into $\mathbb{P}$.

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John Truss, University of Leeds

In [2] it is shown that if p is a partial endomorphism of a finite (simple) graph Δ, then there is a finite graph $\Delta' \supseteq \Delta$ and a (totally defined) endomorphism of Δ' extending p.

Problem: Prove that if Δ is a finite graph, then there is a finite graph $\Delta' \supseteq \Delta$, such that every partial endomorphism of Δ extends to an endomorphism of Δ'.

This would be the analogue of Hrushovski's Lemma for partial automorphisms in the endomorphism context. Here, by "endomorphism" is understood a map such that for any two points of its domain, if they are joined by an edge in the graph, then so are their images (but two points which are not joined are allowed to be mapped to an edge, a non-edge, or collapsed to a point).

Note that there are many examples where the corresponding property is known to hold for partial automorphisms. In fact there is a large literature on it, see for example [1]. For partial endomorphisms, no instances are known (apart from the trivial structure).

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Andrew Zucker, Carnegie-Mellon University

Problem: Let G be a compactly approximable Polish group. Is G uniquely ergodic?

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Presentation Highlights and Scientific Progress Made

The main themes of the workshop lectures were categorized as follows.

- Structural Ramsey theory
- Continuous Fraïssé and topological groups
- Ramsey Properties of Homogeneous Structures
- Homogeneous structures, Morel Theory & Ramsey
- Clones, Reducts, and Constraint Satisfaction Problems

Structural Ramsey theory

Jaroslav Nešetřil (Charles University) opened the meeting with a broad overview on Structural Ramsey theory.

In a joint work with Dana Bartosova, Aleksandra Kwiatkowska (UCLA) followed with the topic in the projective setting. She presented some Ramsey theoretic statements about the dynamics of the homeomorphism group of the Lelek fan. One of the main results is a generalization of the finite Gowers' Ramsey theorem.

Micheal Pawliuk (University of Toronto) provided insight into the amenability and the Hrushovski property for Fraïssé classes of directed graphs (joint work separately with Miodrag Sokic and Marcin Sabok). Building on work of Angel-Kechis-Lyons and Zucker the amenability question has recently been answered for all Fraïssé classes of directed graphs, and using a Mackey-type construction the Hrushovski question has recently been investigated for the same classes. Micheal surveyed the results and the techniques used.

Jan Hubicka (University of Calgary) discussed Ramsey properties of multiamalgamation classes, in joint work with Nešetřil.

Andy Zucker (Carnegie Mellon University), discussed Ultrafilters and Structural Ramsey Theory, considering Ramsey objects and objects of finite Ramsey degree in a Fraïssé class K. He showed that an element of K is a Ramsey object if and only if a certain collection of ultrafilters is nonempty, providing a similar characterization of having finite Ramsey degree. These results apply to the dynamics of the automorphism group of the Fraïssé limit.

Continuous Fraïssé and topological groups

To start the day, Julien Melleray (University Lyon 1) presented results on Polish groups as automorphism groups of metric structures, giving a partial survey of what can be gained by approaching a Polish group as the automorphism group of a homogeneous metric structure. In particular, he used metric Fraïssé limits and the natural topometric structure on the automorphism group of a metric structure.

This was followed by Martino Lupini (California Institute of Technology) on the Poulsen simplex and Fraïssé theory for metric structures. The Poulsen simplex is the unique metrizable Choquet simplex with dense extreme boundary. He explained how one can study the Poulsen simplex from the perspective of Fraïssé theory for metric structures, and how many classical results about the Poulsen simplex can be recovered in this framework. He concluded by mentioning how this point of view allows one to define and construct the noncommutative analog of the Poulsen simplex.

Todor Tsankov (Université Paris 7) presented his joint work with Itaï Ben Yaacov and Tomás Ibarlucía on the Banach representations of dynamical systems and model theory. It is well-known that the automorphism group of an omega-categorical structure encodes all model-theoretic information about the structure. But recently an interesting correspondence has been discovered between properties of the theory (stability, omega-stability, NIP) and classes of Banach spaces on which certain dynamical systems (the automorphism group acting on type spaces over the model) can be represented. In the stable case, those dynamical systems also carry the structure of a semigroup that can be exploited. He discussed what is known about this correspondence as well as some open questions.

Slawomir Solecki (University of Illinois at Urbana-Champaign) discussed Fraïssé limits and topological spaces. He remarked that the pseudoarc is a remarkable compact connected space, in fact, it is the generic compact connected space. He explained the connection between the pseudoarc and projective Fraïssé limits coming for joint work with Trevor Irwin: the pseudoarc is represented as a quotient of such a limit. Further, he described recent work with Todor Tsankov, in which they determined the correct partial homogeneity of the projective Fraïssé limit associated with the pseudoarc. This determination involves combinatorial and basic ”dual” model theoretic arguments (e.g., a notion of dual type). He also described a transfer theorem, through which he recovers Bing's homogeneity of the pseudoarc from our partial homogeneity of the projective Fraïssé limit. He cncluded with recent work with Aristotelis Panagiotopoulos on the Menger curve viewed as a quotient of another projective Fraïssé limit.

Dragan Masulovic (University of Novi Sad) discussed the Kechris-Pestov-Todorcevic correspondence for projective Fraïssé limits. He presented a way to reinterpret the Kechris-Pestov-Todorcevic correspondence in an abstract categorical setting, to then instantiate this abstract setting in several ways. The interpretation in the category of countable structures with embeddings gives the well-known results of K-P-T theory for Fraïssé limits. The interpretation of this setting in categories of arbitrary structures with embeddings yields some recent results of Bartošova in which extreme amenability of automorphism groups of some uncountable structures was established. Finally, the interpretation in op-categories yields duals of some results of K-P-T theory. For example, he showed that if F is a projectively homogeneous structure, then Aut(F) is extremely amenable if and only if the projective age of F has the dual Ramsey property.

Wieslaw Kubis (Academy of Sciences of the Czech Republic) described category-theoretic framework for Fraïssé limits, capturing objects outside of model theory. The basic setting here is a category enriched over metric spaces plus a function measuring the "distortion" of arrows. Within this scheme, and adding some natural axioms, the Fraïssé limit exists, is unique, and has similar properties to classical Fraïssé limits. His approach is parallel to Ben Yaacov's continuous Fraïssé theory, but trying to avoid model-theoretic issues. Within this framework, he captures the Gurarii space, the pseudo-arc, the Poulsen simplex, and some other objects coming from analysis and topology (both new and existing ones).

More on Ramsey Properties of Homogeneous Structures

Norbert Sauer (University of Calgary) started the day discussing partitions of automorphism groups, in particular since in general partition properties of homogeneous structures are properties of their automorphism group. He argued that as it is difficult to characterize homogeneous structures it might be better to try to obtain partition results for subgroups of the symmetric group directly instead of for homogeneous structures. Notions and results and unresolved issues arising in this context were discussed.

Maurice Pouzet (University Lyon 1) presented an overview of the Equimorphy versus Isomorphy problem, in joint work with Claude Laflamme, Norbert Sauer and Robert Woodrow. Here two structures are said to be equimorphic if each embeds into the other. Maurice reported on two conjectures about the number of structures (counted up to isomorphy) which are equimorphic to a given structure; one by Bonato and Tardif asking whether the number of trees equimorphic of a given tree is either 1 or is infinite, the other by Thomassé asking a similar question for relational structures. He presented a positive answer of Thomassé's conjecture for chains and for countable homogeneous structures (whose automorphism group is oligomorphic). He concluded by some results about the hypergraph of copies of a countable homogeneous structure.

Homogeneous structures, Model Theory & Ramsey

David Evans (University of East Anglia) discussed his contributions to Topological dynamics of automorphism groups of Hrushovski constructions. Indeed using Hrushovski's predimension construction, he showed that there exists a countable, omega-categorical structure M with the property that if H is an extremely amenable subgroup of the automorphism group of M, then H has infinitely many orbits on its square. In particular, H is not oligomorphic. This answers a question raised by several authors (including Bodirsky, Pinsker, Tsankov and Nešetřil). It follows that there is a closed, oligomorphic permutation group G whose universal minimal flow M(G) is not metrizable.

David Bradley-Williams (University of Central Lancashire) discussed reducts of primitive Jordan structures, structures which have an automorphism group which is a primitive Jordan group. This means that the automorphism group acting on M is a Jordan group which preserves no non-trivial, proper equivalence relations on M. David first gave a brief survey of examples where results on Jordan groups have been used to obtain results about reducts of primitive Jordan structures. In particular, he discussed the classification of reducts, up to interdefinability, of any relatively 2-transitive semilinear ordering.

Robert Gray (University of East Anglia) discussed set-homogeneous structures. Here a countable relational structure M is called set-homogeneous if whenever two finite substructures U, V of M are isomorphic, there is an automorphism of M taking U to V (but not requiring that every isomorphism between U and V extends to an automorphism). This notion was originally introduced by Fraïssé, although unpublished observations had been made on it earlier by Fraïssé and Pouzet. Clearly every homogeneous structure is set-homogeneous. It is also not too difficult to construct examples of structures that are set-homogeneous but not homogeneous. It is natural to investigate the extent to which homogeneity is stronger than set-homogeneity, and this question has received some attention in the literature. For instance, it was shown by Ronse that any finite set-homogeneous graph is in fact homogeneous. In this talk Robert gave a survey of some of the known results in this area, including results on countably infinite set-homogeneous graphs due to Droste, Giraudet, Macpherson and Sauer, and results on set-homogeneous directed graphs obtained in recent joint work with Macpherson, Praeger and Royle. He concluded with a number of interesting conjectures and open problems that remain about set-homogeneous structures.

John Truss (University of Leeds) discussed countable homogeneous lattices in joint work with Aisha Abogatma. Previously a rather short list of countable homogeneous lattices was known, including, apart from the one-point lattice and the rationals, the countable universal-homogeneous distributive lattice and one or two others arising from amalgamations of certain classes of lattices. They showed that there are in fact uncountably many countable homogeneous lattices. Their examples are all non-modular, and the natural question to ask is whether every countable homogeneous modular lattice is necessarily distributive, a conjecture which has recently been proved by Christian Herrmann. Their method also applies to show that certain other classes of structures also have uncountably many countable homogeneous members.

Gabriel Conant (University of Illinois at Chicago) provided an overview of model theory of generalized Urysohn spaces. He argued that many well known examples of homogeneous metric spaces and graphs can be viewed as analogs of the rational Urysohn space (for example, the random graph as the Urysohn space with distances 0,1,2). In 2007, Delhomme, Laflamme, Pouzet, and Sauer characterized the countable subsets S of non-negative reals for which an "S-Urysohn space" exists. Sauer later showed that, under mild closure assumptions on S, the existence of the S-Urysohn space is equivalent to associativity of a natural binary operation on S induced by usual addition of real numbers. In this talk, Gabriel considered the R-Urysohn space, where R is an arbitrary ordered commutative monoid. He first constructed an extension R* of R, such that any model of the theory of the R-Urysohn space (in a discrete relational language) can be given the structure of an R*-metric space. He then characterized quantifier elimination in this theory by continuity of addition in R*. Finally, he also characterized various model theoretic properties of the R-Urysohn space using natural algebraic properties of R.

Caroline Terry (University of Illinois at Chicago), presented an application of model theoretic Ramsey theory. As Chudnovsky, Kim, Oum, and Seymour recently established that any prime graph contains one of a short list of induced prime subparts, Caroline presented joint work with Malliaris in which they reprove their theorem using many of the same ideas, but with the key model theoretic ingredient of first determining the so-called amount of stability of the graph. This approach changes the applicable Ramsey theorem, improves the bounds, and offers a different structural perspective on the graphs in question.

Matthias Hamann (University of Hamburg) discussed Connected-homogeneous digraphs, which is a directed graph where any isomorphism between every two finite connected subdigraphs extends to an automorphism of the digraph. Matthias discussed the classification of the countable such digraphs, including a description of the main classes of these digraphs as well as a discussion of the main steps in the proof of the classification. In the end he provided arguments showing that their classification is on the one hand complete but on the other hand still incomplete.

David S. Gunderson (University of Manitoba) concluded the day with a discussion on Ramsey arrows for graphs. A simple form of Ramsey's theorem says that for any positive integer m, there exists an n=R(m) so that no matter how the pairs of an n-set are partitioned into two colours, some m-subset has all its pairs the same colour. In terms of graphs, this says if the edges of a Kn are 2-coloured, a monochromatic copy of Km (as a subgraph) can always be found. Such a statement is often expressed in "Ramsey arrow" notation. He then provided a short survey of Ramsey arrows for graphs, culminating in a characterization found with Rodl and Sauer of those triples G,H,r for which there is an F that arrows G when colouring Hs with r colours.

Clones, Reducts, and Constraint Satisfaction Problems

The final day of the workshop started with Manuel Bodirsky (TU Dresden) on applications of homogeneous structures in computer science. Homogeneous structures and their reducts have been used as templates of Constraint Satisfaction Problems (CSPs) to model qualitative reasoning problems in Artificial Intelligence. But this is not the only context in which homogeneous structures arise naturally in CS; Manuel discussed more recent links between homogeneous structures and permutation pattern avoidance classes, and between homogeneous structures and automata theory (for data word languages). He further presented a fragment of existential second-order logic such that the queries that can be formulated in this logic describe (finite unions of) CSPs for reducts of homogeneous structures. This logic is quite powerful and contains MMSNP and most CSPs that have been studied in temporal and spatial reasoning.

Michael Kompatscher (Vienna University of Technology) provided a counterexample on the reconstruction of oligomorphic clones. Two omega-categorical structures are first order bi-interpretable iff their automorphism groups are isomorphic as topological groups. For many well-known omega-categorical structures this statement still holds if we ignore the topology. But in 1990 Evans and Hewitt constructed two omega-categorical structures with isomorphic, but not topologically isomorphic automorphism groups. Similarly two omega-categorical structures are primitive positive bi-interpretable iff their polymorphism clones are topologically isomorphic. Based on the group-counterexample, Michael was able to construct a counterexample for the clones. This is a joint work with Manuel Bodirsky, David Evans and Michael Pinsker.

Michael Pinsker (Charles University Prague) concluded the workshop with a series of conjectures for clones over finitely bounded homogeneous structures. There has been a conjectured criterion, by Manuel Bodirsky and Michael, for when deciding the truth of a primitive positive sentence over a reduct of a finitely bounded homogeneous structure is tractable. This criterion has recently been replaced by a seemingly better criterion, although the equivalence of the two criteria is an open problem. Michael discussed the two conjectures, their relation, and further related conjectures and thoughts.

Outcome of the Meeting

The original goal and ultimately success of the meeting was to bring together the various groups working on the related area of homogeneous structures. In addition to tackling central problems, the workshop contained a substantial training component.

Finally this workshop can be viewed as the fourth international meeting on homogeneous structures, the first three being the following.

- London Mathematical Society Northern Regional Meeting and Workshop on Homogeneous Structures, July 19-22, 2011
- Workshop on Homogeneous Structures, Prague, July 25-27, 2012
- Workshop on Homogeneous Structures, Bonn, October 28-31, 2013

This was also an opportunity to celebrate Professor Sauer's 70th birthday and his contributions to the subject over 45 years.

Participants

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Chapter 28

Approximation Algorithms and Parameterized Complexity (15w5118)

November 29 - December 4, 2015

Organizer(s): Michael Fellows (Charles Darwin University), Klaus Jansen (University of Kiel), Hadas Shachnai (Technion), Roberto Solis-Oba (University of Western Ontario)

The workshop ”Approximation Algorithms and Parameterized Complexity” was organized in cooperation by Michael Fellows, Klaus Jansen, Hadas Shachnai, and Roberto Solis-Oba. Within the 5-day-workshop from the 29th of November until the 4th of December 2015 18 talks were given in Banff center concentrating on the following subjects.

NP-hard problems are optimization problems that are notoriously difficult to solve. They are in fact so difficult that the only known solutions for them require impractically large amounts of time even on the fastest existing computers. There are thousands of these problems arising from a very rich and varied number of applications.

Two approaches that have been independently, but very successfully used to deal with NP-hard problems are approximation algorithms and parameterized algorithms. Approximation algorithms are efficient methods which do not necessarily find optimum solutions; yet, they do guarantee that the output solution achieves a bounded ratio to the optimum. Parameterized algorithms identify and exploit properties of a problem that make it hard to solve to produce efficient solutions for many instances. This workshop explored relationships between techniques used in the design of approximation and parameterized algorithms to gain a better understanding of what makes a problem difficult to solve, with the aim of developing better tools for tackling NP-hard problems.

There are a number of ways in which parameterization and approximation interact mathematically, and ways in which the two approaches could be fruitfully combined to obtain better tools for handling and understanding the computational complexity of hard optimization problems. Some of these have barely been explored; thus, a main goal of the workshop was to provide a venue for experts in parameterized and approximation algorithms to come together and discuss them, furthering our understanding of the complexity of NP-hard optimization problems. Some of the possible ways in which both fields can collaborate and were confirmed by this workshop are listed below.

- EPTAS versus PTAS
- Lower bounds via ETH
- Kernels for approximation algorithms
- Specifically parameterized forms of approximation
- Aggregate parameterizations including approximation

- Parameterized approximations for weighted problems

We had a collection of survey talks, e.g. by Mohammad Taghi Hajiaghayi, by Stefan Kratsch, Mike Fellows, and Klaus Jansen.

- MohammadTaghi Hajiaghayi (University of Maryland at College Park)
  Fixed-Parameter Tractability and Approximability: A Survey of Connections
  In his talk he discussed briefly classes of fixed-parameter tractability as well as approximation algorithms and he surveyed several connections between the two areas in terms of both results and approaches.

- Stefan Kratsch (University of Bonn)
  A brief introduction to kernelization
  Kernelization is a notion from parameterized complexity that captures the concept of efficient preprocessing for NP-hard problems. A kernelization is a polynomial-time algorithm that given an instance (x, k) with parameter k will return an equivalent instance of size bounded only in terms of k. In particular, we were interested in polynomial kernels where the bound depends polynomially on k.

  The talk gave an introduction to core concepts from kernelization. Relations to approximability of the considered problems were discussed.

- Michael Fellows (Charles Darwin University)
  Using Parameterization to Move Approximation into Problem Legislation
  The talk gave a few examples of moving approximation concerns into the definition of parameterized problems — into the modeling of the problem! Which is where, considering the nature of worst-case asymptotic complexity analysis, approximation often realistically belongs. The talk pointed to some large horizons for this approach.

- Klaus Jansen (University of Kiel)
  Lower bounds on the running time for packing and scheduling problems
  Klaus presented lower bounds on the running time for both exact and approximation algorithms based on the exponential time hypothesis (ETH). Then we discussed lower and upper bounds on the running time for exact algorithms for subset sum, partition, knapsack, bin packing, and scheduling on identical machines. Next he gave lower bounds on the running time of approximation schemes for the multiple knapsack, multi-dimensional knapsack and scheduling problem on identical, uniform, and unrelated machines.

The other contributed talks and discussions can be divided in the following research areas.

Packing and Scheduling

Sebastian Berndt (University of Lübeck) studied together with K. Jansen and K.-M. Klein the fully dynamic bin packing problem, where items arrive and depart in an online fashion and repacking of previously packed items is allowed. The goal is to minimize both the number of bins used as well as the amount of repacking. They measured the repacking by the migration factor, defined as the total size of repacked items divided by the size of an arriving or departing item. If one wishes to achieve an asymptotic competitive ration of $1 + \varepsilon$ for the number of bins, a relatively simple argument proves a lower bound of $\Omega(1/\varepsilon)$ for the migration factor. They established a nearly matching upper bound of $O((1/\varepsilon)^4 \log(1/\varepsilon))$ using a new dynamic rounding technique and new ideas to handle small items in a dynamic setting such that no amortization is needed. The running time of their algorithm is polynomial in the number of items n and in $1/\varepsilon$. The previous best trade-off was for an asymptotic competitive ratio of $5/4$ for the bins (rather than $1 + \varepsilon$) and needed an amortized number of $O(\log n)$ repackings (while in the new scheme the number of repackings is independent of n and non-amortized).

Felix Land (University of Kiel) presented (joint work with Klaus Jansen) a fully polynomial $(\frac{3}{2}+\varepsilon)$-approximation for scheduling monotone moldable jobs. A moldable job is a job that can be executed on an arbitrary number of processors, and whose processing time depends on the number of processors allotted to it. He considered the problem of scheduling monotone moldable jobs to minimize the makespan. Most existing approximation algorithms have running time polynomial in the number n of jobs and the number m of processors. He argued that for compact input encodings, such running times are actually exponential in the input size, and that a fully polynomial algorithm has a running time polynomial in n and $\log m$. The best known approximation algorithm with such a running time was due to Mouni, Rapine, and Trystram and achieved approximation ratio $\sqrt{3}+\varepsilon \approx 1.73$. Another algorithm, also due to Mouni, Rapine, and Trystram, had approximation ratio $(\frac{3}{2}+\varepsilon)$, but had running time $\mathrm{O}(nm)$. He described different techniques to improve the running time of the latter to polynomial in n and $\log m$. In particular, they showed how to solve a knapsack problem with n items and capacity m in time $\mathrm{O}(\frac{n^2}{\varepsilon}\log \varepsilon m)$ when items larger than $b=\Theta(\frac{1}{\varepsilon})$ can be compressed by a factor $1-\Theta(\varepsilon)$. For their scheduling problem, the compression increases the makespan by a factor of $1+\varepsilon$, and they expect a wide applicability of their techniques.
Furthermore, they proved that scheduling monotone moldable jobs to minimize the makespan is strongly NP-hard, which was previously known only for the variant without monotony.

Kati Land (University of Kiel) estimated the makespan of the two-valued Restricted Assignment Problem. Together with Klaus Jansen and Marten Maack she considered a special case of the scheduling problem on unrelated machines, namely the Restricted Assignment Problem with two different processing times. They showed that the configuration LP has an integrality gap of at most $5/3 \approx 1.667$ for this problem. This allowed to estimate the optimal makespan within a factor of 5/3 , improving upon the previously best known estimation algorithm with ratio $11/6 \approx 1.833$ due to Chakrabarty, Khanna, and Li.

Monaldo Mastrolilli (IDSIA Instituto Dalle Molle di Studi sull'Intelligenza Artificiale) described research on a Lasserre Lower Bound for the Min-Sum Single Machine Scheduling Problem in joint work with Adam Kurpisz and Samuli Leppanen. The Min-sum single machine scheduling problem (denoted $1||\sum f_j$) generalizes a large number of sequencing problems. The first constant approximation guarantees have been obtained only recently and are based on natural time-indexed LP relaxations strengthened with the so called Knapsack-Cover inequalities (see Bansal and Pruhs, Cheung and Shmoys and the recent $(4+\varepsilon)$-approximation by Mestre and Verschae). These relaxations have an integrality gap of 2, since the Min-knapsack problem is a special case. No APX-hardness result is known and it is still conceivable that there exists a PTAS. Interestingly, the Lasserre hierarchy relaxation, when the objective function is incorporated as a constraint, reduces the integrality gap for the Min-knapsack problem to $1+\varepsilon$. In their paper they studied the complexity of the Min-sum single machine scheduling problem under algorithms from the Lasserre hierarchy. They proved the first lower bound for this model by showing that the integrality gap is unbounded at level $\Omega(\sqrt{n})$ even for a variant of the problem that is solvable in $O(n\log n)$ time by the Moore-Hodgson algorithm, namely Min-number of tardy jobs. They considered a natural formulation that incorporates the objective function as a constraint and proved the result by partially diagonalizing the matrix associated with the relaxation and exploiting this characterization.

Nicole Megow (TU Berlin) talked about an $O(\log m)$-Competitive Algorithm for Online Machine Minimization which was her joint work with Lin Chen and Kevin Schewior. They considered the online machine minimization problem in which jobs with hard deadlines arrive online over time at their release dates. The task was to determine a feasible preemptive schedule on a minimum number of machines. Their main result was an $O(\log m)$-competitive algorithm, with m being the optimal number of machines used in an optimal offline solution. This was the first improvement on an intriguing problem in nearly two decades. To date, the best known result is a $O(\log p_{\min}/p_{\max})$-competitive algorithm by Phillips et al. (STOC 1997) that depends on the ratio of maximum and minimum job sizes, $p_{\max}$ and $p_{\min}$. Even for $m=2$ no better algorithm was known. Their algorithm is in this case constant-competitive. When applied to laminar or agreeable instances, their algorithm achieves a competitive ratio of $O(1)$ even independently of m. The following two key components lead to their new result. Firstly, they derived a new lower bound on the optimum value that relates the laxity and the number of jobs with intersecting time windows. Then, they designed a new algorithm that was tailored to this lower bound and balanced the delay of jobs by taking the number of currently running jobs into account.

Frits Spieksma (K.U. Leuven) presented his joint work with Annette Ficker and Gerhard Woeginger on the topic of the so-called balanced optimization with vector costs. They proposed a framework containing such problems; this framework allowed them to investigate the complexity and approximability of these problems in a general setting. More concretely, each problem in the framework admitted a 2-approximation, and for many problems within the framework this result was best-possible, in the sense that having a polynomial-time algorithm with a performance ratio better than 2 would imply P=NP. Special attention was paid to the balanced assignment problem with vector costs: they showed that the problem remains NP-hard even in case of sum costs.

Andreas Wiese (MPI für Informatik) found with Anna Adamaszek and Giorgi Nadiradze better approximation guarantees for geometric packing problems. A common setting in geometric packing problems is that we are given a set of two-dimensional items, e.g., rectangles, and a rectangular container and the goal is to pack these items or a subset of them items into the container to optimize objective functions like the total profit of the packed items or the necessary height of the container. A typical obstacle in these problem settings is that in the input there are different types of items, i.e., items that are wide and thin, that are high and narrow, or items that are large in both dimensions. Andreas Wiese presented a method to handle this obstacle. In a nutshell, the key was to prove that there are near-optimal solutions in which the given container can be partitioned into few rectangular boxes such that in each box there are only items of one of the mentioned types. This leads to better approximation guarantees for two specific problems: a $(1+\varepsilon)$-approximation algorithm in quasi-polynomial time for the two-dimensional knapsack problem and a $(1.4+\varepsilon)$-approximation algorithm in pseudo-polynomial time for the strip-packing problem. Note that the latter bound is strictly smaller than the lower bound of 3/2 that holds for (non-pseudo-)polynomial time algorithms for the problem.

Guochuan Zhang (Zhejiang University) and Lin Chen studied packing group items. They considered a natural generalization of the classical multiple knapsack problem where instead of packing single items they were packing groups of items. In this setting, they had multiple knapsacks of unit capacity, and a set of items, each of a size within (0,1). These items appeared in groups, where each group was associated with a profit. The profit could be attained if and only if every item of this group was packed into the knapsacks. Such a general model finds applications in delivering bundles of goods. Apart from that, the theoretical issue is of particular interests. It is obvious that no finite bounds are possible, unless P=NP, if a group size (the total size of items in the group) can be arbitrarily large. They thus paid attention to the parameterized version while every group size was bounded by a factor of the total capacity of knapsacks. Along this line, they provided deep insights into the approximability with respect to the factor and derive, respectively, approximation algorithms and inapproximability results.

Graph Problems and Algorithms

In this area of research **Liming Cai** (University of Georgia) talked about Maximum Spanning Backbone k-Tree: Tractability and Approximability. The Maximum Spanning Backbone k-Tree (BkT) problem, for $k \geq 2$, is to find a maximum weight spanning k-tree from the input edge-weighted graph with a designated Hamiltonian path to be desired in the output spanning graph. Originally motivated by research in bio-molecular 3D structure prediction, BkT turns out a typical problem in a new class of languages logic-definable beyond MSOL. They showed that, unlike the Maximum Spanning k-Tree problem that is NP-hard for even $k = 2$, the BkT problem is solvable in time $O(n^{k+1})$, for every fixed $k \geq 2$. While there is evidence of difficulty to improve the polynomial degree $k+1$ to any number lower, they showed that there are $O(n^3)$-time algorithms to approximate the BkT problem to the ratio $k(k-1)$, for every fixed $k \geq 3$. The tractability results also hold with the constraint of a designated spanning tree instead of a designated Hamiltonian path, a scenario that often arises in learning of Markov networks of bounded tree width.

Thomas Erlebach (University of Leicester) pesented his research results on Temporal Graph Exploration in joint work with Michael Hoffmann and Frank Kammer. A temporal graph is a graph in which the edge set can change from step to step. The temporal graph exploration problem TEMPEX is the problem of computing a foremost exploration schedule for a temporal graph, i.e., a temporal walk that starts at a given start node, visits all nodes of the graph, and has the smallest arrival time. They considered only temporal graphs that are connected at each

step. For such temporal graphs with n nodes, they showed that it is NP-hard to approximate TEMPEX with ratio $O(n^{1-\varepsilon})$ for any $\varepsilon > 0$. Thomas also provided an explicit construction of temporal graphs that require $\Theta(n^2)$ steps to be explored. He then considered TEMPEX under the assumption that the underlying graph (i.e. the graph that contains all edges that are present in the temporal graph in at least one step) belongs to a specific class of graphs. Among other results, they showed that temporal graphs can be explored in $O(n^{1.5}k^2 \log n)$ steps if the underlying graph has treewidth k and in $O(n \log^3 n)$ steps if the underlying graph is a $2 \times n$ grid. They also showed that sparse temporal graphs with regularly present edges can always be explored in $O(n)$ steps.
Martin Fürer (The Pennsylvania State University) studied Multi-Clique-Width and defined it as a powerful new width parameter. Multi-clique-width is obtained by a simple modification in the definition of clique-width. It has the advantage of providing a natural extension of tree-width. Unlike clique-width, it does not explode exponentially compared to tree-width. Efficient algorithms based on multi-clique-width are still possible for interesting tasks like computing the independent set polynomial or testing c-colorability. In particular, c-colorability can be tested in time linear in n and singly exponential in c and the width k of a given multi-k-expression. For these tasks, the running time as a function of the multi-clique-width is the same as the running time of the fastest known algorithm as a function of the clique-width. This results in an exponential speed-up for some graphs, if the corresponding graph generating expressions are given. The reason is that the multi-clique-width is never bigger, but is exponentially smaller than the clique-width for many graphs.
In addition **Matthias Mnich** (University of Bonn) showed interesting improved Approximation Algorithm for Minimum Feedback Vertex Sets in Tournaments. His research together with Lászlo A. Végh (London School of Economics and Political Science) considered the minimum feedback vertex set problem in tournaments, which finds applications in ranking scenarios. This problem is NP-hard to solve exactly, and Unique Games-hard to approximate by a factor better than two. They presented an approximation algorithm for this problem that improves on the previously best known ratio 5/2, given by Cai et al. in FOCS 1998 / SICOMP 2001.

Constraint Satisfaction and Voting

Valia Mitsou (Hungarian Academy of Sciences) started with thoughts on Complexity and Approximability of Parameterized CSP which were written down together with Holger Dell, Eunjung Kim, Michael Lampis, and Tobias Mömke. The complexity of various Constraint Satisfaction Problems (CSP) when parameterized by structural measures (such as treewidth or clique-width) is a well-investigated area. They took a fresh look at such questions from the point of view of approximation, considering four standard CSP predicates: AND, OR, PARITY, and MAJORITY. By providing new or tighter hardness results for the satisfiability versions, as well as approximation algorithms for the corresponding maximization problems, they showed that already these basic predicates display a diverse set of behaviors, ranging from being FPT to optimize exactly for quite general parameters (PARITY), to being W-hard but well-approximable (OR, MAJORITY) to being W-hard and inapproximable (AND). Their results indicate the interest in posing the question of approximability during the usual investigation of CSP complexity with regards to the landscape of structural parameters.
She was followed by **André Nichterlein** (Technische Universität Berlin) with his talk *FPT approximation schemes for Shift Bribery* which is a joint work with Robert Bredereck, Jiehua Chen, Piotr Faliszewski, and Rolf Niedermeier. In the Shift Bribery problem, they were given an election (based on preference orders), a preferred candidate p, and a budget. The goal was to ensure that p wins by shifting p higher in some voters preference orders. However, each such shift request came at a price (depending on the voter and on the extent of the shift) and they had to minimize the overall costs. They showed FPT approximation schemes for the Copeland voting rule (the winner is the candidate winning the most head-to-head competitions) with respect to each of the parameters number of voters and number of candidates.

Participants

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Chapter 29

Connecting Network Architecture and Network Computation (15w5158)

December 6 - 11, 2015

Organizer(s): Andrea Barreiro (Southern Methodist University), Maurice Chacron (McGill University), Brent Doiron (University of Pittsburgh), Chris Eliasmith (University of Waterloo), Krešimir Josić (University of Houston), Eric Shea-Brown (University of Washington)

Overview of the Field

The brain solves an astonishing array of computational tasks – from representation, compression, and transformation of highly multivariate and multimodal sensory inputs, to statistical inference based on these inputs, to the expression and control of motor outputs in response. This is accomplished with speed and precision that easily outpaces the most powerful computers and best known algorithms today. Ultimately, the architectures — connectivity and dynamics — of neural circuits are responsible for this success. Understanding the link between architecture and computation is an inescapably mathematical challenge at the core of modern biological science.

Neural circuits achieve their computational power despite operating under severe constraints. Energetic considerations restrict the number of neurons that are available for any given function, together with the number and range of connections among cells. Moreover, communication between each cell incurs a delay. Biological circuits that integrate information from disparate sources need to preserve the relative timing of information arriving along different pathways – a particular challenge given the noise and unreliability of individual synaptic connections in neuronal networks. Finally, the large numbers of inputs that cells receive implies that neuronal networks need to operate in a state of precise balance between excitation and inhibition to prevent runaway synchrony.

Recent Developments and Open Problems

A variety of new experimental techniques provide unprecedented clues into how are neuronal networks structured to meet these constraints. In particular, new multielectrode and imaging techniques are revealing the simultaneous activity of neural ensembles — and, in some cases, entire neural populations — on a huge range of scales. At the same time, new methods allow the precise mapping of microcircuit anatomy (including both cellular dynamics and intercellular connectivity structures). In addition, techniques from microstimulation and optogenetics further allow us to manipulate the activity of specific cell types during meaningful behaviors.

Interpreting and optimizing these new empirical methods and the large data sets that they produce requires the development of new mathematics. There is a dual challenge: properly analyzing data with highly nonlinear dependencies among hundreds to thousands of variables, and developing mathematical and computational models that reveal and explain the relationship between network architecture and function. To understand the structure-function relationship – how neurons that display highly nonlinear and stochastic dynamics in isolation interact cooperatively when arranged in a circuit in order to perform behaviorally relevant computations under biological constraints – the research community must overcome major challenges in applied mathematics.

Three branches of the mathematical sciences must be brought together to enable the next set of advances – (1) dynamical systems, especially high-dimensional dynamics, (2) stochastic analysis, especially coupled random processes, and (3) the theory of network computation. Moreover, the space of possible network architectures and dynamics is so vast that working with experimentalists to guide and constrain the mathematical developments will be essential.

The aim of this workshop was to unite and provide a forum to focus the work of a group of international experts on network dynamics, information theory, and biological computation on the mathematical interface between network structure and function.

1. **Dynamical systems and network oscillations:** Recurrently (feedforward-feedback) coupled networks of spiking neurons often show synchronous activity. Mathematical analysis has revealed the mechanisms by which asynchronous activity loses stability and synchronous population rhythms arise. These mechanisms – and the specific patterns of synchronous rhythms that emerge – depend on rich interactions between network structure, coupling type, and single-oscillator dynamics. Moreover, recent research has shown that rhythms with distinct frequencies appear to interact. Unraveling the dynamical mechanisms of such interactions poses a new set of challenges that are only beginning to be addressed. How synchronous patterns are modified, created, and destroyed when networks are driven by external stimuli (e.g., sensory inputs) is another essential question that is being addressed using these mathematical tools.

2. **Statistical mechanics of network correlations:** Correlations can develop due to overlapping input in purely feedforward networks with irregular, stochastic activity. This is of particular importance for *layered* network architectures ubiquitous in neuroscience, where the propagation and amplification of correlated activity has been studied in systems ranging from cultured neural circuits to intact brains. Here, mathematical analysis seeks to quantify how correlated activity – modeled via multivariate point processes – is transferred among layers. This is a critical challenge; while it is evident from neural recordings that weak correlations are often present and presumably play an important part in normal brain function, excessive correlations are associated with neurological diseases, such as Parkinson's disease and epilepsy. Other current challenges focus on higher-order (beyond pairwise) correlations, and on how these correlation patterns depend on the spatiotemporal structure of stimuli.

3. **Information theory of network coding:** Neuroscience observations from high density electrode arrays are becoming more prevalent, posing the challenge of interpreting data recorded simultaneously from approximately 100 spatial locations. At the same time, results from information theory show that even weak correlations and synchrony can have strong effects on stimulus coding. However, whether these effects improve or degrade coding depends on the spatiotemporal structure of the collective activity. The primary challenge is to develop a systematic framework that predicts the impact of correlations in specific cases, and generalizes to allow an intuitive understanding of the underlying mechanisms of information encoding and decoding.

The question that unites these three areas is:

> **What are the information-theoretic consequences of the correlation and synchrony patterns that arise through the dynamics of prototypical neural circuits?**

Presentation Highlights

The meeting brought together a range of theoretical and computational neuroscientists, and experimental researchers interested in theory to address different aspects of this question. The quality of the talks was high, and – importantly – the presenters heeded the organizer's request to keep the talks accessible. This was particularly important due to the varied backgrounds of the participants. A number of graduate students and postdoctoral fellows also presented their research.

Below we highlight some of the presentations:

Emre Aksay (Weill Cornell Medical College)
Jacob Davidson (University of California Davis)

Network architectures underlying persistent neural activity

Persistent neural activity is important for motor control, short-term memory, and decision making. It is unclear what network processing mechanisms and architectures support this brain dynamic. In this joint presentation an experimental neuroscientist (Aksay), and a computational neuroscientist (Davidson) showed their recent efforts to address this question in the oculomotor integrator, a model system for studying persistent neural activity. They have determined candidate architectures by fitting dynamical network models to population-wide recordings with additional constraints from experiments on cellular morphology, intrinsic excitability, and localized perturbations. They also tested candidate architectures by imaging activity in the dendritic arbor of integrator neurons during persistent firing. These efforts suggest architectures of higher rank than previously assumed. Such architectures may allow persistent activity networks to act as hubs that perform numerous input-output transformations.

John Beggs (Indiana University)

High-degree neurons feed cortical computations

Recent results have shown that functional connectivity among cortical neurons is highly varied, with a small percentage of neurons having many more connections than others. Also, new theoretical work makes it possible to quantify how neurons modify information from the connections they receive. These developments have allowed John Beggs and his lab to investigate how information modification, or computation, depends on the number of connections a neuron receives (in- degree) or sends out (out-degree). John described how his lab uses a high-density 512 electrode array to record spontaneous spiking activity from cortical slice cultures and transfer entropy to construct a network of information flow. They have identified generic computations by the synergy produced wherever two information streams converged, and found that computations did not occur equally in all neurons throughout the networks. Surprisingly, the in-degree of a neuron was not related to the amount of information it computed. Instead, neurons that computed large amounts of information tended to receive connections from high out-degree neurons. To gain insight into these findings, the lab has developed a simple feedforward network model, and found that a degree- modified Hebbian wiring rule best reproduced the pattern of computation and degree correlation results seen in the real data. Interestingly, this rule also maximized signal propagation in the presence of network-wide correlations, suggesting a mechanism by which cortex could deal with common random background input. These are the first results to show that the extent to which a neuron modifies incoming information streams depends on its topological location in the surrounding functional network. Co-authors: Nick Timme and Sunny Nigam

Braden Brinkman (University of Washington)

Crouching tiger, hidden neuron

A major obstacle to understanding population coding in the brain is that neural activity can only be monitored at limited spatial and temporal scales. Inferences about network properties important for coding, such as connectivity between neurons, are sensitive to hidden units: unobserved neurons or other inputs that drive network activity. This problem is important not just for understanding inference from data, but also for which network properties shape spike train statistics as subsampled or pooled signals are transmitted through the brain. Recent computational efforts have been made to fit models to hidden units, but a fundamental theory of the effects of unobserved influences on the statistics of subsampled or pooled network activity remains elusive. Braden Brinkman showed how the methods from statistical physics can be used to develop an analytical framework to begin answering questions about how ground truth properties of neuronal networks are distorted when an experimenter (or downstream neuron) can only observe coarsely resolved activity data. As a specific example, he showed how the coupling filters of a generalized linear model fit to pooled spike train data change as a function of the fraction of spike trains pooled together. Coauthors: Fred Rieke, Eric Shea-Brown, Michael Buice

Nicholas Brunel (University of Chicago)

Statistics of connectivity optimizing information storage in recurrent networks

The rules of information storage in cortical circuits are the subject of ongoing debate. Two scenarios have been proposed by theorists: In the first scenario, specific patterns of activity representing external stimuli become fixed-point attractors of the dynamics of the network. In the second, the network stores sequences of patterns of network activity so that when the first pattern is presented the network retrieves the whole sequence. In both scenarios, the right dynamics are achieved thanks to appropriate changes in network connectivity. Nicolas Brunel described how methods from statistical physics can be used to investigate information storage capacity of such networks, and the statistical properties of network connectivity that optimize information storage (distribution of synaptic weights, probabilities of motifs, degree distributions, etc) in both scenarios. In the final part of the talk Nicolas compared the theoretical results with available data.

Kathleen Cullen (McGill University)

Neural correlates of sensory prediction errors during voluntary self-motion: evidence for internal models in the cerebellum.

The computation of sensory prediction errors is an important theoretical concept in motor control. In this context, the cerebellum is generally considered as the site of a forward model that predicts the expected sensory consequences of self-generated action. Changes in the motor apparatus and/or environment will cause a mismatch between the cerebellum's prediction and the actual resulting sensory stimulation. Thus this mismatch — the sensory prediction error — is thought to be vital for updating both the forward model and motor program during motor learning to ensure that sensory-motor pathways remain calibrated. In addition, through our daily activities, the computation of sensory prediction errors is required to discriminate externally-applied from self-generated inputs. However, direct proof for the existence of this comparison had been lacking.

Kathleen Cullen described how she and her collaborators took advantage of a relatively simple sensory-motor pathway with a well-described organization to gain insight into the computations that drive motor learning. The most medial of the deep cerebellar nuclei (fastigial nucleus), constitutes a major output target of the cerebellar cortex and in turn sends strong descending projections that ensure accurate posture and balance. They carried out a trial-by-trial analysis of these cerebellar neurons during the execution and adaptation of voluntary head movements and found that neuronal sensitivities dynamically tracked the comparison of predictive and feedback signals. When the relationship between the motor command and resultant movement was altered, neurons robustly responded to sensory input as if the movement was externally generated. Neuronal sensitivities then declined with the same time course as the concurrent behavioral learning. These findings demonstrate the output of an elegant computation in which rapid updating of an internal model enables the motor system to sense, and then learn to expect, previously unexpected sensory inputs. This enables both the i) rapid suppression of descending reflexive commands during voluntary movements and ii) rapid updating of motor programs in the face of changes to either the motor apparatus or external environment.

Jeff Dunworth (University of Pittsburgh)

Finite size effects and rare events in balanced cortical networks with plastic synapses

Cortical neuron spiking activity is broadly classified as temporally irregular and asynchronous. Model networks with a balance between large recurrent excitation and inhibition capture these two key features, and are a popular framework relating circuit structure and network dynamics. Balanced networks stabilize the asynchronous state through reciprocal tracking by the inhibitory and excitatory population activity, leading to a cancellation of total current correlations driving cells within the network. While asynchronous network dynamics are often a good approximation of neural activity, in many cortical datasets there are nevertheless brief epochs wherein the network dynamics are transiently synchronized (Buzsáki and Mizuseki (2014), Tan et al. (2014)). Jeff Dunworth described an analysis of paired whole cell voltage-clamp recordings from spontaneously active neurons in mouse auditory cortex slices (Graupner and Reyes (2013)) showing a network where correlated excitation and inhibition effectively cancel, except for intermittent periods when the network shows a macroscopic synchronous event. These data suggest that while the core mechanics of balanced activity are important, we require new theories capturing these brief but powerful periods when balance fails. Traditional balanced networks with linear firing rate dynamics have a single attractor, and fail to exhibit macroscopic synchronous events. Mongillo et al. (2012) showed that balanced networks with short-term synaptic plasticity can depart from strict linear dynamics through the emergence of multiple attractors. Jeff Dunworth and collaborators extended this model by incorporating finite network size, introducing strong nonlinearities in the firing rate dynamics and allowing finite size induced noise to elicit large scale, yet infrequent, synchronous events. They carried out a principled finite size expansion of an associated Markovian birth-death process and identified core requirements for system size and network plasticity to capture the transient synchronous activity observed in experimental data. This model properly mediates between the asynchrony of balanced activity and the tendency for strong recurrence to promote macroscopic population dynamics.

Stefano Fusi (Columbia University)

Computational principles of synaptic plasticity

Memories are stored, retained, and recollected through complex, coupled processes operating on multiple timescales. To understand the computational principles behind these intricate networks of interactions Stefano Fusi and collaborators constructed a broad class of synaptic models that efficiently harnesses biological complexity to preserve numerous memories. In these models the memory capacity scales almost linearly with the number of synapses, which is a substantial improvement over the square root scaling of previous models. This was achieved by combining multiple dynamical processes that initially store memories in fast variables and then progressively transfer them to slower variables. Importantly, the interactions between fast and slow variables are bidirectional. The proposed models are robust to parameter perturbations and can explain several properties of biological memory, including delayed expression of synaptic modifications, metaplasticity, and spacing effects.

Julijana Gjorgjieva (Brandeis University)

Optimal sensory coding by neuronal populations

In many sensory systems the neural signal splits into multiple parallel pathways, suggesting an evolutionary fitness benefit of a very general nature. For example, in the mammalian retina, ~ 20 types of retinal ganglion cells transmit information about the visual scene to the brain. What factors drove the evolution of such an early and elaborate pathway split remains elusive. Gjorgjieva and collaborators test the hypothesis that pathway splitting enables more efficient encoding of sensory stimuli, in the context of a specific prominent instance of sensory splitting: the emergence of ON and OFF pathways that code for stimulus increments and decrements, respectively. They developed a theory of optimal coding for a population of sensory ON and OFF neurons and computed the coding efficiency for different mixtures of ON and OFF cells. They found that optimal ON-OFF ratio in the population can be related to the statistics of natural stimuli, resulting in set of predictions for the optimal response properties of the neurons.

Cameron Harris (University of Washington)

Role and limits of inhibition in an excitatory burst generator

The pre-Bötzinger complex (preBötC) is now recognized as the essential core of respiratory rhythm generation, where it generates the inspiratory phase. Rhythmogenesis occurs through network synchronization. Harris and collaborators use a biophysical model of the entire preBötC to investigate several questions: What is the role of inhibitory cells in the preBötC? How does changing the sparsity of connections and synaptic strengths affect the resulting rhythm? These modeling results were compared to *in vitro* slice experiments in which inhibitory and excitatory synaptic transmission were progressively blocked. Harris found that too much sparsity or inhibition disrupts rhythm generation, yet highly connected networks without inhibition also produce non-biological rhythms. Their slice experiments suggest that the real preBötC lies within the partially synchronized region of network parameter space. As inhibitory neurons are added to the network, some cells fire out-of-phase with the main population rhythm, which offers an explanation for the out-of-phase cells observed in preBötC. However, Harris also reported that it is not possible to produce a two-phase population rhythm in their model, without adding further structure to the network. The preBötC and Bötzinger complexes therefore require structured networks in order to produce alternating inspiratory and expiratory rhythms. Finally, Harris presented preliminary stages of a spin model for oscillator phases which reproduces the qualitative features of the synchronization transition.

Kathryn Hedrick (Southern Methodist University)

Megamap: Flexible representation of a large space embedded with nonspatial information by a hippocampal attractor network

The problem of how the hippocampus encodes both spatial and nonspatial information at the cellular network level remains largely unresolved. Spatial memory is widely modeled through the theoretical framework of attractor networks, but standard computational models can only represent spaces that are much smaller than the natural habitat of an animal. Hedrick and collaborators propose that hippocampal networks are built upon a basic unit they call a *megamap*, or a cognitive attractor map in which place cells are flexibly recombined to represent a large space. Its inherent flexibility gives the megamap a huge representational capacity and enables the hippocampus to simultaneously represent multiple learned memories and naturally carry nonspatial information at no additional cost. On the other hand, the megamap is dynamically stable, as the underlying network of place cells robustly encodes any location in a large environment given a weak or incomplete input signal from the upstream entorhinal cortex. Hedrick's results suggest a general computational strategy by which a hippocampal network enjoys the stability of attractor dynamics without sacrificing the flexibility needed to represent a complex, changing world.

Zachary Kilpatrick (University of Houston and University of Colorado, Boulder)

Learning the volatility of a dynamic environment

Humans and other animals make perceptual decisions based on noisy sensory input. Recent studies focus on ecologically realistic situations in which the correct choice or the informative features of the stimulus change dynamically. Importantly, optimal evidence accumulation in changing environments requires discounting prior evidence at a rate determined by environmental volatility. To explain these observations, Zachary Kilpatrick showed how to extend previous accumulator models of decision making to the case where the correct choice changes at an unknown rate. An ideal observer can optimally infer these transition rates and accumulate evidence to make the best decision. He also discussed a neural implementation for this inference process whereby Hebbian plasticity shapes connectivity between populations representing each choice.

Ashok Litwin-Kumar (Columbia University)

Learning associations with both pure and randomly mixed representations

To make decisions and guide actions based on sensory information, neurons that mediate behavior must learn to respond appropriately to combinations of previously experienced stimuli and/or contexts. Many models of learning assume this is accomplished by a feedforward hierarchy of layers of neurons leading from input to desired output. However, sensory information is often relayed by multiple convergent and divergent pathways, each of which may have different representations of the input. Ashok Litwin-Kumar discussed the ability of output neurons that receive both pure stimulus information and randomly mixed stimulus/context information via an indirect pathway to perform associative learning. He showed that for realistic input-output mappings, the optimal pattern of connectivity is an intermediate one that includes input from both pure and mixed representations converging on the output layer. He also discussed the optimal level of mixing to maximize behavioral performance, finding, surprisingly, that sparse connectivity improves performance compared to the fully connected case. These results shed light on the principles governing learning from random representations, a strategy employed in many areas of the brain.

Artur Luczak (University of Lethbridge)

Neuronal activity packets as basic units of neuronal code

Neurons are active in a coordinated fashion, for example, an onset response to sensory stimuli usually evokes a $\sim 50-200$ms long burst of population activity. Recently it has been shown that such 'packets' of neuronal activity are composed of stereotypical sequential spiking patterns. The exact timing and number of spikes within packets convey information about the stimuli. Artur Luczak presented evidence that packets can be a good candidate for basic building blocks or 'the words' of neuronal coding, and can explain the mechanisms underlying multiple recent observations about neuronal coding, such as: multiplexing, LFP phase coding, and provide a possible connection between memory preplay and replay. This presentation summarized and expanded on the opinion paper: Luczak et al. (2015, Nature Rev. Neurosci.; doi:10.1038/nrn4026)

Cheng Ly (Virginia Commonwealth University)

Firing Rate Statistics with Intrinsic and Network Heterogeneity

Heterogeneity of neural attributes has recently gained a lot of attention and is increasing recognized as a crucial feature in neural processing. One outstanding question is how two distinct sources of heterogeneity — synaptic weights at the network level, and intrinsic excitability at the single-cell level — interact and alter neural activity. Cheng Ly used a recurrent spiking neural network model to study how these two forms of heterogeneity lead to different distributions of firing rates. Even in an uncoupled network, intrinsic heterogeneity naturally leads to firing rate heterogeneity. Coupling can lead to amplification or attenuation of firing rate heterogeneity, depending on both the relationship between intrinsic and network heterogeneity and the operating regime of the recurrent network: in particular, whether it is firing asynchronously or rhythmically. To analytically characterize these observations, Ly employed dimension reduction methods and asymptotic analysis to derive compact analytic descriptions of the phenomena. These descriptive formulas show how these two forms of heterogeneity determine the firing rate heterogeneity in various settings.

Dani Marti (École Normale Superieure)

Structured connectivity as a source of slow dynamics in randomly connected networks

Cortical networks exhibit dynamics on a range of timescales. Slow dynamics at the timescale of hundreds of milliseconds to seconds carry information about the recent history of the stimulus, and can therefore act as a substrate for short-term memory. How networks composed of fast units, like neurons, can generate such slow dynamics is still an open question. One possible mechanism is based on positive feedback: in randomly connected networks, the collective timescale can be set arbitrarily long by balancing the intrinsic decay rate of individual neurons with recurrent input. This type of mechanism relies however on fine-tuning the synaptic coupling.

Another possibility is that slow dynamics are induced by structured connectivity between neurons. In fact, the connectivity of cortical networks is not fully random. The simplest and most prominent deviation from randomness found in experimental data is the overrepresentation of bidirectional connections among pyramidal cells. Dani Marti argued that symmetry in the connectivity can act as a robust mechanism for the generation of slow dynamics in networks of fast units.

Using numerical and analytical methods, Dani Marti investigated the dynamics of networks with partially symmetric structure. He considered the two dynamical regimes exhibited by random neural networks: the weak-coupling regime, where the firing activity decays to a single fixed point unless the network is stimulated, and the strong-coupling or chaotic regime, characterized by internally generated fluctuating firing rates. He determined how symmetry modulates the timescale of the noise filtered by the network in the weak-coupling regime, as well as the timescale of the intrinsic rate fluctuations in the chaotic regime. In both cases symmetry increases the characteristic asymptotic decay time of the autocorrelation function. Furthermore, or sufficiently symmetric connections the network operating in the chaotic regime exhibits aging effects, by which the timescale of the rate fluctuations slowly grows as time evolves. Such history-dependent dynamics might constitute a new mechanism for short-term memory storage in random networks.

Michael G Metzen (McGill University)
Volker Hofmann (McGill University)

The role of neural correlations in information coding

The role of correlated neural activity in neural coding remains controversial. The two speakers showed that correlated neural activity can provide information about particular stimulus features independently of single neuron activity using the weakly electric fish, Apteronotus leptorhynchus as an animal model. These fish generate an electric organ discharge (EOD) surrounding their body, the amplitude of which is encoded in the discharge of electroreceptors (P-units) that synapse onto pyramidal neurons in the hindbrain electrosensory lateral line lobe (ELL) that in turn synapse onto neurons within the midbrain torus semicircularis (TS). When two conspecifics come into close proximity, each fish experiences a sinusoidal amplitude modulation (i.e. beat) with a frequency that is equal to the difference between both EOD frequencies. The beat can be further modulated due to movements of the animals, thus creating an envelope. Furthermore, these fish can generate communication signals or chirps (i.e. electrosensory objects) that consist of transient increases in EOD frequency and always occur simultaneously with the beat under natural conditions. The pairwise correlation coefficient but not single neuron spiking activity at the periphery: 1) can reliably be used to predict the stimulus envelope and 2) allows for the emergence of a feature invariant representation of natural communication stimuli that is actually exploited by the electrosensory system. Moreover, information carried by correlated neural activity at the periphery is decoded and further refined in downstream brain areas. This gives rise to similar behavioral responses to stimulus waveforms associated with a given electrosensory object. As such, correlated activity codes for stimulus attributes that are distinct from those coded by firing rate and provide a novel role for neural variability. Furthermore, correlated neural activity is invariant to identity preserving transformations of natural stimuli. This reveals how a sensory system exploits this fact in order to implement the emergence and refinement of invariant neural representations of natural stimuli and how these mediate perception and behavior. Herewith we show that neural correlations can serve as an extra channel of information coding independently of single neuron firing rate. The associated neural circuits are generic and thus likely to be found across systems and species.

Stephan Mihalas (Allen Institute of Brain Sciences)

Cortical circuits implementing optimal cue integration

Neurons in the primary visual cortex (V1) predominantly respond to a patch of the visual input, their classical receptive field. These responses are modulated by the visual input in the surround. This reflects the fact that features in natural scenes do not occur in isolation: lines, surfaces are generally continuous. There is information about a visual patch in its surround. This information is assumed to be passed to a neuron in V1 by neighboring neurons via lateral connections. The relation between visual evoked responses and lateral connectivity has been recently measured in mouse V1. In this study we combine these three topics: natural scene statistics, mouse V1 neuron responses and their connectivity. Stephan Mihalas addressed the question: Given a set of natural scene statistics, what lateral connections would optimally integrate the cues from the classical receptive field with those from the surround?

To do so he assumed a neural code: the firing rate of the neuron maps bijectively to the probability of the feature the neuron is representing to be in the presented image. He next generated a database of features these neurons represent by constructing a parameterized set of models from V1 electrophysiological responses, and used the Berkeley Segmentation Dataset to compute the probabilities of co-occurrences of these features. He computed the relation between probabilities of feature co-occurrences and the synaptic weight which optimally integrates these features. The relation between evoked responses and connectivity which leads to optimal cue integration is qualitatively similar to the measured one, but several additional predictions are made. Finally, he hypothesized that this computation: optimal cue integration is a general property of cortical circuits, and the rules constructed for mouse V1 generalize for other areas and species.

Ruben Moreno-Bote (University Pompeu Fabra)

Causal Inference in Spiking Networks

While the brain uses spiking neurons for communication, theoretical research on brain computations has mostly focused on non-spiking networks. The nature of spike-based algorithms that achieve complex computations, such as object probabilistic inference, is largely unknown. Here Ruben Moreno-Bote demonstrated that a family of high-dimensional quadratic optimization problems with non-negativity constraints can be solved exactly and efficiently by a network of spiking neurons. The network infers the set of most likely causes from an observation using explaining away, which is dynamically implemented by spike-based, tuned inhibition.

Katie Newhall (University of North Carolina-Chapel Hill)

Variability in Network Dynamics

Mathematical models of neuronal network dynamics, such as randomly connected integrate-and-fire model neurons, typically create homogeneous dynamics in the sense that a single neuron in the network is representative of the ensemble behavior, and dynamics in time are statistically repeatable. Katie Newhall will discuss work in progress on experimental data in which neither is true, looking at a statistical method to answer biological questions, and pondering the existence of simple model network motifs capable of producing such variability.

Alex Reyes (New York University)

Homeostatic control of neuronal firing rate and correlation: scaling of synaptic strength with network size

Features of sensory input are represented as the spatiotemporal activities of neuronal population. This network dynamics depends on the balance of excitatory (E) and inhibitory (I) drives to individual neurons. Maintaining balance in the face of continuously changing nervous system is vital for preserving the response properties of neurons and preventing neuropathologies. While homeostatic processes are in place to maintain excitatory level, the conditions for maintaining stable responses are yet unclear. Alex Reyes used a culture preparation to systematically vary the density of the network. Using optogenetic techniques to stimulate individual neurons in the network with high spatial and temporal resolution, he was able to systematically vary the number and correlation of external inputs. He found that the average firing rate and the correlation structure of synaptic inputs are invariant with network size. Finally, he showed how paired recordings could be used to measure the synaptic strengths and connection probability between excitatory (E) and inhibitory (I) neurons to confirm experimentally a long standing theoretical assumption that synaptic strength scales with the number of connections per neuron (N) closer to $N^{-1/2}$ than to N^{-1} .

Robert Rosenbaum (University of Notre Dame)

Correlations and dynamics in spatially extended balanced networks

Balanced networks offer an appealing theoretical framework for studying neural variability since they produce intrinsically noisy dynamics with some statistical features similar to those observed in cortical recordings. However, previous balanced network models face two critical shortcomings. First, they produce extremely weak spike train correlations, whereas cortical circuits exhibit both moderate and weak correlations depending on cortical area, layer and state. Second, balanced networks exhibit simple mean-field dynamics in which firing rates linearly track feedforward input. Cortical networks implement non-linear functions and produce non-trivial dynamics, for example, to produce motor responses. We propose that these shortcoming of balanced networks are overcome by accounting for the distance dependence of connection probabilities observed in cortex.

Robert Rosenbaum generalized the mean-field theory of firing rates, correlations and dynamics in balanced networks to account for distance-dependent connection probabilities. He showed that, under this extension, balanced networks can exhibit either weak or moderate spike train correlations, depending on the spatial profile of connections. Networks that produce moderate correlation magnitudes also produce a signature spatial correlation structure. A careful analysis of in vivo primate data reveals this same correlation structure. Finally, he showed that spatiotemporal firing rate dynamics can emerge spontaneously in spatially extended balanced networks. Principal component analysis reveals that these dynamics are fundamentally high-dimensional and reliable, suggesting a realistic spiking model for the rich dynamics underlying non-trivial neural computations. Taken together our results show that spatially extended balanced networks offer a parsimonious model of cortical circuits.

Woodrow Shew (University of Arkansas)

Functional implications of phase transitions in the cerebral cortex

A long-standing hypothesis at the nexus neuroscience, physics, and network science posits that a network of neurons may be tuned through a phase transition. Originally, this idea was motivated by intriguing analogies between the brain and physical systems which undergo phase transitions including Ising and percolation models. More recently, this idea has graduated from an appealing analogy to an experimentally supported and biophysically important fact. Woodrow Shew reviewed recent experiments and models, which have now established not only that phase transitions can occur in cerebral cortex, but also that neural information processing crucially depends on what phase the cortex is in. Cortical phase can be controlled by myriad biophysical mechanisms including tuning the balance of excitation and inhibition and adaptation to sensory input. Importantly, multiple aspects of information processing, such as sensory dynamic range and discrimination are optimized when the network operates nearby (but not exactly at) the critical point of a phase transition. These studies suggest that by operating in the vicinity of criticality the cerebral cortex may be tuned to accommodate changing information processing needs depending on behavioral context.

Aubrey Thompson (Carnegie Mellon University)

Relating spontaneous dynamics and stimulus coding in competitive networks

Understanding the relation between spontaneously active and stimulus evoked cortical dynamics is a recent challenge in systems neuroscience. Recordings across several cortices show highly variable spike trains during spontaneous conditions, and that this variability is promptly reduced when a stimulus drives an evoked response (Churchland, Yu, et al., Nat Neuro, 2010). Aubrey Thompson showed how networks of spiking neuron models with clustered excitatory architecture capture this key feature of cortical dynamics (Litwin-Kumar and Doiron, Nat. Neuro, 2012). In particular, clusters show stochastic transitions between periods of low and high firing rates, providing a mechanism for slow cortical variability that is operative in spontaneous states. She expanded on past work and explored a simple Markov neural model with clustered architecture, where spontaneous and evoked stochastic dynamics can be examined more carefully. She modeled the activity of each cluster in the network as a birth-death Markov process, with positive self feedback and inhibitory cluster-cluster competition. Her Markov model allows a calculation of the expected transition times between low and high activity states, yielding an estimate of the invariant density of cluster activity. Using this theory, she explored how the strength of inhibitory connections between the clusters sets the maximum likelihood for the number of active clusters in the network during spontaneous conditions. This work relates two disparate aspects of cortical computation—lateral inhibition and stimulus coding.

Jochen Triesch (Frankfurt Institute for Advanced Studies)

Wheres the noise? Key features of spontaneous activity and neural variability arise through learning in a deterministic network

Even in the absence of sensory stimulation the brain is spontaneously active. This background noise seems to be the dominant cause of the notoriously high trial-to-trial variability of neural recordings. Recent experimental observations have extended our knowledge of trial-to-trial variability and spontaneous activity in several directions: 1. Trial-to-trial variability systematically decreases following the onset of a sensory stimulus or the start of a motor act. 2. Spontaneous activity states in sensory cortex outline the region of evoked sensory responses. 3. Across development, spontaneous activity aligns itself with typical evoked activity patterns. 4. The spontaneous brain activity prior to the presentation of an ambiguous stimulus predicts how the stimulus will be interpreted. At present it is unclear how these observations relate to each other and how they arise in cortical circuits.

Jochen Triesch demonstrated that all of these phenomena can be accounted for by a deterministic self-organizing recurrent neural network model (SORN), which learns a predictive model of its sensory environment. The SORN comprises recurrently coupled populations of excitatory and inhibitory threshold units and learns via a combination of spike-timing dependent plasticity (STDP) and homeostatic plasticity mechanisms. Similar to balanced network architectures, units in the network show irregular activity and variable responses to inputs. Additionally, however, the SORN exhibits sequence learning abilities matching recent findings from visual cortex and the networks spontaneous activity reproduces the experimental findings mentioned above. Intriguingly, the networks behaviour is reminiscent of sampling-based probabilistic inference, suggesting that correlates of sampling-based inference can develop from the interaction of STDP and homeostasis in deterministic networks. He concluded that key observations on spontaneous brain activity and the variability of neural responses can be accounted for by a simple deterministic recurrent neural network which learns a predictive model of its sensory environment via a combination of generic neural plasticity mechanisms.

Aaron Russel Voelker (University of Waterloo)

Computing with temporal representations using recurrently connected populations of spiking neurons

The modeling of neural systems often involves representing the temporal structure of a dynamic stimulus. Aaron Voelker extended the methods of the Neural Engineering Framework (NEF) to generate recurrently connected populations of spiking neurons that compute functions across the history of a time-varying signal, in a biologically plausible neural network. To demonstrate the method, he proposed a novel construction to approximate a pure delay, and use that approximation to build a network that represents a finite history (sliding window) of its input. Specifically, he solved for the state-space representation of a pure time-delay filter using Pade-approximants, and then map this system onto the dynamics of a recurrently connected population. This construction is robust to noisy inputs over a range of frequencies, and can be used with a variety of neuron models including: leaky integrate-and-fire, rectified linear, and Izhikevich neurons. Furthermore, he extended the approach to handle various models of the post-synaptic current (PSC), and characterize the effects of the PSC model on overall dynamics. Finally, he showed that each delay may be modulated by an external input to scale the spacing of the sliding window on-the-fly.

Joel Zylberberg (University of Colorado School of Medicine)

Correlated stochastic resonance

Even when repeatedly presented with the same stimulus, sensory neurons show high levels of inter-trial variability. Similarly high levels of variability are observed throughout the brain, leading us to wonder how variability affects the function of neural circuits.

One the one hand, prior work on stochastic resonance (SR) has shown that random fluctuations can enhance information transmission by nonlinear circuit elements like neurons. Specifically, the thresholding inherent in spike generation means that much of the information contained within the membrane potential can fail to propagate downstream. Random membrane potential fluctuations soften spike thresholds, allowing more information to survive the spike-generation process. This phenomenon reflects a tradeoff between the positive effects of threshold-softening, and the negative effects of corrupting signals by noise.

While membrane potential fluctuations are often correlated between neurons in vivo, the role of this collective behavior in SR is largely unknown. Concurrently to the SR studies, other work investigated the impact of correlations on signal encoding by noisy non-spiking populations. For these non-spiking models, coding performance is highest when the noise is absent altogether: the noise is always a hindrance to the population codes. Consequently, those studies cannot reveal conditions under which collective variability enhances information coding. Despite these limitations, the prior studies of non-spiking models show that depending on the patterns of inter-neural correlation correlations can mitigate corruption of signals by noise. Combining ideas about correlations, and about SR, Joel Zylberberg showed that correlated membrane potential fluctuations can soften neural spiking thresholds without substantially corrupting the underlying signals with noise, thereby significantly enhancing spiking neural information coding.

Scientific Progress Made

Understanding the mechanisms by which the nervous system represents and processes information is a fundamental challenge for mathematical neuroscience. The goal of this meeting was to bring together experimental and theoretical neuroscientists to discuss the questions that will drive research in mathematical and computational neuroscience for years to come. Large initiatives, like the BRAIN Initiative in the US, and the Human Brain Project in Europe are driving the development of new experimental tools to study neural circuits, and large scale computational models of the brain. However, all participants agreed that an understanding of the resulting data and simulation results will require the development of new mathematical, and statistical approaches.The participants presented a range of such approaches.

One common theme that emerged was the necessity to extend the mathematical tools that have been developed to describe populations of statistically independent neurons. New experimental approaches have revealed that the joint activity of neural populations can carry information, and that neural computations are carried out collectively by many cells. A number of presentations pointe to how correlations, synchrony, collective oscillations and state transitions in population activity are essential to understanding brain function. Mathematical methods have been essential for this understanding, and for the development of computational model that will emulate the brain. It also became clear that, as of yet, no overarching theory or approach seemed to be sufficient to achieve these goals. Participants thus presented a range of techniques and approaches.

Outcome of the Meeting

The meeting succeeded in bringing together a number of leaders in related areas of computational and experimental neuroscience. The participants discussed different approaches to common central questions in neuroscience research: How neural activity subtends neural computation, and how both emerge from the patterns of connections and interaction in neural tissue.

The week was spent seeking bridges among mathematical disciplines, computational and experimental approaches. The ample time that we allowed for discussions, and the excellent facility at BIRS, resulted in discussions that continued far after the end of each talk. Many meeting participants reported that they have made new connections, learned about new techniques and ideas, and received valuable feedback and comments on their research.

A number of graduate students and postdoctoral fellows participated in the meeting. Nearly all of these participants also gave talks during the meeting. While many of these students came from either side of the mathematics/neuroscience divide, they had no trouble in communicating their ideas to the diverse audience attending the workshop. All talks contained non-trivial mathematics, but presented in a way understandable to the participating experimentalists (admittedly, a selected group). We also observed, that while some of the presented research made use of fairly sophisticated mathematical ideas, all of it was well motivated by questions pertinent to neuroscientists.

Participants

Aksay, Emre (Weill Cornell Medical College)
Barreiro, Andrea (Southern Methodist University)
Beggs, John (Indiana University)
Best, Janet (Ohio State University)
Brinkman, Braden (University of Washington)
Brunel, Nicolas (University of Chicago)
Buice, Michael (Allen Institute of Brain Science)
Chacron, Maurice (McGill University)
Cullen, Kathleen (McGill University)
Davidson, Jacob (University of California, Davis)
Diesmann, Markus (Jlich Research Centre)
Doiron, Brent (University of Pittsburgh)
Dragoi, Valentin (University of Texas at Houston)
Dunworth, Jeff (University of Pittsburgh)
Eliasmith, Chris (University of Waterloo)
Fusi, Stefano (Columbia University)
Gjorgjieva, Julijana (Brandeis University)
Harris, Kameron (University of Washington)
Hedrick, Kathryn (Southern Methodist University)
Hofmann, Volker (McGill University)
Josic, Kresimir (University of Houston)
Kilpatrick, Zachary (University of Colorado)
Litwin-Kumar, Ashok (Columbia University)
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Reyes, Alex (New York University)
Rosenbaum, Robert (University of Notre Dame)
Sharpee, Tatyana (The Salk Institute for Biological Studies)
Shea-Brown, Eric (University of Washington)
Shew, Woodrow (University of Arkansas)
Stolarczyk, Simon (University of Houston)
Thompson, Aubrey (University of Pittsburgh)
Triesch, Jochen (Frankfurt Institute for Advanced Studies)
Voelker, Aaron (University of Waterloo)
Zylberberg, Joel (University of Colorado)

Two-day Workshop Reports

Chapter 30

Incorporating 'Computational Thinking' into the Grade-school Classroom (15w2187)

January 16 - 18, 2015

Organizer(s): Tim Bell (University of Canterbury), Sean Graves (University of Alberta), Geri Lorway (Thinking 101)

Introduction

The goal of this project was to bring together educators and researchers from the fields of computer science, applied mathematics and Education K to 12 as well as University and College levels, to consider questions, concerns and issues that surround the international attention being paid to developing a computer science curriculum for the elementary grades. There are several current frameworks for thinking, learning and teaching that might make the integration of topics and skills from computer science more manageable in terms of classroom teachers and their planning for including computer science and computational thinking in their teaching and in their assessment of student learning.

The Computer Science Unplugged materials, co-authored by Tim Bell, Mike Fellows and Ian Witten, offer a starting point for teachers who wish to investigate ideas around computer science and the trending themes of "computational thinking" and "coding". Jeanette Wing's challenge to include computational thinking as the fourth R when considering curriculum in the 21st Century offers another possible starting point. (See article in appendix)

Teaching for Thinking, Critical Thinking, Design Thinking, STEM Sciences are all terms that teachers are hearing, reading about and trying to make sense of as they respond to the constantly evolving dialogue around what constitutes the skills, attitudes and knowledge base we need our students to develop. The goal of this weekend was to consider ways to begin to connect these "terms" and "trends" into a connected whole. Finding Common Ground across the Disciplines might be one way to describe it. The questions that emerge for teachers: Where will we? How do we? Why should we? position Computational Thinking and Computer Science within our evolving conceptions of curriculum and competencies for 21st Century Learners?

Themes

Computational thinking

It is a difficult term to define but we all have a better idea of what it is about after the weekend.

Do we need to define it?

A starting point for the discussion of computational thinking is offered by Woollard and Selby as follows:

> "As supported by the preceding arguments, computational thinking is an activity, often product oriented, associated with, but not limited to, problem solving. It is a cognitive or thought process that

reflects

- the ability to think in abstractions,
- the ability to think in terms of decomposition,
- the ability to think algorithmically,
- the ability to think in terms of evaluations, and
- the ability to think in generalizations.

This proposed definition attempts to incorporate only those terms for which there is a consensus in the literature or those terms that are well defined across disciplines. The intent is to focus on the thinking aspect of the original phrase. In other words, computational thinking is a focused approach to problem solving, incorporating thought processes that utilize abstraction, decomposition, algorithmic design, evaluation, generalizations."

Selby,C. Woollard, J. , Computational Thinking: The Developing Definition, SIGCSE 2014, 5-8 March, Atlanta GA

Computer science isn't about computers

One of the themes of the workshop was that *computer science is mainly about people, not about computers*. This comes up in several ways: first, that computer software is developed essentially to make the world a better place for people; second, that there are exciting jobs available for graduates, yet many don't get a taste for what is involved and therefore don't find their calling; and third, that all interfaces need to take account of the human using them, such as keeping response times under 0.1 seconds to appear instant. A related topic that came up was sustainability; wasting time on computation creates more power usage and requires more cooling, which has a direct impact on the environment. Also, good algorithms for planning can reduce the distance travelled by vehicles, and again reduce the impact on the environment, improve the safety for the people involved, and reduce the amount of human time required for a delivery. These connections seemed to resonate with the educators at the workshop, and made them realize both the importance of the subject, and also the value of having a more diverse range of students access ideas from computer science through school curriculum.

A growing theme among those who study thinking: What do we mean by Community Thinking and how does it relate here? Community thinking, Sustainability,

Design Thinking How and where does design thinking fit? How does computational thinking support, link to, develop from, merge with design thinking and the current move to Maker/Thinker Explorations in the classroom? Susan Crichton offered some potential sources of insight:

- http://blogs.ubc.ca/centre/2013/11/18/maker-day-tool-kit/
- http://theelearningcoach.com/elearning_design/design-thinking-for-instructional-desig/

Visual Spatial Reasoning

Nora Newcombes articles offer an interesting and engaging entry point for teachers to consider the role that visual spatial reasoning plays in the learning of science, math, social studies. The design of the Computer Science Unplugged materials definitely meshes with many of the key aspects of learning in math and science that Newcombe outlines.

The group was challenged to consider how the "packaging" of knowledge and content might help keep the focus on process not product when we are designing and delivering curriculum.

Picture This: Available at www.aft.org/periodical/american-educator/summer-2010/picture

Seeing Relationships: Available for download at

http://www.aft.org/sites/default/files/periodicals/Newcombe_0.pdf

STEM vs STEAM What about the link to the ARTS and the role of the arts in conceptions of the STEM disciplines...

Connecting to 21st Century Competencies Currently many education systems are re-aligning curriculums to better reflect a need to focus on process, not product. In Alberta, the Ministerial Order for Student Learning, May, 2013 identifies the competencies that Alberta teachers need to expect their students to develop. Teachers who participated in the weekend indicated that building a framework, with specific examples linked to actual curriculums currently in use is a project they would like to pursue and plans are underway to re-connect this group in the near future to continue that work.

Feedback

- I want to thank you again for making arrangements for me to attend the Computational Thinking workshop this weekend. I found it truly inspiring, not the least of which was watching all the light bulbs go off for this dedicated group of elementary teachers from across the province. Thank you for your dedication to mathematics education and for gathering us all together for this important and timely event.
- You were definitely right when you said we would barely have time to scratch the surface on this topic! I took away a great deal of information, and I'm excited to share with my group tomorrow the perspectives and insight I received.
- I just wanted to say thanks again for including me in the weekend. I really appreciated the opportunity to learn with Tim and the other participants to get a better understanding of computational thinking. The resources are terrific and I introduced binary numbers to my class yesterday. It was pretty exciting to have one of my grade one students recognize the pattern and be able to tell what the next number in the pattern was. You could really see the brains working as we worked out how to show numbers from 1-25 with the cards! I can see so many possible extensions with these ideas.
- Thank you very much for inviting me. I loved having the opportunity to talk to primary teachers
- I have already been contacted by a (non-participant) teacher who heard that the weekend was extremely successful and is asking about when the next one will be!
- I have been sharing the materials, the topics and the insights from the weekend with colleagues and they are anxious to know when there will be a follow-up as they would like to attend.

Participants

Arndt, Linda (Teacher Pembina Hills)
Ayres, Trinity (Calgary Catholic School District)
Bell, Tim (University of Canterbury)
Brown, April (Peace Wapiti School Division No. 76)
Bullock, Jordan (Evergreen Elementary School)
Easton, Dianna (Teacher Calgary Board of Education)
Graves, Sean (University of Alberta)
Greer, Annie (Grande Prairie Public Schools)
Hohn, Tiina (MacEwan University)
Kotyk, Nicole (Evergreen Elementary School)
Krasnikoff, Melissa (Teacher Parkland Schools)
Lambert, Lynn (Christopher Newport University)
Layton, Ryan (President of ATA Ed Technology Council)
Lemay, Julie (Alberta Education)
Lomax, Bill (Career and Technology Studies Alberta Education)
Lorway, Geri (Thinking 101)
McNabb, Gail (Teacher Pembina Hills)
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Rader, Cyndi (Colorado School of Mines)
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Rodriguez, Brandon (Colorado School of Mines)
Sauerborn, Mardelle (Scool District 5)
Simmons, Brian (Calgary Board of Education)
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Warr, Johnathan (PWSD76 teacher)

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Chapter 31

Convex and Combinatorial Geometry Fest (15w2177)

February 13 - 15, 2015

Organizer(s): Abhinav Kumar (MIT), Daniel Pellicer (Universidad Nacional Autonoma de Mexico), Konrad Swanepoel (London School of Economics and Political Science), Asia Ivić Weiss (York University)

Karoly Bezdek and Egon Schulte are two geometers who have made important and numerous contributions to Discrete Geometry. Both have trained many young mathematicians and have actively encouraged the development of the area by professors of various countries.

Karoly Bezdek studied the illumination problem among many other problems about packing and covering of convex bodies. More recently he has also contributed to the theory of packing of equal balls in three-space and to the study of billiards, both of which are important for physicists. He has contributed two monographs [1, 2] in Discrete Geometry.

Egon Schulte, together with Ludwig Danzer, initiated the study on abstract polytopes as a way to enclose in the same theory objects with combinatorial properties resembling those of convex polytopes. An important cornerstone on this topic is his monograph with Peter McMullen [11]. In the latest years he explored the link between abstract polytopes and crystallography.

In this Workshop, which is a continuation of the preceding five-day one, we explored the ways in which the areas of abstract polytopes and discrete convex geometry has been influenced by the work of the two researchers. The last three decades have witnessed the revival of interest in these subjects and great progress has been made on many fundamental problems.

Abstract polytopes

Highly symmetric combinatorial structures in 3-space **Asia Ivić Weiss** kicked off the workshop, discussing joint work with Daniel Pellicer [15] on the combinatorial structure of Schulte's chiral polyhedra. An abstract polyhedron is a partially ordered set that mimics the face lattice of convex polyhedra. It is regular whenever its automorphism group (as partially ordered set) acts transitively on the flags (maximal chains), and it is chiral whenever its automorphism group induces two orbits on flags with the additional property that adjacent flags are in distinct orbits. The original motivation of this work was to determine the existence or non-existence of infinite abstract chiral polyhedra.

No finite abstract chiral polyhedron can be geometrically realized in 3-space with automorphisms corrresponding to geometric symmetries. Egon Schulte found all (infinite) polyhedra in 3-space (that is, symmetric embeddings of abstract polyhedra) that are geometrically chiral. There are three infinite families of chiral polyhedra with finite faces, denoted by $P(a,b)$, $Q(c,d)$ and $Q_1(c,d)$, and three infinite families with infinite faces, denoted by $P_1(a,b)$, $P_2(a,b)$, $P_3(a,b)$. The question now becomes whether any of these is abstractly chiral.

Weiss and Pellicer proved that every polyhedron with finite faces that is geometrically chiral is also combinatorially chiral, hence providing an affirmative answer to the original question. On the other hand, they showed that the polyhedra with infinite faces are all abstractly regular, and that all chiral members in any of these three families are combinatorially isomorphic.

Undine Leopold talked about vertex-transitive polyhedra, where a polyhedron is here meant to be a closed, connected, orientable surface embedded in 3-space which is face-to-face tiled by finitely many plane, simple polygons (that are allowed to be non-convex and coplanar). Leopold has recently completed the classification of vertex-transitive polyhedra for tetrahedral rotation symmetry in her thesis [8], with no new examples besides the previously known polyhedron of genus 3.

For octahedral rotation symmetry there are three known examples, with the combinatorially regular Grünbaum polyhedron of genus 5 among them [6]. It is shown that the genus of any candidate map to be realizable as a polyhedron with this particular symmetry must belong to a finite set, bounded above by 31. However, several hundred polyhedral maps with a vertex-transitive action of the octahedral rotation group can be enumerated on surfaces of genus up to 31. Leopold presented an overview of the recent progress she made on closing this gap.

Constructions of highly symmetric abstract polytopes

Mark Mixer discussed recent progress on the problem of finding abstract regular polytopes of high rank with respect to the degree of their automorphism groups as transitive permutation groups. This is product of joint work with Peter Cameron, Maria Elisa Fernandes and Dimitri Leemans [4].

String C-groups of rank r are groups generated by r involutions satisfying some commutativity property and an intersection condition that hold for Coxeter groups with string diagram. They are in a one-to-one correspondence with the automorphism groups of regular r-polytopes [11, Section 2E]. Every string C-group can be represented by permutation graphs, where the number of vertices corresponds to the degree of the group. These graphs are used to show that if the group is neither alternating nor symmetric, and the degree of the group is n then the rank of the C-group (and of the polytope) must be at most $n/2 + 1$ except for finitely many exceptions, which are completely classified.

Daniel Pellicer gave a new, purely combinatorial construction of abstract polyhedra from abstract 4-polytopes. This generalizes a geometric construction that has been previously used to describe Petrie's skew regular polyhedron with square faces in 3-space as well as two of Coxeter's regular skew polyhedra in 4-space [5]. It was shown that the automorphism group of a polyhedron constructed in this way is the extended automorphism group (the group of automorphisms and dualities) of the initial 4-polytope. Some examples of regular and chiral polyhedra obtained in this way were shown.

Discrete convex geometry

The structure of polytopes

Ted Bisztriczky gave a talk on recent progress that he has made on the combinatorial structure of neighbourly 4-polytopes. A neighbourly d-polytope is a polytope in which any k vertices form a face, for all k up to $d/2$. Thus, in the 4-dimensional case the graph of the polytope is complete. It is known that there is, up to combinatorial equivalence, a unique neighbourly 4-polytope with 6 or 7 vertices, exactly 3 with 8 vertices, 23 with 9 vertices, and in the range from 333 to 432 on 10 vertices. Shemer [16] introduced a sewing construction that enabled the construction of many neighbourly polytopes. Bisztriczky introduced a generalization of the class of totally sewn neighborly polytopes, called k-linked polytopes, and studied their structure.

Nicholas Matteo, in his talk on four-orbit convex polytopes, presented a classification of all the convex polytopes with four flag orbits. They exist only in dimension 7 or less.

The polytopes with one flag orbit are of course the regular polytopes. The polytopes with two flag orbits are as close to being regular as possible, without being regular. These exist only in two or three dimensions, and are the cuboctahedron, the icosidodecahedron, and their duals, together with two infinite families of polygons [9]. The polytopes with three flag orbits have also recently been classified by the speaker [10] (and exist only in dimensions 8 or less).

Tilings and quasicrystals

Muhammad Khan presented his findings on the enumeration of polyomino tilings. He introduced two hypergraph polynomials, the edge cover polynomial and the edge decomposition polynomial. There is a natural way of associating a hypergraph to a tiling problem and he showed that the edge decomposition polynomial of this associated hypergraph is the generating function for the tilings. In this way edge decomposition polynomial can be used to count the number of tilings of a rectangular region by any finite set of polyominoes. He also gave a deletion-based recursive procedure for calculating the edge cover and edge decomposition polynomials of a given hypergraph.

Marjorie Senechal talked about tiling models of matter starting with the ancient Greeks. In the early 19th century tiling models became the paradigm for both crystal form and growth. The Crystallographic Restriction Theorem, stating that a rotation about a point of a 2- or 3-dimensional lattice is either 2-, 3-, 4-, or 6-fold, seemed to "outlaw" symmetry of order 5 in crystals.

Around 1974, Penrose (foreshadowed by Kepler) found tilings with no lattice structure, although having 5-fold symmetry in a generalized sense. His tilings were subsequently explained in a simpler way by De Bruijn as sections of a 5-dimensional lattice. Alan Mackay predicted the existence of quasicrystals with 5-fold symmetry in 1981. Dan Schechtman and his coworkers discovered quasicrystals in 1984. For this Schechtman received the Nobel prize in Chemistry in 2011. What was originally thought of as a minor upheaval in crystallography became a major shift. It might be that quasicrystals are a type of 3-dimensional Penrose tiling, but this is a problematic model, requiring too many tile shapes and too complicated matching rules. Also, the only quasicrystal so far whose structure has been determined (by Takakura et al. 2007), does not behave as a tiling. This creates doubts on the validity of the crystallographic restriction, opening several interesting problems for mathematicians and crystallographers.

Packing and covering by convex bodies **Alexander Litvak** gave a talk on his work with Karoly Bezdek on covering and packing convex bodies by cylinders, following up from [1]. This work contributes to the generalizations and extensions of the celebrated plank problem of Tarski, originally solved by Bang, and later extended by Keith Ball and others. Here a question posed by Bang is investigated.

For $1 \le k \le d-1$, define a k-codimensional cylinder C in $\mathbb{R}^d$ to be a set of the form $C = H + B$, where H is a k-dimensional linear subspace of $\mathbb{R}^d$ and B a measurable subset of $H^\perp$. Define the cross-sectional volume of C with respect to a convex body K by $\mathrm{crv}_K(C) = |B|/|PK|$, where P is the orthogonal projection onto $H^\perp$. This quantity is affine invariant.

It is shown that if K is an ellipsoid and is covered by 1-codimensional cylinders $C_1, \ldots, C_N$, then $\sum_{i=1}^N \mathrm{crv}_K(C_i) \ge 1$. It then follows by the Rogers-Shepard Theorem that if K is an arbitrary convex body covered by 1-codimensional cylinders $C_1, \ldots, C_N$, then $\sum_{i=1}^N \mathrm{crv}_K(C_i) \ge \binom{d}{k}^{-1}$. In the case $k = d-1$, Keith Ball proved the best possible lower bound of 1 in case K is centrally symmetric. When $d = 3$ and $k = 1$, Bang conjectured that the lower bound should be $1/2$. The above result gives $1/3$.

The k-codimensional cylinders $C_i = B_i + H_i$ are defined to form a packing of K if B_i is contained in the orthoginal projection of K onto $H_i^\perp$ and if $C_i \cap C_j \cap K = \emptyset$ for any distinct i and j. It is shown that if K is an ellipsoid in $\mathbb{R}^d$ and $C_1, \ldots, C_N$ are 1- or 2-codimensional cylinders that form a packing of K, then $\sum_{i=1}^N \mathrm{crv}_K(C_i) \le 1$.

Marton Naszódi talked about the classical illumination problem from convex geometry, due to Hadwiger and others. It is easy to see that the d-dimensional Euclidean ball can be covered by $d+1$ translates of its interior. On the other hand, the d-dimensional cube needs 2^d translates of its interior, since no translate of the interior can cover more than one vertex. The Hadwiger conjecture states that each convex body can be covered by at most 2^d translates of its interior. This difficult conjecture is open already in dimension 3.

Naszódi shows that arbitrarily close to the Euclidean ball there are bodies needing a number of translates that is exponential in d. His example is a probabilistic construction [12].

Combinatorial geometry **Konrad Swanepoel** talked about his work with János Pach [13, 14] on a problem asked by Martini and Soltan [11]. Given a finite set S of n points in Euclidean space $\mathbb{R}^d$, we say that two points a and b from S form a double-normal pair if S lies in the closed slab bounded by the two hyperplanes orthogonal to the segment ab passing through a and b. Martini and Soltan asked for the determination of the largest number of double-normal pairs that can occur in a set of n points in $\mathbb{R}^d$. In dimension 2 this maximum equals $3\lfloor n/2 \rfloor$. In dimension 3 there is a construction that places half of the points on a circular arc, and the other two points on another circular arc, in such a way to create a complete bipartite graph, resulting in $\lfloor n^2/4 \rfloor$ double-normal pairs. This turns out to be asymptotically sharp. For points in $\mathbb{R}^3$ that lie on a sphere there is an upper bound of $17n/4 - 6$, which is sharp for infinitely many n. In higher dimensions we show that asymptotically, the maximum number of double-normal pairs is $\frac{1}{2}(1 - 1/k(d))n^2 + o(n^2)$, where $d - O(\log d) \leq k(d) \leq d - 1$.

This work has recently been greatly improved by Andrei Kupavskii [7].

Participants

Berman, Leah (University of Alaska Fairbanks)
Bezdek, Karoly (University of Calgary)
Bisztriczky, Ted (University of Calgary)
Bracho, Javier (UNAM)
Carrancho Fernandes, Maria Elisa (University of Aveiro)
Conder, Marston (University of Auckland)
Foerster, Melanie (University of Calgary)
Hubard, Isabel (UNAM)
Ivic Weiss, Asia (York University)
Khan, Muhammad (University of Calgary)
Leemans, Dimitri (Universit Libre de Bruxelles)
Leopold, Undine (Technische Universitaet Chemnitz)
Litvak, Alexander (University of Alberta)
Matteo, Nicholas (Northeastern University)
Mixer, Mark (Wentworth Institute of Technology)
Monson, Barry (University of New Brunswick)
Naszodi, Marton (cole polytechnique fdrale de Lausanne)
O'Reilly-Regueiro, Eugenia (Universidad Nacional Autonoma de Mexico)
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Senechal, Marjorie (Smith College)
Swanepoel, Konrad (London School of Economics and Political Science)
Williams, Gordon (University of Alaska Fairbanks)

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Chapter 32

Integer Sequences K -12 (15w2178)

February 27 - March 1, 2015

Organizer(s): Gordon Hamilton (MathPickle), Neil Sloane (OEIS Foundation)

The On-Line Encyclopedia of Integer Sequences (OEIS) has many pedagogic gems that remain undiscovered by K–12 educators. These sequences need to be lifted out of obscurity and become a part of every child's experience of mathematics.

The primary objective of the Integer Sequences K–12 conference was to help unearth these gems by finding 13 integer sequences—one for each grade K–12. Here is the list we came up with:

Kindergarten

A034326 Hours struck by a clock.

$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$

Grade 1

A030227 Number of n-celled polyominoes with bilateral symmetry.

$1, 1, 2, 3, 6, 10, 20, 34, 70, 121, 250, 441, 912, 1630, 3375, \ldots$

Grade 2

A243205 The Nasty Mr. Sneeze: consider the $n \times n$ Go board as a graph; remove i nodes and let j be the number of nodes in the largest connected subgraph remaining; then $a(n) = minimum(i + j)$.

$1, 3, 5, 9, 12, 16, 20, 25, 29, 36?, 41?, 47?, \ldots$

Grade 3

A254873 Recamán $[\div, -, +, \times]$: starting at the seed number, 14, the sequence continues by dividing, subtracting, adding or multiplying by the step number, 2. Division gets precedence over subtraction which gets precedence over addition which gets precedence over multiplication. The new number must be a positive integer and not previously listed. The sequence terminates if this is impossible.

$14, 7, 5, 3, 1, 2, 4, 6, 8, 10, 12, 24, 22, 11, 9, 18, 16, 32, 30, 15, 13, 26, 28, 56, 54, 27, 25, 23, 21, 19, 17,$
$34, 36, 38, 40, 20.$ ⟨terminated⟩

Grade 4

A071983 Square chains: the number of permutations (reversals not counted as different) of the numbers 1 to n such that the sum of any two consecutive numbers is a square. Starting with $a(15) = 1$, the sequence is:

$1, 1, 1, 0, 0, 0, 0, 0, 3, 0, 10, 12, 35, 52, 19, 20, 349, \ldots$

OR (some of us want to try these in the classroom)

A253472 Numbers n such that $1, 2, ..., 2n$ can be partitioned into n pairs, where each pair adds up to a perfect square.

$4, 7, 8, 9, 12, 13, 14, 15, 16, \ldots$

Grade 5

A256174 Boomerang Fractions: starting with 1, on the first step add $1/n$, and on subsequent steps either add $1/n$ or take the reciprocal. $a(n) =$ fewest number of steps required to return to 1. (The sequence starts with $a(2)$.)

$4, 9, 7, 20, 6, 33, 13, 23, 16, 62?, 8, 75?, 18, 17, 25, \ldots$

Grade 6

A125508 Integral Fission: a prime factorization tree in which every pair of children is chosen so they are as equal as possible and the largest child goes on the right. $a(n)$ are the lowest numbers for which a new tree shape is encountered.

$2, 4, 8, 16, 20, 32, 40, 64, 72, 88, 128, 160, 176, 200, 220, 256, 272, 288, 320, 336, 360, 400, 420, 460, 480, 512, 540, 544, 640, 704, 864, 880, 920, \ldots$

Grade 7

A039834 Fibonacci numbers (A000045) extended to negative indices. $F(0) = 0$ in the sequence below.

$1, 1, 0, 1, -1, 2, -3, 5, -8, 13, -21, 34, -55, 89, -144, \ldots$

Grade 8

Similar to A226595 Lengths of maximal non touching increasing paths in $n \times n$ grids.

$0, 2, 4, 7, 9, 12, 15, 17, 20, \ldots$

$\ldots$ but actually

"Lengths of maximal non touching increasing paths in $n \times n$ grids starting at the upper left and ending at the lower right."

$0, 2, 4, 6, 9, 12?, 15?, 17?, 20, \ldots$

Grade 9

A069283 The number of ways that n can be written as the sum of at least two consecutive positive integers.

$0, 0, 0, 1, 0, 1, 1, 1, 0, 2, 1, 1, 1, 1, 1, 3, 0, 1, 2, 1, 1, 3, 1, 1, 1, 2, 1, 3, 1, 1, 3, 1, 0, 3, 1, 3, 2, 1, 1, 3, 1, 1, 3, 1, 1, 5, \ldots$

Grade 10

A225745 Smallest k such that n numbers can be picked in $\{1, \ldots, k\}$ with no four in arithmetic progression.

$1, 2, 3, 5, 6, 8, 9, 10, 13, 15, 17, 19, 21, 23, 25, 27, 28, 30, 33, 34, 37, 40, \ldots$

Grade 11

A000108 Catalan numbers: $C(n) = \text{binomial}(2n, n)/(n + 1) = (2n)!/(n!(n + 1)!)$.

$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$

Grade 12

A000127 Maximal number of regions obtained by joining n points around a circle by straight lines. Also number of regions in 4-space formed by $n - 1$ hyperplanes.

$1, 2, 4, 8, 16, 31, 57, 99, 163, 256, \ldots$

These are classroom ambassadors for Integer Sequences. The list took three weeks of emails after the conference before everyone was happy, but in the end we had consensus or near consensus and where differences of opinion remain, they are mild.

The secondary and more time intensive goal—soon to be started—is to develop and distribute teacher-friendly resources so that these integer sequences get the wide exposure they deserve.

Working documents around the Integer Sequences K-12 conference have been set up by Neil Sloane on OEIS: `https://oeis.org/w/index.php?title=Integer_Sequence_K-12_(Banff,_2015)`

Participants

Alekseyev, Max (George Washington University)
Anajao, Rosa (University of Alberta)
Cavers, Mike (University of Calgary)
Chan, Vincent (University of Calgary)
Chapman, Olive (University of Calgary)
Cherkowski, Gina (STEM Alberta)
Edgar, Tom (Pacific Lutheran University)
Guy, Richard (The University of Calgary)
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James, Gael (River Valley School)
Jungic, Veselin (Simon Fraser University)
Okemakinde, Seun (University of Ibadan)
Picciotto, Henri (Henri Picciotto's Math Education Page)
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Robichaud, Zaak (Bearspaw Christian School)
Saarnio, Lora (NuevaSchool)
Serenevy, Amanda (Riverbend Community Math Center)
Vikairaghavan, Rakhee (Calgary Board of Education)
Woodrow, Robert (University of Calgary)
Zucker, Joshua (American Institute of Mathematics)

Chapter 33

2015 Ted Lewis Math Fair Workshop (15w2195)

April 24 - 26, 2015

Organizer(s): Sean Graves (University of Alberta), Tiina Hohn (MacEwan University), Ted Lewis (SNAP Mathematics Foundation)

Introduction

The SNAP Foundation is a non-profit organization whose mandate is to encourage the development of mathematics learning resources at the classroom level with very little retraining of the teaching staff, with very flexible budgets, and by utilizing the energy and natural curiosity of the students themselves. The main theme of the BIRS workshop was, What is a SNAP math fair and how to organize a math fair in your classroom. The speakers mostly consisted of teachers/educators who shared their math fair experiences and success stories.

The first SNAP type math fair was designed in Edmonton by Mike Dumanski and Andy Liu in 1997-1998. Since then, a large number of schools in Alberta and beyond have adapted the SNAP math fair to their needs. The SNAP program has been spread through similar workshops and conferences, and mainly by teachers themselves.

SNAP received its initial funding from the Canadian Mathematical Society and from private donations. PIMS, the Pacific Institute for the Mathematical Sciences, has been a long time financial supporter of our math fairs. BIRS, the Banff International Research Station, has provided funding for the BIRS math fair workshops that have been held in Banff on a regular basis. Currently, our major supporter is Thinkfun - a company that develops a variety of excellent puzzles.

Puzzle Resources

- www.mathfair.com
- www.puzzles.com
- www.nrich.maths.org
- www.galileo.org
- www.mathpickle.com

2015 Workshop Highlights

The workshop started out with a presentation by Tiina Hohn of MacEwan University and Janice Hoffman of Belmont School in Edmonton. Tiina introduced the topic of ”What is a Math Fair?”, and Janice shared the experiences that she and the other teachers at Belmont school had this year while organizing a school-wide (K-6) Math Fair. This was followed by a presentation from Nicole Kotyk of Evergreen Elementary in Drayton Valley sharing her experiences in organizing her second Math Fair. These presentations are very important for teachers who are thinking of having a Math Fair but are unsure how to start. Here are some of the issues Janice and Nicole addressed: where they found their puzzles, how and when they introduced the puzzles to their students, how student groups were chosen and how the puzzles were allocated to the groups, the importance of dress rehearsals, when and where their Math Fairs were held, who was invited to their Math Fairs, how they assessed their students.

A presentation by Brian Simmons of Twelve Mile Coulee School in Calgary followed. Brian shared his experiences in using a 3D printer in his classroom. It was very eye-opening to see how a puzzle-solving mind-set can lead into this exciting application.

2015 Workshop Feedback

”As a pre-service secondary math teacher I was very interested in the benefits of incorporating problem solving and puzzles into the classroom and will strive to teach with these ideas in mind. Thanks again for the invite, it was a wonderful experience.”

”It was really great to meet so many people who are passionate about math fairs. I loved the atmosphere, and the discussions were illuminating. Thanks for a great weekend.”

”I always leave the Banff conference feeling rejuvenated and loving math. There were many different presentations this year that related to teaching and maybe thinking more globally as a teacher (the 3-D printer presentation) I feel that I have achieved a goal in actually getting my whole school involved in a Math Fair... and was interested to hear how other schools collaborate and assess their math fairs. Next year I already know how I can change and improve ours!! I feel very fortunate to have been invited and enjoyed all the great discussions and meeting new people.”

Participants

Armstrong, Maura (Telus World of Science Edmonton)
Bedard, Jaclyn (teacher Edmonton Catholic)
Congdon, Sarah (Teacher Edmonton Catholic)
Dahl, Tiffany (teacher Calgary Board of Education)
Graves, Sean (University of Alberta)
Hildebrandt, Maxine (Mother Earth’s Children’s Charter School)
Hoffman, Janice (Edmonton Public Schools)
Hohn, Tiina (MacEwan University)
Kotyk, Nicole (Evergreen Elementary School)
Lewis, Ted (SNAP Mathematics Foundation)
Locke, Jennifer (teacher Calgary Board of Education)
Lorway, Geri (Thinking 101)
Manz, Kirsten (student teacher UofA)
Olsen, Shelby (Education student University of Alberta)
Ottaway, Paul (Capilano University)
Pasanen, Trevor (University of Alberta)
Schmidt, Hilary (student teacher UofA)
Simmons, Brian (Calgary Board of Education)
Simmons, Sarah (Calgary board of education)
Summers, Ashley (Teacher (Calgary))
Tahmasebi, Nazanin (University of Alberta)

Chapter 34

Alberta Number Theory Days VII (15w2198)

June 12 - 14, 2015

Organizer(s): Nathan Ng (University of Lethbridge), Manish Patnaik (University of Alberta), Ander Steele (University of Calgary)

This conference is the unique annual opportunity for the Albertan Number Theory community to regroup and discuss the latest research progress in the field as well hear about the advances of Alberta's own researchers. The conference also provides an ideal environment for younger researchers to introduce their work and connect with other well-established researchers in the province and outside of their own universities.

This year we had 24 participants from Alberta, one from Montreal, two from China (visiting UBC), and two distinguished plenary speakers: William Casselman (UBC) and Julia Gordon (UBC). Moreover, nearly one third of the participants were female this year and two of nine invited speakers were female.

Overview of the Field

Number theory is a broad and central area of research with many connections and applications to other areas of mathematics and science. It is also an extremely active and diverse area of research. The subject may be divided roughly into several subdisciplines that range from pure mathematics, such as algebraic number theory, arithmetic geometry, analytic number theory, and automorphic forms and representation theory, to more applied areas such as computational number theory and mathematical physics.

Alberta Number Theory Days allows for face to face discussion between peers and facilitates collaboration between researchers within the province as well as distinguished out of province participants. It allows the community of Alberta number theorists to discuss the important recent advances in the field and in their own research. New connections are made and old associations are renewed, and these personal interactions often lead to the conception of new projects. The meeting aims to allow for the exchange of knowledge among researchers in number theory, which will improve the progress of current projects.

Presentation Highlights

There were many exceptional talks this year and numerous informal discussions throughout the friendly two day meeting.

Our first plenary speaker, **William Casselman** from the University of British Columbia, a founding figure in the field of automorphic forms and leading expert on many areas of mathematics, gave a fascinating talk on "Newton polygons and ramification." In this talk, he revisited a classical (and increasingly important) topic of ramification and explained a new, elementary perspective on the topic focusing on the geometry of Newton polygons. While this work was inspired by a recent article of Jonathan Lubin (which in turn followed an old idea of Tate), it aimed to provide more elementary and conceptual proofs of certain somewhat inaccessible topics in the literature.

Our other plenary speaker **Julia Gordon** (UBC) is a young expert on the representation theory of p-adic groups and its connections with model theory. She explained how an old technique due to Langlands and Kottwitz relating orbital integrals to isogeny classes of elliptic curves could be used to give an explanation of recent surprising heuristics of Geckler on the number of rational points on elliptic curves over finite fields. The talk was very well received by the Alberta audience, several members of whom had worked on closely related topics in the past.

Anna Puskas, a new face on the Alberta number theory scene, is currently a postdoctoral fellow at the University of Alberta. She explained a connection between a famous formula of William Casselman and Joseph Shalika for Whittaker functions on p-adic groups and the theory of crystal bases which arise in combinatorial representation theory. It was a particular highlight to have her speak on a result of one of our distinguished plenary speakers, and Dr. Puskas benefitted a greal deal from her interactions with Dr. Casselman during the weekend.

Fitting of the ever increasing purview of number theory, this year we invited the mathematical physicist **Thomas Creutzig** from the University of Alberta to deliver a lecture on "Logarithmic Hopf links and modular forms." In his lecture, Dr. Creutzig first explained how the representations of a regular vertex operator algebra (or conformal field theory) form a modular semi-simple tensor category, so that the categorical $SL(2,\mathbb{Z})$-action is generated by the Hopf link with twist corresponding to the elments S and T of the modular group. He then moved to the case of vertex operator algebras that have indecomposable but reducible modules, where nearly nothing is known about a similar modular story. In this setting, he introduced what he called logarithmic Hopf links and explained how they relate to representations of the corresponding fusion ring of the module category.

We were also fortunate this year to also have **Stephan Ehlen**, a postdoctoral fellow at McGill University, deliver a lecture on "On Two Arithmetic Theta Lifts". Theta lifts are an important tool in the modern theory of automorphic forms but somewhat under-represented in the Alberta number theory community, so we were happy to have Dr. Ehlen explain how to generalize part of Stephen Kudla's older work on central derivatives of Eisenstein Series and height pairings from the setting of Shimura curves to higher dimensional orthogonal and unitary Shimura varieties. He showed how to obtain the required Green's currents in the theory as regularized theta lifts of certain 'truncated' Poincare series. He also explained one interesting application in arithmetic geometry, namely that his results (which were joint with Siddharth Sankaran) imply the modularity of the difference of two arithmetic generating series for Kudla-Rapoport cycles on unitary Shimura varieties.

Also on the subject of automorphic forms, Alberta's own **Clifton Cunningham** from the University of Calgary delivered a lecture on "Lifting Hilbert modular forms to spin modular forms." In this talk, he explained how Hilbert modular forms for totally real fields of degree n determine automorphic representations of forms of the group $GSpin(2n+1)$ over $\mathbb{Q}$. Using examples, he explained how the automorphic representations obtained in this way may or may not be holomorphic. This was joint work with Lassina Dembele.

A veteran of many Alberta Number Theory days **Amir Akbary** from the University of Lethbridge delivered a lecture "On the greatest prime factor of some divisibility sequences". If $P(m)$ denotes the greatest prime factor of m, Dr. Akbary first recalled a result of M. Ram Murty and S. Wong who proved, under the assumption of the ABC conjecture,

$$P(a^n-1) \gg_{\varepsilon,a} n^{2-\varepsilon}$$

for any $\varepsilon > 0$. He studied analogous results for the corresponding divisibility sequence over the function field $\mathbb{F}_q(t)$ and for some divisibility sequences associated to elliptic curves over the rational field $\mathbb{Q}$. We note here that divisibility sequences were a topic of last year's meeting with one of the experts on this area Katherine Stange a distinguished plenary speaker during Alberta Number Theory Days 2014.

Another frequent participant of Alberta Number Theory Days past **Mark Bauer** from the University of Calgary delivered a lecture on "Cubic Irrationalities and a Ramanujan-Nagell Analogue." In his talk, he considered a cubic analogue of the Ramnujan-Nagell equation and showed that by constructing explicit, restricted irrationality measures for the cube root of two that are, in some very real sense, better than expected, it was possible to derive meaningful and interesting bounds on the difference between the cube of an integer and powers of 2. These measures naturally yield results about certain Diophantine equations. This was joint work with Michael Bennett.

Finally, we mention that we had another postdoctoral fellow speak at our meeting: **James Parks** from the University of Lethbridge delivered a lecture on "The asymptotic constant for amicable pairs of elliptic curves." If E is an elliptic curve over $\mathbb{Q}$ with primes p, q of good reduction, he recalled the notion introduced by Silverman and Stange, for a pair of such primes (p, q) to be amicable: $|E(\mathbb{F}_p)| = q$ and $|E_q(\mathbb{F}_q)| = p$, where $|E(\mathbb{F}_\ell)|$ denotes cardinality of the group of points of the reduction of E over the finite field of ℓ-elements $\mathbb{F}_\ell$. He then considered a function that counts such pairs on average over a family of elliptic curves and discussed new results related to the constant obtained in the asymptotics of this function.

Objectives Achieved

This was the seventh edition of Alberta Number Theory Days. Previous conferences took place in Lethbridge (2008), Calgary (2009), and BIRS (2010, 2011, 2013, 2014). This friendly meeting gathers the number theorists of the Alberta Universities to interact and exchange ideas once a year. This year, the plenary speakers were William Casselman (UBC) and Julia Gordon (UBC). Professor Casselman is a leading authority on automorphic forms and representation theory and Dr. Gordon is a leading expert in the representation theory of p-adic groups and motivic integration.

During this conference there were a total of nine talks: three external speakers, two from Calgary, two from Lethbridge, and two from Edmonton. We have an increasing number of young female researchers and it was important to reflect this in both the schedule and the list of participants. As mentioned earlier, nearly one third of the participants this year were female and two of nine speakers were female. Another goal of the conference was to give the opportunity to young researchers to present their research. In particular, three of the talks were given by postdocs: Stephan Ehlen (McGill) , James Parks (U. Lethbridge), and Anna Puskas (U. Alberta). Although this conference is mainly for Alberta researchers we have tried in recent years to have more outside participation. This year there were two participants from B.C., one from Quebec, and two from China. Although there were fewer talks than last year, participants had more opportunities to discuss and exchange ideas during the breaks. In particular, the Alberta researchers in automorphic forms and representation theory (Clifton Cunningham (Calgary), Manish Patnaik (Edmonton), and their students and postdocs) benefited greatly from interacting with the plenary speakers. We also tried to showcase some of the high quality research being done in Alberta, as evidenced by the talks by Mark Bauer (Calgary) and Amir Akbary (Lethbridge).

Participants

Akbary, Amir (University of Lethbridge)
Ali, Abid (University of Alberta)
Aryan, Farzad (University of Lethbridge)
Bauer, Mark (University of Calgary)
Bose, Arnab (University of Lethbridge)
Casselman, Bill (University of British Columbia)
Creutzig, Thomas (University of Alberta)
Cunningham, Clifton (University of Calgary)
Ehlen, Stephan (McGill University)
Fenton, Diane (University of Calgary)
Francis, Forrest (University of Lethbridge)
Gordon, Julia (University of British Columbia)
Guy, Richard (The University of Calgary)
Jacobson, Mike (University of Calgary)
Kadiri, Habiba (University of Lethbridge)
Lindner, Sebastian (University of Calgary)
Lumley, Allysa (University of Lethbridge)
Ng, Nathan (University of Lethbridge)
Parks, James (University of Lethbridge)
Patnaik, Manish (University of Alberta)
Pusks, Anna (University of Alberta)
Rezai Rad, Monireh (University of Calgary)
Scheidler, Renate (University of Calgary)
Shahabi, Majid (University of Calgary)
Siavashi, Sahar (University of Lethbridge)

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Yang, Hai (University of British Columbia)
Zhu, Huilin (University of British Columbia)
Zvengrowski, Peter (University of Calgary)

Chapter 35

Global Rigidity (15w2199)

July 17 - 19, 2015

Organizer(s): Robert Connelly (Cornell), Steven Gortler (Harvard), Tibor Jordán (Eotvos Lorand), Tony Nixon (Lancaster), Walter Whiteley (York)

Overview of the Field

The rigidity and flexibility of a structure, either man-made in buildings, linkages, and lightweight deployable forms, or found in nature ranging from crystals to proteins, is critical to the form, function, and stability of the structure. A strong form of rigidity is "Global Rigidity" when the given lengths permit only one realization, up to congruence. The mathematical theory of "Global Rigidity" is developing methods for the analysis and design of man-made structures , such as sensor networks, as well as for natural structures such as proteins.

We live in 3-dimensions, and a fundamental problem is to develop results for global rigidity in 3-dimensions which are as good, and as efficient, as the recently developed theory for global rigidity of structures in 2-dimensions. The mathematical methods also give insights into fundamental mathematical systems of constraints and computations, with even wider application in areas of computer aided design and manufacturing, CAD/CAM.

Recent Developments and Open Problems

Connelly-Gortler [2] extended the characterisation of universal rigidity in terms of PSD stress matrices [5] from generic frameworks to arbitrary frameworks.

Jordán, Kiraly and Tanigawa [7] extended the, d-dimensional, characterisation of global rigidity for body-bar frameworks [4] to body-hinge frameworks. Resulting from this work they identified a new family of counterexamples to Hendrickson's conjecture [6] in 3-dimensions. Previously only a single counterexample, $K_{5,5}$, was known [1]. Following this a key substantial open problem is to develop an appropriate conjecture for 3D global rigidity.

Tanigawa [8] gave a new sufficient condition for generic global rigidity in terms of vertex redundant rigidity. He used his more elaborate geometric arguments to simplify the combinatorial steps in the characterisation of generic global rigidity in the plane.

Presentation Highlights

Shin-Ichi Tanigawa presented his sufficient condition for global rigidity mentioned above.

There were two interesting talks on universal rigidity. Steven Gortler talked about the universal rigidity of complete bipartite graphs [3] and Anthony Man-Cho So discussed facial reduction techniques and degree of singularity.

There were two prominent student talks. Katie Clinch talked about progress towards characterising global rigidity of direction-length frameworks and Hakan Guler presented a new necessary condition for global rigidity in terms of covers.

There were also a number of 5 minute talks and presentations of open problems.

Scientific Progress Made

The session on Sunday Morning on points at infinity was a direct result of extending the workshop. It was a great way to pull together a number of connections among these topics. In particular it was a very clear connection between the 5-day workshop and the 2-day.

Bob Connelly, Steven Gortler and Tony Nixon had an interaction about global rigidity for frameworks in the plane with the l^p metric following on from Stephen Powers presentation of this problem. It may happen that rather than use a stress matrix to determine global rigidity, it may be more useful to some of the more combinatorial methods of Szabadka and Jordán and Tanigawa.

Connelly and Gortler also had an interaction with Simon Guest and Louis Theran concerning the global rigidity of complete bipartite graphs, where some subset of the edges have been removed. Again many of these graphs can be proven to be generically globally rigid by some standard edge splitting methods.

Tony Nixon and Bill Jackson had some discussions about their ongoing work on global rigidity on surfaces. They also benefitted from a discussion with Steven Gortler about a stress matrix characterisaton in this context.

Outcome of the Meeting

Tibor Jordán and Walter Whiteley talked for several hours, using information from the workshop to work on a revision on the Handbook Chapter on Global Rigidity.

Participants

Clinch, Katie (Queen Mary London)
Connelly, Robert (Cornell University)
Cruickshank, James (National University of Ireland)
Eftekhari, Yaser (York University)
Gortler, Steven (Harvard University)
Guler, Hakan (Queen Mary, University of London)
Jackson, Bill (University of London)
Jordan, Tibor (Eotvos University, Budapest)
Karpenkov, Oleg (University of Liverpool)
Kirly, Csaba (Eotvos University)
Kitson, Derek (Lancaster University)
Lam, Wai Yeung (Technische UniversitŁt Berlin)
Nixon, Anthony (Lancaster University)
Power, Stephen (University of Lancaster)
Schulze, Bernd (Lancaster University)
Serocold, Hattie (Lancaster University)
Servatius, Herman (Worcester Polytechnic Institute)
Sitharam, Meera (University of Florida)
So, Anthony Man-Cho (The Chinese University of Hong Kong)
Tanigawa, Shin-ichi (Kyoto University)
Theran, Louis (Aalto University)
Thorpe, Michael (Arizona State University)
Trelford, Ryan (York University)
Whiteley, Walter (York University)

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Chapter 36

Prairie Discrete Math Workshop (15w2206)

August 7 - 9, 2015

Organizer(s): Joy Morris (University of Lethbridge)

Overview

The eleventh Prairie Discrete Math Workshop was held at BIRS on the weekend of August 7–9. There were 30 participants, including 5 graduate students, 3 post-doctoral fellows, 2 early-career researchers (within the first few years of employment after post-doctoral fellowships), and 20 more-senior faculty members. Eleven of the participants were women (including one of the students, one of the post-doctoral fellows, and one of the early-career researchers). We had broad representation from Manitoba (11 participants from 2 universities), Saskatchewan (7 participants from 2 universities), and Alberta (9 participants from 2 universities), as well as two participants from BC and one invited speaker who was visiting from Australia. Although these gatherings have been held almost annually since 2003, the 10th workshop had been held in 2013 at Thompson Rivers University in Kamloops. A combination of factors including low attendance from prairie provinces at that meeting led to no gathering being organised in 2014, so it was our hope that holding this workshop at BIRS would revitalise the gathering. Indeed, plans are already underway for the 2016 workshop to be held in Winnipeg, hosted by the University of Manitoba.

Discrete Mathematics is a broad field that covers a wide variety of research areas within departments of Mathematics and of Computer Science. The goal of this workshop was to gather researchers from the prairie provinces who specialise in Discrete Mathematics, for the purposes of facilitating collaboration, and introducing students and post-doctoral fellows to the expertise available within the region.

Specialties of researchers who work in the prairie provinces include cryptography; graph theory; optimisation; and design theory and finite geometries. Within each of these areas, we have researchers who bring tools from other mathematical fields to bear on problems in that area, including tools from number theory, linear algebra, and abstract algebra. The diversity is so broad that we are often only dimly aware of developments in other parts of discrete mathematics, and yet it is quite common to find that someone who has been working in one field of discrete mathematics is able to use ideas from their field to make significant progress on a question in some other field, once it has been brought to their attention.

In the program for the workshop, we included 4 invited talks of 50 minutes each, and 5 contributed talks of 25 minutes each. We tried to include quite a bit of free time for informal discussions, also. All of the lectures were videotaped and should be made available through BIRS.

Presentation Highlights

Michael Doob from the University of Manitoba gave the first invited talk. In addition to his expertise in discrete mathematics, he has considerable technological expertise, and it was here that he focussed his presentation. He talked about the use of technology to enhance research presentations. He concentrated on beamer (for creating slides), tikz (for creating graphics), and sage (for symbolic computation), and showed tips and tricks that are useful for any mathematician who uses these tools, particularly for presenting results. He showed numerous examples of code and the corresponding output. His examples of animations related to algebraic graph theory [10, 14]. Most participants learned from this talk and were excited to have these ideas for improving their use of these tools in the future.

Renate Scheidler's invited talk came second on the schedule. She is an expert in number-theoretic cryptography, working at the University of Calgary. She introduced us to hyperelliptic curve arithmetic, and its applications to public key cryptography [7]. Her talk began by explaining the basics of public key cryptography. She then explained how the points on an elliptic curve (together with the point at infinity) form an abelian group that can be used to encrypt communications based on the difficulty of the discrete log problem. She noted that this form of encryption is being used. She went on to explain how these concepts can be generalised to hyperelliptic curves. She explained that even though the operation used to form the group for hyperelliptic curves is more complicated and harder to compute than for elliptic curves, this is offset by the fact that the base field involved can be half as large without compromising the level of security provided (as long as the hyperelliptic curve has genus 2). This form of encryption is also practical, and its use does not thus far require a licensing fee. This was an excellent introduction to this extremely practical and important branch of discrete mathematics that lies on the boundary with number theory.

Gabriel Verret gave an invited talk that related to the use of permutation groups in graph theory. He is a young researcher with an impressive research record, originally from the Ottawa area, who currently holds a post-doctoral fellowship at the University of Western Australia. The actions of permutation groups on graphs is an extremely active area of discrete mathematics, with permutation group-theoretic techniques often leading to advances in our understanding of graph theory, as well as graph-theoretic techniques leading to advances in our understanding of permutation group theory. This talk was an example of the latter. He spoke about some advances he had made (with a coauthor) [15] in understanding digraphs that have vertices with almost equal neighbourhoods, when a permutation group is acting primitively on the vertices. The techniques involved counting certain properties of the graphs, but led to some results about so-called "synchronising" permutation groups [1].

The final invited talk was given by **Richard Brewster**. He is an expert on graph homomorphisms, from Thompson Rivers University. He discussed the problem of recolouring a graph (moving from one colouring to another by successively recolouring one vertex at a time, ensuring that every colouring is proper) and other closely-related problems [6, 16]. He focussed on the computational complexity of this problem, which is known to be polynomial for 3-colourings, but becomes PSPACE-complete for k-colourings when $k \geq 4$ [4]. He described the generalisation of this problem to circular (p, q)-colourings, and explained work of his (with coauthors) [5] showing that here, too, there is a complexity dichotomy, since the problem is polynomial when $p/q < 4$ and PSPACE-complete if $p/q \geq 4$.

Of the five contributed talks, three were given by a student, a post-doctoral fellow, and an early career researcher. The other two described work that had involved students or post-doctoral fellows. The contributed talks focussed on specific work by the researchers, often as part of a collaboration. Most of them related to graphs or hypergraphs. More specifically, topics touched on in contributed talks included random and self-avoiding walks (**Nicholas Beaton**, University of Saskatchewan) [3], the percolation time for certain cellular automata on graphs (**Karen Gunderson**, University of Manitoba) [2, 11], recent developments on finding Erdös-Gallai type inequalities for degree sequences of hypergraphs and score sequences of hypertournaments (**Muhammad Khan**, University of Calgary) [12, 13], polygons and discrete models for polymers and entangled DNA (**Chris Soteros**, University of Saskatchewan), and games played on graphs (**Boting Yang**, University of Regina) [8, 9].

Outcome of the Meeting

We received numerous appreciative comments from participants indicating that the eleventh Prairie Discrete Math Workshop was a success. It gathered discrete mathematicians from across the prairies, from senior researchers through students. It strengthened professional and personal bonds, and forged new connections and collaborations across the prairies. It made us all aware of expertise amongst our neighbours that we will be able to call upon, and of recent developments in a variety of fields. There were many informal discussions that may lead to collaborations, as well as work that went on between existing collaborators during breaks in the program. A specific example of a new collaboration that developed during the meeting was between researchers from the University of Regina and Thompson Rivers University, on a problem involving posets.

As previously noted, this workshop did serve (as hoped) to revitalise these annual gatherings, and the next one is already being planned. Although it was a great pleasure to hold this workshop in Banff, and we hope to return from time to time, the limit of 30 participants is too small for us to use the BIRS facilities every year, as our target participants are quite numerous and we would not wish these workshops to become exclusive.

The organisers would like to thank BIRS and the Banff Centre for their hospitality and financial support in the form of accommodation for participants. We would particularly like to thank the staff who were extremely patient and helpful through the planning and preparation phases, as well as during the workshop itself. We would also like to acknowledge the generousity of PIMS, enabling us to provide breakfasts and lunches for participants, as well as some travel support. Finally, we would like to thank all of the participants, without whom no gathering can be successful, and in particular the speakers for their excellent presentations.

Participants

Arman, Andrii (University of Manitoba)
Atapour, Mahshid (University of Saskatchewan)
Beaton, Nicholas (University of Saskatchewan)
Breen, Jane (University of Manitoba)
Brewster, Richard (Thompson Rivers University)
Cavers, Mike (University of Calgary)
Craigen, Robert (University of Manitoba)
Currie, James (The University of Winnipeg)
Doob, Michael (University of Manitoba)
Gosselin, Shonda (University of Winnipeg)
Gunderson, Karen (University of Manitoba)
Gunderson, David S. (University of Manitoba)
Herman, Allen (University of Regina)
Kharaghani, Hadi (University of Lethbridge)
Kirkland, Stephen (University of Manitoba)
Linek, Vaclav (University of Winnipeg)
Liprandi, Max (University of Calgary)
McGuinness, Sean (Thompson Rivers University)
Meagher, Karen (University of Regina)
Morris, Joy (University of Lethbridge)
Purdy, Alison (University of Regina)
Sands, Bill (University of Calgary)
Sasani, Sara (University of Lethbridge)
Scheidler, Renate (University of Calgary)
Seyffarth, Karen (University of Calgary)
Soteros, Christine (University of Saskatchewan)
Verret, Gabriel (University of Western Australia)
Visentin, Terry (University of Winnipeg)
Yang, Boting (University of Regina)

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Chapter 37

Positivity in Algebraic Combinatorics (15w2208)

August 14 - 16, 2015

Organizer(s): Angèle Hamel (Wilfrid Laurier University), Stephanie van Willigenburg (University of British Columbia)

Overview of the Field

Symmetric functions form a significant area of research, especially within algebraic combinatorics, in particular because of the distinguished basis of Schur functions. These functions also arise naturally in representation theory, enumerative combinatorics, algebraic geometry, quantum physics and are of vital interest in geometric complexity theory in computer science. More precisely, in representation theory every Schur function corresponds to an irreducible character under the Frobenius characteristic isomorphism. Meanwhile in enumerative combinatorics they are generating functions for tableaux, and in algebraic geometry they arise when studying the cohomology ring of the Grassmannian. In quantum physics they are the images of the basis of semi-infinite wedges under the Boson-Fermion correspondence. Lastly, in geometric complexity theory they are key to resolving whether $P \neq NP$.

Therefore a deeper understanding of this basis, and its properties, enables many of the beautiful and useful results in these areas, and many applications via connections between diverse areas from pure mathematics to applied mathematics and beyond. A key aspect of such a deeper understanding is knowing exactly how an arbitrary symmetric function can be expressed in the basis of Schur functions. Indeed, it is a long-standing and important open question to classify whether an arbitrary function is a positive linear combination of Schur functions, that is, Schur-positive. This is because, for example, if a homogeneous symmetric function is Schur-positive, then it arises as the Frobenius image of some representation of the symmetric group.

The positivity question is a timely one—it was the focus of a Research Note in the CMS Notes in September 2014. Furthermore, advances have recently been made regarding positivity for a diverse range of closely related functions including quasisymmetric functions, Schur Q and P functions and Schubert polynomials. We thus believed the time is ripe for a brief but focused workshop to exchange ideas and tools on various aspects of this problem. The key goal of this two-day workshop was to create the intense atmosphere that will facilitate new collaborations and jumpstart new research.

Objectives of the Workshop Proposal

In mathematical terms, the Schur function, s_λ, and skew Schur function, $s_{\lambda/\mu}$, can be defined combinatorially as a weighted sum over tableaux. By the Littlewood-Richardson rule, skew Schur functions and products of Schur functions can be expressed in terms of Schur functions via $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$, or, equivalently, $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ where the $c^\lambda_{\mu\nu}$ are positive integers.

Generalizing this, what are the properties that ensure the Schur-positivity of the difference of skew Schur functions, $s_{\lambda/\mu} - s_{\sigma/\tau}$? This is easily seen to be equivalent to inquiring about conditions necessary to guarantee the equality of skew Schur functions, $s_{\lambda/\mu} = s_{\sigma/\tau}$. Similarly, we can inquire as to the conditions under which $s_\lambda s_\mu - s_\nu s_\rho$ is Schur-positive.

Despite the straightforward nature of the first question, it is still wide open after a century of active research in the area, which includes the work of Fields medallists Tao and Okounkov. Consequently, the Schur-positivity question has spawned many research results and reinterpretations, both in algebra and combinatorics, and our goal for this workshop was to make progress on a number of fronts, taking advantage of some new avenues and tools in the area. In this regard we would like to make special mention of the following specific questions, along with some very recent references:

- Apply quasisymmetric Schur function positivity and F-positivity to Schur-positivity (e.g. McNamara, 2014)
- Explore Schur-positivity of other function such as Schur P and Q functions, and cylindrical Schur functions (e.g. Ardila and Serrano, 2012)
- Determine conditions for equality of symmetric functions (McNamara and Ward, 2014)
- Discuss combinatorial formulae for Schubert and Schur functions (e.g. Lam and Rietsch, 2012; Assaf, N. Bergeron and Sottile, 2014)
- Determine necessary and sufficient conditions for Schur-positivity of symmetric function differences (e.g. McNamara and van Willigenburg, 2012)
- Develop tools for specific functions to be Schur-positive (e.g. Ballantine and Orellana, 2014; Assaf, 2014).

Our overarching objectives were to share the latest results on the subject, search for common ground between the various perspectives in algebra, algebraic geometry, representation theory, quantum physics and combinatorics, and facilitate a cross-pollination of results. This was facilitated by gathering a diverse group from around the world from areas impacted by Schur functions, ranging from graduate students to leaders of the fields.

Presentation Highlights

Our two day workshop featured nine half hour presentations and one joint 45 minute presentation. We invited specific participants to give these talks, aiming for good coverage and breadth in the topics, and for an emphasis on open problems. Given that the time frame of the workshop was short, we felt that the focus had to be on the future, including on facilitating new collaborations through open problems. An important part of this was the one hour open problem session. As is evident from the comments received from the participants and quoted in the next section, the open problem session alone led to fruitful collaborations.

Here are short summaries of each of the talks. Early career researchers are indicated by a *:

Victor Reiner, (University of Minnesota) Thrall's problem and two coarsenings.

This opening talk concerned Thrall's problem, a problem concerning finding a combinatorial interpretation of the coefficients $a(\lambda, \mu)$ in the Schur function expansion

$$\sum_w F_{Des(w)} = \sum_\mu a(\lambda, \mu) s_\mu.$$

Although this original problem remains difficult, the talk focused on two variations of it that are expected to be more tractable. Partial results and connections to other algebraic results were also discussed.

Peter McNamara, (Bucknell University) Conjectures concerning the difference of two skew Schur functions.

This talk was a round up of a number of conjectures concerning the difference $s_A - s_B$ of two skew Schur functions. In particular the presenter explored the following questions: 1) What are necessary and sufficient conditions on the structure of A and B for $s_A = s_B$?, 2) Among the connected skew shapes, which ones are maximal in Schur-positivity order (i.e. $A \le B$ if $s_B - s_A$ is Schur positive)?, 3) Row-overlap conditions give necessary, but not sufficient, conditions for $A \le B$—is there another algebraic inequality that is equivalent to the row-overlap conditions?, 4) the most obvious generalization of the Saturation Theorem of Knutson and Tao fails when A is a skew shape—does a less obvious, but still natural, generalization hold?

Vasu Tewari*, (University of British Columbia) Littlewood-Richardson rules for symmetric skew quasisymmetric Schur functions.

The Littlewood-Richardson rule shows how to expand a skew Schur function in terms of Schur functions, and is an important classical result in symmetric function theory. A natural question is to find an analogous rule for quasisymmetric Schur functions. This talk gave two such quasisymmetric Littlewood-Richardson rules and showed how they reduce to the classical result in special cases, and how they can be used to solve an outstanding conjecture.

Olga Aznehas and Ricardo Mamede*, (Universidade de Coimbra) Skew-shapes with interval support in the domination lattice.

Given a Schur function, s_λ, its support is the set of partitions μ such that the Schur function, s_μ, appears with positive coefficient in the Schur expansion of s_λ. This talk focused on conditions in which Schur functions have, or do not have, interval support. In particular, the main results explored conditions for classifying when multiplicity-free skew shapes, shapes that are disjoint unions of ribbons, and products of Schur functions have interval support.

Louis Billera, (Cornell University) Some positivity questions for Coxeter groups.

This talk focused on positivity questions for coefficients of Kazhdan-Lusztig polynomialals for Bruhat intervals in Coxeter groups, in particular on some open problems, including a conjecture on the nonnegativity of the coefficients of the complete cd-index (defined by the presenter and Brenti), and a direct enumerative proof of the nonnegativity of the a-vector.

Alejandro Morales*, (UCLA) A q-analogue of Naruse's hook-length formula for skew shapes.

This talk gave in an introduction to Naruse's 2014 result which gives a skew version of the hook-length formula for symmetric functions. It followed this introduction with proofs of two q analogues of Naruse's formula and explained how they relate to well-known symmetric function theory.

Sara Billey, (University of Washington) Trees, tanglegrams, and tangled chains.

This talk focused on a number of new enumerative formulas for tanglegrams, which are graphs with application in computational biology. These enumerative formulas include ones to count the number of distinct binary rooted tanglegrams with n matched vertices, the number of tangled chains of binary trees of any length, and a new formula for the number of binary trees with n leaves.

Angela Hicks*, (Stanford University) Reflections on Gessel's fundamental basis.

This tablet-based talk explored the true meaning of particular form of result. From the abstract: "Many recent conjectures and theorems in algebraic combinatorics have the following setup: a (sometimes graded) module M has a clearly defined S_n action. A (not obviously) related set S of combinatorial objects has an associated permutation (call it $\sigma(s)$ for $s \in S$), which is used to define a quasisymmetric function (with an associated theory for monomials and objects indexed by words). The statement is then made that

$$\text{Fchar}(M) = \sum_{s \in S} F_{\text{ides}(\sigma(s))}$$

(or a graded version, with additional variables), frequently followed by an indirect proof that the sum is symmetric. What do we mean by this?" The talk explored various answers and frameworks for answers to this question.

Arthur Yang, (Nankai University) Schur positivity arising form log-concavity problems.

This talk looked at Schur positivity in the context of log-concavity problems, using the fact that log-concavity can be derived from Schur positivity of certain symmetric functions. In particular the talk focused on questions involving q-Narayana numbers and Narayana polynomials, along with questions regarding the log-concavity of the generating functions for permutations with respect to the length of the longest increasing subsequences.

Sami Assaf, (University of Southern California) Schur positivity.

This chalk and talk presentation that rounded out the workshop and presented the application of the method of dual equivalence for obtaining Schur expansions of quasisymmetric functions.

Scientific Progress Made and Outcome of the Meeting

We organized this meeting on reasonably short notice a few short months before the workshop date, and were pleased to have such good attendance by both early career and established researchers from all over North America as well as some from China and Portugal. The fact that they were willing to travel so far for such a short time is testament to the regard with which they held the workshop, the fellow participants, and, presumably, the location. The short time frame of the meeting was clearly no object to progress either, and we were gratified to see how many of our participants responded positively and specifically to outcomes that had been achieved during that time. With such a strong foundation, the expectation is that these nascent results will multiply, and we are inspired to organize a follow up meeting at some point in the future.

Below is a selection of feedback from our participants. Although the comments vary in terms of details, two common themes emerge from them, themes expressed in the words of our participants as follows: “All in all, I learnt a surprising amount of mathematics in the course of two days!” and “It goes to show that even with all the new electronic gadgetry, there is no substitute for sustained face-to-face discussions, even over a short period like a weekend.”

Edited selection of participant comments on outcomes and progress:

The workshop ”Positivity in Algebraic Combinatorics” was an invaluable opportunity to learn about exciting new results in combinatorics. Furthermore, it provided me an opportunity to present my work in front of the leading experts in the field. All in all, I learnt a surprising amount of mathematics in the course of two days!

Some of the questions that were raised after my talk opened new avenues of research, and Stephanie van Willigenburg and I hope to address them in the near future. In particular, inspired by a question of Vic Reiner, we plan to investigate whether symmetric skew quasisymmetric Schur functions are obtained as characteristics of certain projective 0-Hecke modules. Another question was on the existence of skew Pieri rules in the spirit of Assaf-McNamara’s for skew quasisymmetric Schur functions. We already have results in this direction and are in the process of writing them up. Finally, drawing motivation from Lou Billera’s talk, we plan to investigate the cd-index and Schur positivity of a poset arising naturally from skew composition tableaux. Additional interest comes from the fact that this poset can be seen as a subposet of the strong Bruhat order on the symmetric group.

Amongst the exciting things I learnt from the talks were the use of dual equivalence graphs to establish Schur positivity, Lie representations and associated combinatorics of the free Lie algebra, q-deformations of Naruse’s hook-length formula for counting Young tableaux of skew shape, combinatorics of tanglegrams, and log-concavity and Schur positivity of some interesting functions arising naturally in combinatorics. The open problem session also featured a host of interesting questions and ideas.

The workshop gave us the opportunity to share our research with other participants as well as learning from their contributions. This turned out into new insight and possible new directions of our results. In particular, we have benefited of new inputs from Sara Billey and Vic Reiner. It is really enjoyable to learn that one can go further in our work.

At the workshop after my talk I heard from that one student of S. Assaf had a related result. In the near future I want to get in contact with this student to discuss their work (we are in the same city). I also heard interesting suggestions from two other participants (V. Reiner and S. Billey) to generalize the results in the direction of forests and to see if the approach generalizes to certain identities of reduced works in Schubert calculus. I agreed to exchange preprints with the second participant once our respective projects are done.

The meeting was indeed very helpful for me to attend. I learned useful things from the talks, but most notably I found one of the open questions that Lou Billera mentioned in the problem session very interesting, a question of understanding some rather fundamental looking symmetric functions that have arisen in his work. Both Lou and I were still in Banff all of Sunday afternoon, and we both took the same shuttle back to Calgary on Sunday evening, and this allowed us to spend several hours on Sunday afternoon and evening working together on his question, and we seem to have made substantial headway. We are continuing to email each other about his question and hope this will develop into a joint project. I also had quite helpful discussions with Sami Assaf and Vic Reiner, but definitely this possible project with Billera was the most valuable outcome of the meeting for me.

I very much enjoyed this workshop and thank you for having me! I am very happy to know more people who are working on Schur positivity and related subjects. I also found some interesting open problems like: the existence of Naruse type analogue for the hook-content formula of skew partitions, a quasisymmetric skew Saturation Theorem, and more. You did a really good job in organizing the workshop!

I came to the workshop with an open mind, expecting to start a new collaboration and find a new direction for my research. I was not disappointed. Without exception all the talks were engaging, and were distinguished by the presenters' enthusiasm and by their generosity with open problems. With respect to new directions, while many avenues presented themselves, my collaborator and I were drawn most to tanglegrams (discussed by Sara Billey) as being a promising mix of new and old, since they represent both a new direction for us but are also related to phylogenetic trees with which I am already familiar. Since leaving Banff we have actively been working on developing a research direction in this area.

Unusually for me, this meeting did spur two collaborations and ideas leading to possibly one more.

Ive begun discussions with both Sara Billey and Patricia Hersh about the symmetric function question I gave at the problem session. There are already some partial results and new conjectures; hopefully the three of us can push this to something more substantial.

Also, based on the question Vic Reiner asked me at my talk, we have formulated two very plausible new conjectures on a topic that I have been working on for decades.

Strangely enough, all of these involve some aspects of positivity. It goes to show that even with all the new electronic gadgetry, there is no substitute for sustained face-to-face discussions, even over a short period like a weekend.

Participants

Ahlbach, Connor (University of Washington)
Assaf, Sami (University of Southern California)
Azenhas, Olga (Universidade de Coimbra)
Benedetti , Carolina (Fields Institute)
Bergeron, Nantel (York University)
Billera, Louis (Cornell University)
Billey, Sara (University of Washington)
Hamel, Angele (Wilfrid Laurier University)
Hersh, Patricia (North Carolina State University)
Hicks, Angela (Stanford University)
Mamede, Ricardo (Universidade de Coimbra)
McNamara, Peter (Bucknell University)
Morales, Alejandro (UCLA)
Pang, Chung Yin Amy (LaCIM, UQAM)
Pawlowski, Brendan (University of Minnesota)
Reiner, Victor (University of Minnesota)

Schneider, Lisa (University of California Riverside)
Swanson, Josh (University of Washington)
Tewari, Vasu (University of British Columbia)
van Willigenburg, Stephanie (University of British Columbia)
Wang, Larry (Nankai University)
Yang, Arthur (Nankai University)

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Chapter 38

First Canadian Summit on Applications of Partial Differential Equations in the Sciences (15w2186)

August 28 - 30, 2015

Organizer(s): Thomas Hillen (University of Alberta), Jun-Cheng Wei (University of British Columbia)

Schedule of Summit

Friday, August 28, 2015: Arrival of participants. 19:00 onward: Informal meeting in the BIRS lounge (2nd Floor, Corbett Hall)

Saturday, August 29, 2015: 9:00-12:00 and 14:00-17:00 Short presentations of delegates; 15 talks arranged (see below)

Sunday, August 30, 2015: 9:00-12:00 Discussion of CRG activities.

First Day of Talks

On August 29th, 15 talks were arranged.

As part of the PIMS Collaborative Research Group on Applied PDEs, the purpose of this summit meeting is to get to know each other and their current research activities in order to identify common interests and to discuss upcoming events and activities.

Each participant was invited to present a 20 minute talk. The focus of these talks was to provide an introduction of their interests, a summary of typical methods used or developed and a discussion of interesting open questions.

There are seven talks in the morning, starting from 9am and ending at 12:30pm. This was chaired by Thomas Hillen. There are eight additional talks in the afternoon, starting from 2pm and ending at 5:20pm. This was chaired by Jun-Cheng Wei.

Nathan Kutz (University of Washington): Data-driven dimensionality-reduction in PDEs

Nathan introduced several basic data-driven methods in PDEs and discussed applications in nonlinear Schrodinger equations, compressive sensing for PDEs in fluids, etc. This is a completely new inter-displinary field and creates enormous opportunity for understanding data.

Razvan Fetecau (Simon Fraser University): Nonlocal aggregation models: dynamics and equilibria

Razvan presented aggregation model of self-collective behavior with potentials having short attractive range and long replusive range. He gave a complete characterzation of steady states and their stability in some parameter regions. He mentioned several open problems including anisotropic model as well as nonlocal models with nonlinear diffusion.

Rachel Kuske (University of British Columbia): Stochastic facilitation through interactions of delays and noise

Pattern formations with white noise have gain widespread interests in recent years. Rachel described noise-driven patterns and oscillations and sustained transients "stabilized" by noise. She gave an example of Swift-Hohenberg model with Paragas controls and noise. Then she mentioned several challenging open problems including understanding the effect of the noise on bifurcation structure of SPDES, sustained transients, stochastic bifurcations, etc.

David Iron (Dalhousie University): Models of processes in cell physiology

David proposed reaction-diffusion systems to model polarity formation in nematode development and aggregation of cell surface receptors. He showed several numerical and partial results on possible patterns. Some open conjectures are raised.

Paul Muir (St. Mary's University):B-spline adaptive Gaussian collocation error control software for the numerical solutions of PDEs

Paul discussed several old and new numerical PDE-solvers: BACOL, HPNEW, MOVCOL, EPDCOL, BACOLR, BACOLI, etc. He mentioned future work on spatial adaptivity/error control based on transformations involving Moving Mesh PDEs.

Ray Spiteri (University of Saskatchewan):Efficient time-stepping

Ray's topics is an interface talk between computer science, math and numerical analysis. He introduced time-stepping method in differential equations and applications in optimization and optimal control. Then he mentioned current projects on heart simulation, optimal design and operation of PEM fuel cells, etc.

Jacek Szmigielski (University of Saskatchewan): Wave breaking and shocks: an integrable systems perspective

Jacek started introduced a shallow water wave equation known as the Camassa-Holm PDE, and related equations, and and discussed the role of peakons in existence and blowing ups. He then discussed the Hamiltonian structure of peakons and perspectives from integrable systems. He mentioned generalized $b-$systems and several open problems associated with it.

Theodore Kolokolnikov (Dalhousie University): Vortex nucleation in a dissipative GPE with rotation

Theodore analyzed the stability and instability of boundary layer solutions of GPE with large rotations. A weakly nonlinear analysis is performed for the ground states and the instability mode is analyzed. He mentioned several open problems on the analysis of nonlinear vortex patterns.

Yingfei Yi (University of Alberta): PDE method in stochastic dynamics

Yingfeng introduced an ODE system with Ito white noise perturbation. Using the distribution-based approach, the transition probability density is shown to satisfy Fokker-Planck equation which can be solved using geometric and PDE methods. General open questions on dynamical systems with noise perturbations are raised.

Jeff Williams (Simon Fraser University): Proof by construction: an overview of rigorous numerics

Jeff gave an overview of rigorous numerics computing proofs. He started with the general scheme of such methods and then gave several examples of this scheme. Applications of this new method include the existence and nondegeneracy of nonlinear Schrodinger equations, and the energy cascade of diblock copolymer models.

Chun-Hua Ou (Memorial University of New Foundland): Research on two fields: applied dynamical systems and PDE

Chun-Hua investigated the traveling wave problems for nonlocal population models. The new characteristics of the model is the combination of delays and nonlocal interactions. He introduced on-going work on traveling wave solutions for SI model. Another topic of his interest is the asymptotics of PDEs in higher dimensions.

Ralf Wittenberg (Simon Fraser University): PDEs with spatiotemporal complex behavior

Ralf discussed patterns and spatiotemporal chaos on stabilized Kuramoto-Sivashinsky equation. He gave several interesting behavior of the so-called Matthew-cox equation. A challenging question is to rigorous analysis.

Alexei Cheviakov (University of Saskatchewan): Systematic construction of conservation laws, symmetries for PDEs and ODEs

Alex started with general definitions of local conservation laws and the associated Euler-Lagrange equation to find multipliers. He discussed several applications in mathematical modelling. He expressed willingness to co-supervise graduate students.

Michael Ward (University of British Columbia): Pattern foormations: new directions and challenges

Michael presented two new directions in pattern formations for reaction-diffusion systems. The first is the dynamics of localized spots on the sphere. A new DAE system is derived and its equilibria, dynamics and Fekette points are discussed. The second direction is diffusion induced synchronous oscillations. This is an ODE model coupled with PDE bulk diffusion. Many new features of this system including Hopf bifurcations are discovered.

Thomas Hillen (University of Alberta): Mathematical modelling with fulling anisotropic diffusion

Thomas started with an introduction of two kinds of anisotropic diffusions–Fick diffusion and myopic diffusion. He discussed three applications of anisotropic diffusion models: transport model for mesenchymal motion, diffusion glioma brain cancer model and sea turtles

Second Day of Discussions

On Sunday morning (Aug. 30), Michael Ward, representing the organizers of the CRG, presented an overview of the CRG proposal and proposed activities in the next three years. The participants had extensive discussions on possible collaborations and suggested several new initiatives including possible graduate visiting and co-supervision.

Scientific Progress Made and Outcome of the Meeting

All of the work that was presented was very recent. After each talk there was always long discussions and interactions. Many participants saw new ideas arising from interaction with others who brought their own perspectives. A number of new collaborations appear to have arisen from the meeting, and the work of other collaborations was advanced. For example Michael mentioned possible link of Thomas's sea turtle model with the calculation of the first mean pass time. Theodore took deep interest in Razavan's nonlocal models. Ray was interested in numerical computations of Michael's DAE systems with large number of particles. Jeff's rigorous computing may solve several open problems of Wei. Nathan's data-driven methods bring new ideas to pattern formations. Alexei's conservation laws may be applied to Toda system studied by Wei. Yingfeng's dynamical system with white noise resonates with Rachel's noise-induced pattern formation. Jasek's peakon analysis resonates with spiky patterns and singularity formation studied by Michael and Wei.

Participants

Cheviakov, Alexei (University of Saskatchewan)
de Vries, Gerda (University of Alberta)
Fetecau, Razvan (Simon Fraser University)
Hillen, Thomas (University of Alberta)
Iron, David (Dalhousie University)
Kolokolnikov, Theodore (Dalhousie University)
Kuske, Rachel (University of British Columbia)
Kutz, J. Nathan (University of Washington)
Muir, Paul (Saint Mary's University)
Ou, Chunhua (Memorial)
Spiteri, Ray (University of Saskatchewan)
Szmigielski, Jacek (University of Saskatchewan)
Ward, Michael (University of British Columbia)
Wei, Juncheng (University of British Columbia)
Williams, JF (Simon Fraser University)
Wittenberg, Ralf (Simon Fraser University)
Yi, Yingfei (University of ALberta)

Chapter 39

Postdoctoral Retreat in Stochastics (15w2216)

September 18 - 20, 2015

Organizer(s): Mike Kouritzin (University of Alberta), Ed Perkins (UBC)

Overview

This was a first meeting of the PIMS Postdoctoral Training Centre in Stochastics in which 4 of the 5 postdoctoral fellows in the first year of the program, together with their advisors and 6 Ph.D. students from UA, U. Calgary, UBC and U. Washington met to discuss research in a variety of areas in pure and applied probability. All of the students and postdocs attending made 40 minute presentations on their current research. The one postdoc (at U. Washington) who did not attend could not obtain his visa in time in spite of his very best efforts. The purpose of the meeting was for the postdocs, faculty and senior students to get acquainted with each other and give the young researchers an opportunity to explain their work to the Western Canadian probability community.

The Retreat in Stochastics was a huge success from both mathematical and collaborative points of view. It became apparent that our postdoctoral fellows and senior students are already attacking challenging problems at the forefront of probability and stochastics.

Presentation Highlights

The mix of the latest theoretical developments in the subject and some more applied modelling results was very healthy for the young researchers. For example, in the regime of heavy tailed distributions there were two quite different sets of results presented. Mathav Murugan (UBC) spoke on his recent work on getting precise on-diagonal asymptotics for heavy-tailed random walks on general graphs. The interesting part of this work was that the key hypothesis concerned a sub-Gaussian estimate for the simple random walk on the graph–something that is relatively well-understood now. Getting off-diagonal estimates was presented as an important open question. In a more applied direction, recent PhD graduate Samira Sadeghi (UA) presented optimal convergence results on stochastic approximation and outer products of two-sided linear processes in the heavy-tailed and long-range dependent setting, while demonstrating their use in big-data parameter estimation.

Khoa Le (UC/UA) presented some very recent results on the parabolic Anderson model driven by space-time noise, which is white in time and coloured (so long range dependent) in space. There has been much activity recently on the asymptotic behaviour in t of the nth moments. Xia Chen recently confirmed a conjecture of Mueller and Khoshnevisan and calculated the precise exponential growth rate of the nth moments in the white noise setting in one spatial dimension when the initial data is constant. Le studies the long term rate of propagation of the moments when the initial data is compact and gives a variational characterization of the critical propagation rate for a quite general setting and goes on to develop large deviation results. Also in the realm of stochastic pde, PhD student Chi Dong (UA) presented a purely topological technique (as opposed to classic Ray-Knight theorems) for replicating measures and processes living on “bad spaces” (e.g., the tau topology, pseudopath topology) or non-compact spaces onto compact metric spaces. As applications, several filtering equation, stationary filter, and martingale problem solution results (that were only known for compact or Polish spaces) were all extended to these more general settings.

Richard Balka (UBC) gave a complete and elegant proof of his recent result with Yuval Peres that a.s. Brownian motion can only be of bounded variation on time sets of dimension at most $1/2$. Hence the zero set and points where the path equals its past maximum attain the maximal dimension. The result has recently been extended to include fractional Brownian motions of any Hurst index.

In the mathematical finance area, Jonathan Chavez (UC) presented a new (more realistic) limit order book model and considered the resulting price process. He gave a weak invariance principle for this price process and supported his result with simulations. PhD student Ilnaz Asadzadeh (UC) presented a time series model for energy prices that was tested on real data. The model was based upon a copula between the current price and most recently prior price that handled atypical joint tail behaviour. Evidence was given that there was little gain in including earlier prices in the model.

PhD student Brett Kolesnik (UBC) gave a wonderful introduction to the Brownian map, the random surface counterpart of Brownian motion which was featured in J.F. Le Gall's plenary talk at the 2014 ICM. He went on to describe his work with Angel and Miermont on the geometry of the cut loci of the Brownian map. Last but certainly not least, PhD student Matt Junge (UW) presented his recent proof of equidistribution of the max-2 process, answering an open problem of Benjamini, Maillard and Paquette. The max-2 process is an inductively defined sequence of points in the unit interval where at the nth step two points are chosen at random and the one falling in the larger interval determined by the previously chosen points is kept. The result states that the empirical distribution of the resulting set of points converges to the uniform distribution.

Open Problems

During an open problem session, a number of senior and junior researchers presented problems. These included classical problems on competing species models in measure-valued diffusions (Ed Perkins, UBC) and on the Martin boundary for super-Brownian motion (Deniz Sezer), and questions of current interest such as the probability of human extinction due to an asteroid collision (Matt Junge, UW). Brett Kolesnik (UBC) posed an interesting question on coexistence of competing binary branching Brownian motions in one dimension (the two populations annihilate upon contact) which generated interested discussion among participants.

Scientific Progress Made

Chi Dong made progress on recovering the classic Ray-Knight theorem from his replication technique. Ilnaz Asadzadeh started investigating particle filters and discussion with the UA group. Brett Kolesnik's problem prompted further discussion at UBC where Nathanael Berestycki, an international expert on branching Brownian motion, was visiting. Some simulations suggest coexistence should be possible and it has evolved into a working project for Brett with Nathanael and Omer Angel.

Outcome of the Meeting

An enjoyable aspect of the meeting was the interest of all participants in the wide range of topics discussed. It confirmed the high level of the PTCS postdoctoral fellows. There was also an opportunity to discuss future plans for the program including broadcast seminars, outside speakers and short courses. Participants clearly enjoyed the informal atmosphere and plans were made for visits to other PIMS sites. Everyone was happy to learn that this will be an annual event for the program.

Participants

Angel, Omer (University of British Columbia)
Asadzadeh, Ilnaz (University of Calgary)
Balka, Richard (University of British Columbia)
Barlow, Martin (University of British Columbia)
Chavez Casillas, Jonathan (University of Calgary)
Dong, Chi (University of Alberta)
Junge, Matthew (University of Washington)
Kolesnik, Brett (University of British Columbia)

Kouritzin, Michael (Univerisity of Alberta)
Le, Khoa (Mathematical Sciences Research Institute)
Murugan, Mathav (University of British Columbia)
Perkins, Ed (UBC)
Sadeghi, Samira (INVIDI Technologies & University of Alberta)
Sezer, Deniz (University of Calgary)
Swishchuk, Anatoliy (University of Calgary)
Wallace, Ben (University of British Columbia)
Ware, Tony (The University of Calgary)

Chapter 40

2015 Canadian Math Kangaroo Contest Workshop (15w2191)

December 4 - 6, 2015

Organizer(s): Rossitza Marinova (Concordia University of Edmonton), Mariya Svishchuk (Mount Royal University)

The Third Canadian Math Kangaroo Contest (CMKC) workshop event was generously supported by BIRS and CMKC. BIRS provided the venue for the workshop, CMKC supported the workshop with $5000 for travel and meals of participants. Contest representatives and volunteers from the Calgary AB, Edmonton AB, London ON, Montreal QC, and Ottawa ON attended the workshop. In total, the number of participants was seven (7).

Overview of the Field

Math Kangaroo is an annual international math competition for school students. This is the world's largest math competition, with more than six million participants worldwide. The main purpose of Math Kangaroo is to introduce participants to math challenges in an enjoyable way, thus, inspiring their further interest and advancement in mathematics. It provides participating students with great and valuable experience in competitive math.

Math Kangaroo contest is unique to Canada. Students can participate in Math Kangaroo independently of their home school's involvement. Math Kangaroo Centres in Canada typically are universities. Almost all other competitions run through schools. It is still one of the very few math contests available for Canadian elementary students.

While the reputation, the merit, and the quality of inspired learning are at a very high level, the atmosphere on the contest day is unique compared to most of the other contests. The Canadian Math Kangaroo program contributes to the science, engineering and education communities through its activities that revolve around the contest but go far beyond its organization.

In 2015, Math Kangaroo was offered in 33 location. We had 4197 contestants (26% increase compared to 2014) , 7 new centres have been opened to administer the contests. Hundreds of students were involved in various training and learning activities prior to the contest day.

Organization of such event requires well planned preparation that includes problems selection and verification with Canadian math curriculum, development of training materials and legal regulations for the centres, and much more.

Recent Developments and Open Problems

The workshop was intended for mathematicians and mathematics educators involved with the Canadian Math Kangaroo Contest, the organization administering Math Kangaroo in Canada.

The main objective of the workshop was to efficiently finalize the packages of English version problem sets for all levels of the 2016 contest.

Additional objectives were:

1. Verification of problems with Canadian math curriculum.
2. Completing solutions to the problems.
3. Decisions on matters relating to pre-contest activities, contest day, post-contest activities;
4. Dealing with the recent growth of the competition locally and nationally.

Discussion Highlights

The workshop consisted of several discussion sessions on topics of interest to the organization and the workshop participants.

- Friday, December 4: the evening session was devoted to the organizational problems. We discussed and analyzed the 2015 contest issues at the national and regional levels.
- Saturday, December 5: the morning and afternoon session were devoted to problems selection, solution and verification process for all levels. During the evening session, the participants discussed some legal issues in connection with the administration of regional centres.
- Sunday, December 6: the morning and afternoon sessions were used for finalizing the problems selection.

Meeting Progress Made

The two-days BIRS workshop facilitated discussions and decisions on how to further improve the organization of the Math Kangaroo contest and accompanying programs.

Outcome of the Meeting

The workshop is another significant milestone for the Canadian Math Kangaroo Contest organization. Representatives from various cities and provinces exchanged ideas and discussed issues. The major meeting outcomes include:

- Improving the quality of the mathematical content offered by the Math Kangaroo contest.
- Increasing the visibility of the contest and its accompanying activities.
- Connecting volunteers to work together towards inspiring K-12 students about mathematics.

Sharing information and ideas is crucial for maintaining a program of such scope, diversity, quality and continuity. The BIRS workshop facilitated efficient collaboration, coordination, and knowledge transfer among Math Kangaroo coordinators.

Participants

Marinova, Rossitza (Concordia University of Edmonton)
Pandeliev, Todor (AVG Technologies)
Pandelieva, Valeria (Canadian Math Kangaroo Contest)
Pelczer, Ildiko (Concordia University Montreal)
Petterson, Josey (Canadian Math Kangaroo Contest)
Sendov, Hristo (University of Western Ontario)
Svishchuk, Mariya (Mount Royal University)

Focused Research Group Reports

Chapter 41

Localization of Eigenfunctions of Elliptic Operators (15frg188)

March 27 - April 4, 2015

Organizer(s): Doug Arnold, University of Minnesota, Guy David, Université de Paris Sud, Marcel Filoche, École Polytechnique, David Jerison, Massachusetts Institute of Technology, Svitlana Mayboroda, University of Minnesota

A short reminder of the subject area of the workshop Vibrations and waves play a major role in many fields of physics and of engineering, whether they are of acoustic, mechanical, optical or quantum nature. In many cases, the understanding of a vibrational system can be brought back to its spectral properties, i.e. the properties of the eigenvalues and eigenfunctions of the related wave operator. In disordered systems or in complex geometry, these eigenfunctions may exhibit a very peculiar characteristic called localization: the spatial distribution of the eigenfunctions can be strongly uneven, most of the energy being stored in a very small subregion of the entire domain. This phenomenon can have important consequences on the macroscopic behavior of physical systems, as for instance the metal-insulator transition in disordered alloys or the enhanced damping of waves achieved by complex geometries.

In terms of mathematics, the central question is the spatial behavior of the eigenfunctions of a divergence form elliptic operator, e.g., $L = -\mathrm{div} A\nabla + V$, in a bounded domain Ω, with various types of boundary conditions. In particular, one aims to predict and to quantify localization of the eigenfunctions triggered by irregularities of the coefficients of the elliptic matrix $A = A(x)$, of the potential V, or of the domain Ω. Given an obviously extended range of applications, many ad hoc approaches have been developed to treat particular instances of this problem, in particular, Laplacian on various peculiarly shaped domains (e.g., fractals), or Hamiltonian $L = -\Delta + V$ with a disordered potential (e.g., semiclassical theory and the studies surrounding Anderson localization). However, there has been no overreaching theory which would address occurrence and frequency of localized eigenfunctions, their specific spatial location and severity of localization in the general scenario. In particular, interplay between the influence of A, V, and Ω, seemed largely out of reach.

A more precise description of the specific program for Research in Teams The recent years brought several breakthroughs elucidating the profound mechanism of localization of eigenfunctions.

In 2009, Filoche et Mayboroda have shown in [Phys. Rev. Letters, 2009] that even an apparently very simple modification of the boundary of the domain can lead to a drastic alteration of the properties of its low frequency vibrations. In their example, clamping one point inside the domain could split a vibrating plate into two almost independent subregions in which the stationary waves would localize. In particular, the example indicated the strength and the peculiarities of localization of eigenfunctions of the biLaplacian.

This preliminary work has been followed by the discovery in 2011 of a universal mechanism of localization for the eigenfunctions of elliptic operators [Filoche, Mayboroda, PNAS 2012]. In this theory, one single function, obtained in the entire domain as the solution of a simple Dirichlet problem (or, more generally, as an L^1 norm of the Green function in one of the variables), defines a landscape in which the valleys are the boundary lines (in 2D) or surfaces (in 3D) separating neighboring localization subregions. This theory therefore splits every complex vibrating system into a hidden partition of disjoint subregions in which the low frequency eigenfuntions are localized.

Further development of the collaboration indicated that the boundaries separating the localization subregions can also be seen as free boundaries of a certain minimization problem. The functional defining minimization is comprised of two terms. The first one is the energy naturally associated to the aforementioned landscape on each localization subregion. For instance, for a Hamiltonian, this energy is $\int |\nabla u|^2 + Vu^2 - 2u$. The second term is a convex function of the volumes of the localization subregions, used, in particular, in order to eliminate trivial solutions but its exact role is yet to be understood. We have been able to prove [David, Filoche, Jerison, Mayboroda, 2014, submitted] that the minimizers exist, are Lipschitz regular, and that the free boundaries of all (nondegenerate) subregions are (n-1)-Ahlfors regular and uniformly rectifiable. Actually, we demonstrate a stronger result, that the subregions satisfy interior and exterior big balls condition. All these results pertain to a slight generalization of the case of the Hamiltonian and treating functionals with variable coefficients or higher order differential operators remains a challenging open problem. We have, however, already obtained some preliminary results for the case of the bilaplacian.

In the workshop, we mostly focused on a completely different direction. We have recently showed that the landscape function, or rather its reciprocal, $W = 1/u$, determines an Agmon distance. Hence, not only it governs the decay of eigenfunctions at a linear scale, as described above, but also is responsible for their *exponential decay* under favorable circumstances. We have worked out several key examples illuminating the advantages that the landscape function offers compared to the original potential of the system (if the latter is present) in terms of the precise estimates for exponential decay.

What we did in Banff

The week in Banff allowed us to finalize a description of the some of the results mentioned above (a paper called “The effective confining potential of quantum states in disordered media” just submitted). We also plan to write a paper that describes the mathematical aspects of the results concerning W (that is, exponential decay and Agmon distance, etc.), and the Banff meeting greatly helped to crystallize our ideas in this regard. In addition we spent some time testing predictions for the localization of eigenmodes and the values of the coresponding eigenvalues for a variety of random operators (mainly Schrödinger operators with a random periodic potential on the line), and imagine automatic ways to compute the first eigenvalues in higher dimensions. We believe that the algorithms developed in Banff give a base for a future paper in numerical analysis, concerning the approximation of the lower energy eigenvalues. Finally we think we made progress on the reciprocal to the localization statements hinted above, i.e., find reasonable conditions under which, if the Schrödinger operator with potential V has strong localization properties, than this is seen by the Agmon distance computed with the help of our effective potential $W = 1/u$.

We also tried to set reasonable tasks, to be pursued in the foreseeable future. We don’t give the list here.

We are very happy with the results of the workshop. Obviously discussions with the whole group present at the same time helped us make substantial progress in the understanding of the relation between the wells of the effective potential W and the localization properties of the initial operator. It was particularly helpful that D. Arnold could come, because this way we were able to do quite a number of experiments which otherwise would probably have taken forever.

The material conditions of the meeting were nearly perfect.

Participants

Arnold, Douglas (University of Minnesota)
David, Guy (Universite Paris XI, France)
Filoche, Marcel (Ecole Polytechnique; France)

Jerison, David (Massachusetts Institute of Technology)
Mayboroda, Svitlana (University of Minnesota)

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Chapter 42

Classification of Multiplicity-Free Kronecker Products (15frg193)

May 10 - 17, 2015

Organizer(s): Christine Bessenrodt (Leibniz University Hannover), Jeffrey Remmel (University of California - San Diego), Stephanie van Willigenburg (University of British Columbia), Sami Assaf (University of Southern California), Christopher Bowman (City University London), Angela Hicks (Stanford University), Vasu Tewari (University of British Columbia)

Overview of the Field

Schur functions have been a central area of research dating from the time of Cauchy in 1815 (bialternants [7]) to the present day work of Field's medallists such as Tao (the hive model [8]) and Okounkov (Schur positivity [9]). This is due to their ubiquitous nature in mathematics and beyond.

Schur functions are labelled by weakly decreasing sequences of positive integers, called partitions. Given two Schur functions, they can be 'multiplied' in three different ways: outer product, inner product, and plethysm. We wish to understand the coefficients that arise in expanding such a product with respect to the basis of Schur functions.

The outer product is the most well understood, the coefficients arising are the Littlewood–Richardson coefficients and there is an efficient combinatorial description of these coefficients, called the Littlewood–Richardson rule. In the case of the inner and plethysm products, the coefficients are called the Kronecker and plethysm coefficients respectively, and they are substantially more difficult to understand. The determination of the classical Clebsch-Gordan coefficients arising in the decomposition of tensor products amounts in the case of the symmetric groups to a search for an efficient combinatorial description of the Kronecker coefficients; it is one of the central problems of algebraic combinatorics today.

Additionally, outside of pure mathematics, this Kronecker problem has gained great traction of late due to its deep connections with quantum information theory and the central role it plays within Geometric Complexity Theory, an approach that seeks to settle the celebrated P versus NP problem via Geometric Complexity Theory. The P versus NP problem is one of the several \$1,000,000 Millennium Prize Problems set by the Clay Mathematics Institute. Geometric Complexity Theory poses several conjectures on the computation of Kronecker coefficients, even for special partitions, and on deciding whether a given coefficient is non-zero. It is conjectured that the decidability problem is in $\sharp P$ (is 'easy'), whereas calculation of a given coefficient is in $\sharp NP$ (is 'hard'). It is known that the problem of computing Kronecker coefficients is $\sharp P$-hard [6].

Recent Developments and Open Problems

The search for an efficient combinatorial description of the Kronecker coefficients (of a similar flavour as the Littlewood-Richardson rule) has formed a very active research area for several decades. Along the way, many beautiful formulas have been discovered for specific cases, including

- products indexed by a pair of hooks or two-part partitions [10], [11], [12], [13], and [5];
- constituents indexed by partitions of small depth [16], by partitions of small Durfee size [1], or by hooks [4];
- products with few homogenous components [2] and [3];
- products indexed by partitions of near-rectangular shape [15].

These formulas serve both as inspiration, and as a reminder that there is still a great amount of work to do.

One further avenue, which has yet to be pursued, is to classify Kronecker products whose coefficients satisfy certain bounds. In this direction, in 1999 Bessenrodt conjectured a complete classification of multiplicity-free Kronecker products of Schur functions (or equivalently, of irreducible characters of the symmetric groups), that is, pairs of Schur functions whose Kronecker product only yields coefficients equal to 0 or 1.

During the recent American Institute of Mathematics workshop Combinatorics and complexity of Kronecker coefficients our group made progress towards Bessenrodt's conjecture for the case where one Schur function was indexed by a hook, and the other Schur function was indexed by an arbitrary partition.

Therefore the main aim for our Focused Research Group was to completely resolve Bessenrodt's 1999 conjectured classification of multiplicity-free Kronecker products.

Scientific Progress Made

Our Focused Research Group was completely successful in proving the following theorem, which had been a conjecture of Bessenrodt, dating from 1999; at that time, it was known that all the products on the list were multiplicity-free and the problem remained to show that the list was complete.

Theorem 42.0.1 *Let λ, μ be partitions of $n \in \mathbb{N}$. Then the Kronecker product $s_\lambda \otimes s_\mu$ of the Schur functions s_λ, s_μ is multiplicity-free if and only if the partitions λ, μ satisfy one of the following (up to conjugation of one or both of the partitions):*

1. *One of the partitions is (n), and the other one is arbitrary;*
2. *one of the partitions is $(n-1, 1)$, and the other one is a fat hook;*
3. *$n = 2k+1$ and $\lambda = (k+1, k) = \mu$, or $n = 2k$ and $\lambda = (k, k) = \mu$;*
4. *$n = 2k$, one of the partitions is (k, k), and the other one is one of $(k+1, k-1), (n-3, 3)$ or a hook;*
5. *one of the partitions is a rectangle, and the other one is one of $(n-2, 2), (n-2, 1^2)$;*
6. *the partition pair is one of the pairs $((3^3), (6, 3))$, $((3^3), (5, 4))$, and $((4^3), (6^2))$.*

It is worth pointing out that a classification of multiplicity-free outer products of Schur functions was obtained by Stembridge [14]; the application of his result (and a generalization due to Gutschwager) plays an important role in the proof of Theorem 42.0.1.

A key facet of our proof was to deal first with some special situations which are close to the products on the list and which arise as critical situations in the final induction proof. Crucial aspects are a detailed case analysis and a delicate application of both a recursion formula for Kronecker products due to Dvir and a semigroup property of Kronecker coefficients observed by Manivel; for small size, also computer calculations have been used. Towards the classification result we showed first:

- Let λ be a hook or a 2-part partition. Then the Kronecker product $s_\lambda \otimes s_\mu$ is multiplicity-free if and only if it appears on the list given by Theorem 42.0.1.

As part of the induction argument in the proof of Theorem 42.0.1, we next dealt with products $s_\lambda \otimes s_\mu$ where λ was assumed to be a rectangle, and then more generally with the case of a fat hook λ.

A further important feature of the inductive approach to the classification result stated in Theorem 42.0.1 was to employ its close connection to a more general classification conjecture on multiplicity-free products of skew Schur functions (formulated by Bessenrodt some years ago).

With these results and insights as our foundation, we were able to extend our case analysis and use our techniques once more to eventually prove Theorem 42.0.1. As just indicated, this then also led to the confirmation of the more general classification conjecture for multiplicity-free products of skew Schur functions.

Outcome of the Meeting

Further to completing our project, from which a journal article will result, we were also able to become fully versed in the strengths and weaknesses of powerful techniques such as Dvir recursion and Manivel's semigroup property, and learn new techniques related to quasisymmetric functions. These were facilitated by informal lectures given by Bessenrodt and Remmel, respectively.

We would like to thank BIRS for this indispensable Focused Research Group opportunity, which was crucial to the success of our project.

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Chapter 43

Geophysical Viscoplastic Flows (15frg215)

October 25 - November 1, 2015

Organizer(s): Neil J. Balmforth (UBC)

Preamble Mud, lava and ice are examples of complex geophysical fluids, in the sense that they have a complex constitution at the microscopic scale that controls their macroscopic material properties and complicates their flow behaviour in comparison to simple viscous fluids like air and water. Under the right physical conditions, all three geophysical materials can collapse and flow, creating natural hazards and triggering environmental and economic disasters. The unifying feature of the flow behaviour of these three materials is that they possess a yield stress and nonlinear rheology - the hallmarks of viscoplastic fluids.

A recent example relevant locally to Banff was the failure of the Mount Polley mine tailings dam. In August 2014, this failure released millions of cubic metres of water and toxic mine waste products in a few hours, prompting the government to declare a local state of emergency and leaving a devastated area in which the long-term environmental impacts remain unclear. Fluid mechanics is key to understanding several aspects of the disaster, ranging from the catastrophic erosional incision of the dam breach at the initiation of the release, to the sediment plume entering Quesnel lake at its terminus. More relevantly to viscoplastic fluid mechanics, the outburst likely took the form of a concentrated mud flow over part of its duration. Such flows are similar to dense snow avalanches and other types of landslides and debris flows, but the detailed mathematical modelling of such phenomenon remains in its infancy.

In October of 2015, BIRS hosted the *Sixth Workshop on Viscoplastic Fluids: from Theory to Experiment.* The first meeting of this series was held at Banff in 2005, and the sixth installment returned to BIRS a decade later. The Focussed Research Group ran concurrently with this workshop to gain a topical perspective on the status of the subject and to consider in particular the mathematical modelling of geophysical applications of viscoplastic fluid mechanics. The focussed group remained together beyond the 5-day workshop, dedicating more time to existing collaborations, to develop new ones, and, overall, to discuss a range of more specific problems in more depth than were presented the workshop.

Group members

- Neil Balmforth, University of British Columbia
- Richard Craster, Imperial College
- Duncan Hewitt, University of Cambridge
- Sarah Hormozi, University of Ohio
- Andy Hogg, University of Bristol
- Jim McElwaine, University of Durham
- David Pritchard University of Strathclyde
- Anthony Wachs, University of British Columbia

The group members all had experience in the modelling of complex geophysical flows. However, they also had a spectrum of expertise ranging from numerical computation to asymptotic analysis and analogue experimentation. Several had already forged existing collaborations; we built upon these and established more, exploiting the different, but complementary approaches taken by different group members

Existing collaborations between the group members include some fostered by some of the earlier workshops on viscoplastic fluids. From the perspective of techniques, numerical expertise is provided by Hormozi, Hewitt, McElwaine and Wachs, asymptotic analysis by Balmforth, Craster, Hogg and Pritchard, experimental expertise by McElwaine. In terms of areas of application, Hormozi, Balmforth and Wachs are experts in viscoplastic fluids, McElwaine is leading researchers in granular materials, Hormozi, Pritchard and Wachs have research experience in engineering problems, and Balmforth, Hewitt and Hogg have expertise in complex geophysical fluids.

Agenda

The original mandate of the group members was focussed on the mathematical modelling of environmental hazards like mud flows, outbursts from tailing ponds and granular avalanches. However, as the 5-day workshop progressed, it became clear what other topical and interesting problems might be profitably discussed by the FRG over the days that followed. The major step forward was to identify and outline some specific problems to look at, either collectively as a whole group, or in smaller collaborations of group members. Since the meeting, progress has already been made on some of these problems. For the future, our intent is to maintain the FRG as a productive collective and to try to meet regularly on similar occasions to the Banff meeting.

Specific topics that were identified and prompted in depth discusions:

Stopping times: Owing to the nature of the fluid, and specifically the yield stress, viscoplastic fluids are often observed to approach a state of rest in finite time when the driving forces are insufficient to sustain motion. By contrast, asymptotic theories for free surfaces flows, of the kind typical in complex geophysical gravity currents, suggest that the stopping time is infinite (as shown by Hogg, Balmforth and Craster and others). Computations by some of the group members of the full flow problem, with any asymptotic approximation, do in fact suggest the stopping time is finite. Evidently, there is some feature of the asymptotics that is incorrect. One possibility is that the asymptotic approximation is not uniform in time and breaks down as the fluid comes to rest, demanding a more refined treatment, a more faithfull perturbation theory.

The effect of thixotropy: A recent recognition in viscoplastic fluid mechanics is that the material structure of real fluids is rather more complicated than captured by standard models. Notably, ageing effects resulting from the relatively swift destruction of the microstructure, but its much lengthier healing, result in a persistent time-dependent rheology that is poorly captured by current models which assume an instantaneous relation between stress and strain rate. Many clays and other geophysical materials are thixotropic, but a systematic analysis of how this might affect their flow behaviour in environmental settings has yet to be provided (Balmforth, Hewitt and Pritchard have made initial steps in this direction). From cruder perspective, the FRG speculated on whether related ideas might be relevant to more varied problems including the surges seen on ice-covered rives and lakes, and to the mechanism of mucus clearance during a cough.

Failure mechanisms: Problems in geotechnical engineering are sometimes posed in terms of the calculation of critical factors dictating the onset of flow, or failure. In the collapse of a vertical embankment, for example, one requires the critical height or cohesive strength of the soil. Such problems are often posed using the limit-point analysis of plasticity theory. In fact, the ideal plastic material is closely related to the perfect viscoplastic fluid, which the two potentially behaving identically near the onset of flow. Thus the failure criteria plasticity theory may be relevant in viscoplastic fluid mechanics. Despite this, viscoplastic fluids often appear to fail due to the appearance of narrow viscoplastic boundary layers which are not necessarily features of the plasticity problem (as computations by Balmforth, Hormozi and Wachs have shown). Classical discussion of such boundary layers dates back to Oldroyd in 1947. However, his boundary layer equations are complicated nonlinear partial differential equations that have never been solved. A major question is whether they do indeed characterize the boundary layer, or whether they do not actually have a solution and a different theory is needed. There is also an interesting connection with granular mechanics (the failure of sand piles and so forth) which McEwlaine and Hogg have studied extensively, both experimentally and with discrete element simulations. A key question is whether the analysis of the granular problem can inform the viscplastic theory, or vice versa. For all these contexts, the FRG discussed the idea that significant progress might be made by looking at specific, idealized flow problems. For such problem, one can provide numerical computations of the full flow problem to compare with boundary-layer asymptotics. Such computations require detailed care in view of the small but finite velocities that characterize failure near the critical conditions (the limit point), and there is a distinct possibility that new tools of numerical analysis are needed to deal accurately with the situation.

Suspension mechanics: Recent studies of concentrated suspension have proposed different rheological characterizations than the traditional viscoplastic constitutive laws. In part, these are based on the granular nature of the solid constituents, opting for a two-phase approach rather than the single-phase theories common in viscoplasticity. Nevertheless, both approaches lead to constitutive laws that describe fluid yield stress and nonlinear viscous behaviour. This raises the question of how to relate the two theories, and in what situations one might be preferable over the other. The applications include the flow of slurries through cracks in hydrofracture and mud hydrology, or the disposal of mine tailings. Also related is the compressional rheology of mineral slurries, which is used to characterize these materials and is necessary to understand their de-watering characteristics during disposal. De-watering is essential in other types of suspensions, such as wood pulp (as recently explored by Hewitt and others), with potentially significant industrial impact. Geophysical currents are usually dominated by shear stresses, whereas de-watering problems more typically involve compression. This raises the interesting question of whether the material behaves differently under compression than in shear, and if it does, how to characterize that. A critical effect that poses many natural hazards and which results from suspesnion rheology is liquification. As far as the FRG was aware, the modelling of this process was rudimentary, and much could be gained from improving this situation. The FRG considered approaching the problem from the theoretical side. In particular, we discussed how to formulate model flow problems that could be explored using existing empirical constitutive models. The results can be compared with solutions from computational simulations at the particle level (as pioneered by Wachs and Hormozi), thereby providing a demanding test of the empirical models.

Slip: Recent experimental work has demonstrated conclusively that many viscoplastic materials slip over smooth surfaces. This is not real slip, but a manifestation of the development of spatial inhomogeneity within the fluid: the microstructure which dictates the non-Newtonian behaviour becomes depleted near the surface due to the migration of the constituents that form the microstructure out of a thin layer adjacent to the surface. This apparent slip is well known for polymer solutions, and has now been demonstrated clearly for many viscoplastic fluids. Despite this, slip is often ignored in theoretical modelling, primarily because one needs a characterization of the slip; *i.e.* a slip law. Until recently, little information on such laws was available. However, careful experiments and micromechanical models have now proposed a number of realistic, calibrated slip laws that can be incorporated into theoretical flow models. This now places us in a position where we can model wall slip and revise many of the flwo solutions to problems in which this phenomenon should be taken into account and previously been incorrectly modelled by its omission. The slip issue certainly applies to geophysical gravity currents for which the slip over the underlying surface may well be a controlling factor in the flow dynamics, including when materials become mobile and when they return to rest. There is also the interesting question of whether the modelling in comparison to observations or experiments could inform the slip model in situations in which the callibration or the slip law itself is not known.

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The best sources of references for recent work in viscoplastic fluid dynamics is the series of proceedings that have followed the bi-annual workshops:

Visoplastic Fluid Mechanics: from theory to application

I: *Journal of Non-Newtonian Fluid Mechanics*, bf vol. 142

II: *Journal of Non-Newtonian Fluid Mechanics*, bf vol. 158

III: *Rheologica Acta*, bf vol. 50

IV: *Journal of Non-Newtonian Fluid Mechanics*, bf vol. 193

V: *Journal of Non-Newtonian Fluid Mechanics*, bf vol. 220

A further issue of the *Journal of Non-Newtonian Fluid Mechanics* has been set aside for the upcoming sixth proceedings.

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Chapter 44

Current Challenges for Mathematical Modelling of Cyclic Populations (15frg202)

November 8 - 15, 2015

Organizer(s): Rebecca Tyson (University of British Columbia Okanagan), Frédéric Barraquand (Université de Bordeaux), Jonathan Sherratt (Heriot-Watt University), Don DeAngelis (University of Miami)

Overview

The 8 members of the focussed research group met with the express purpose of writing a paper based on the successful 2013 BIRS 5-day workshop of the same title (BIRS 13w5151). During the Focussed Research Group, we submitted a paper proposal to Ecology Letters, a widely-read and high impact journal, which was accepted. It is therefore to this journal that we have submitted our paper (submission date: February 5th, 2016).

The paper is co-authored by virtually all of the attendees of the 2013 workshop, which include both mathematicians and ecologists. The goal of the original workshop was to look for mathematical innovation at the interface of theory and data. This interdisciplinary approach naturally carried over into the writing of the paper, which is thus a document that marries both mathematical and ecological points of view. From a mathematical perspective, the paper achieves two very important goals. First, the paper contains a wealth of information on cyclic population dynamics and models thereof, gathered in one place, that will serve as an invaluable resource to mathematicians interested in developing new theory. Second, as a publication in the journal Ecology, the paper will receive a wide readership within the theoretical ecology community. Consequently, the mathematical aspects of the paper will have the widest possible impact among ecologists, thus creating bridges between mathematicians and ecologists.

The following report is a summary of some of the highlights of the paper.

Abstract Population cycling is a widespread phenomenon in ecology, observed across taxa in both lab and field conditions. Despite the theoretical and practical relevance of population cycles to our appreciation of ecological dynamics, both the mechanisms underlying such phenomena and their consequences for whole ecosystems remain incompletely understood. Our paper reviews recent theoretical work in this area. We explore how cycle loss or gain arises from environmental changes, as well as links between population cycles and eco-evolutionary dynamics, spatial synchrony, biodiversity maintenance, and from a management perspective, the control of outbreaks. In particular, we emphasise the importance of stochasticity in modulating or creating cyclic behaviour, and suggest that future developments in the field will need to be intrinsically tied to a stochastic point of view.

Introduction Almost a century after the publication of Elton's seminal paper on population cycles [1], we now understand and can recognize many different causes of oscillatory behavior [2, 12]. While the many successes in the study of population cycles warrant celebration, empirical research continues to reveal areas where our knowledge is far from complete. In our paper, we review how the interplay between data and mathematical theory has led and continues to lead to new insights into the mechanisms behind, and consequences of, cyclic population dynamics.

Although many populations show annual cycles driven by cyclic weather patterns, we focus here on sustained oscillations in population abundance that are not a mere reflection of seasonality. In particular, we highlight cases where connecting observation to theory has proven particularly challenging. For example, the observation that cycle amplitudes of many populations decrease along North-South gradients in the Northern Hemisphere has puzzled ecologists for decades [3]. Earlier mechanistic models suggested that this pattern is due to a geographical gradient in the predator community with a stabilizing influence of generalist predators in the South [4, 5]. However, several alternative explanations are possible, including changes in seasonality and prey life history [3, 6], partly because generalist predators do not always have a stabilising effect in model communities [7] and partly because spatial patterns are not clear-cut [6]. Recent mechanistic modeling shows that incorporating well-documented seasonal changes in prey fecundity can explain these patterns [8]. This example illustrates how empirical work guided theoretical development and how the use of varied assumptions in theoretical studies broadened our understanding of how and why population cycles arise.

Apart from uncertainties regarding the mechanisms causing population cycles, understanding the effects of cycles on ecosystems poses its own challenges. These effects can be rather dramatic as cyclic populations can be momentarily very abundant, and such highly fluctuating communities may play a role in biodiversity maintenance [9]. Furthermore, many open questions remain on the response of cycling populations to environmental changes [10] and, reciprocally, the control of pest outbreaks [11]. Understanding the ecosystem-level consequences of cycles is particularly important for populations that historically cycled but have recently become non-cyclic, and vice versa.

Arguably the best known cause of population cycles is the delayed negative feedback loop between a specialist predator and its prey [13, 12]. Historically, predator-prey theory has had a central role in the study of population cycles. Hence, we begin our review with this basic paradigm, and use it to subsequently introduce alternative causes of cycles. The basic structure of most predator-prey models is a set of differential equations for the prey density, N, and the predator density, P:

$$\frac{dN}{dt} = \underbrace{f(N)}_{\substack{\text{prey}\\ \text{pop. growth}}} - \underbrace{g(N,P)}_{\substack{\text{functional}\\ \text{response}}} P \tag{44.0.1}$$

$$\frac{dP}{dt} = \underbrace{E(g(N,P))}_{\substack{\text{numerical}\\ \text{response}}} P - \underbrace{\mu P}_{\substack{\text{predator}\\ \text{death}}} . \tag{44.0.2}$$

Here, g is the so-called functional response, which describes prey consumption rates as a function of prey and predator densities, E is the numerical response, which describes the conversion of consumed prey into predator population growth, and μ is the predator's per capita death rate. For certain functions E and g, sustained predator-prey oscillations are possible. These oscillations emerge because temporary increases in the prey population support a growing number of predators until over-predation causes both populations to crash, initiating a new cycle.

In our review, we emphasize successes that come from integrating iterative gains made in empirical and theoretical ecology, using modeling approaches that relax classical assumptions made in eqns (44.0.1)-(44.0.2) to yield novel insights. When discussing causes of populations cycles, we make a distinction between so-called qualitative and quantitative causes. We define qualitative causes as structural features of the dynamical system (e.g., predators, parasites) whose presence allows cycles to exist through a delayed negative feedback loop. On the other hand, quantitative features are quantifiable properties such as physiological or life history traits (e.g., high prey fecundity, saturating predator functional response), that make the system more or less prone to exhibit sustained oscillations.

Summary

The paper follows with a discussion of work on the snowshoe hare cycles, higher-dimensional systems including food webs and large communities, age or stage structure and maturation delays, interactions between evolution and population cycles, the role of stochasticity, and the problems of cycle gain and loss as well as ecosystem function and management, especially under global change.

Our paper synthesises recent research and promising avenues for modeling population cycles. Because some of the best-studied cycles involve consumer-resource interactions, we chose this mechanism as a starting point. However, our paper points out that cycles can often be influenced, or even originate, from processes other than the popular time-delayed negative feedback between consumers and resources [13]. More than two interacting species may sometimes be necessary for cycling to develop, as in some tritrophic systems or in non-transitive competition networks. Conversely, stage structure in a single species can induce additional delays leading to cycling [2]. Furthermore, random environmental forcing may induce quasi-periodic cycles in otherwise damped systems, or cause irregular fluctuations between alternative stable states. Cyclic patterns can also be altered by eco-evolutionary dynamics or age and food web structure. These examples illustrate the broad perspective that must be taken in studying the causality of population cycles.

The richness and intricacy of recent empirical observations point to a multitude of challenges in our theoretical understanding of population cycles. These challenges suggest several important directions for mathematical modeling. First, cyclic populations are embedded into large communities that can both affect cycles in non-intuitive ways and be affected by an abundant cyclic species (e.g., altered coexistence mechanisms). Therefore, analysis of multi-dimensional ecological time series and high-dimensional dynamical models are needed to understand cycles in their broader ecosystem-level context.

Second, consideration of behavioral responses, demographic structure and short-term evolution in individual species will be needed in order to explain cyclic population dynamics with greater accuracy. All of these factors can be influenced by environmental perturbations that are generally neither white noise nor purely periodic, but a mixture of both. Future mechanistic models for cyclic populations will therefore likely be demographically structured (perhaps down to the individual level), statistically fitted to rich data sets, and forced by stochastic environmental variation.

Third, spatial or temporal changes in cyclicity, and ways to control such changes, are vibrant areas of research. Much might be learned about mechanisms behind cycling by trying to both explain the variation in population dynamics of cyclic species and, in models or laboratory systems, control such dynamics.

Ecological modeling is not yet at the stage where it can forecast the response of cyclic populations to environmental changes in the field, except in a few cases, but a constant feedback between theory and empirical research will certainly help us move forward in that direction.

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Research in Teams Reports

Chapter 45

Holomorphic Functions on Products on ℓ_∞ (15rit180)

May 3- 10, 2015

Organizer(s): R. M. Aron (Kent State University), P. M. Gauthier (Université de Montréal), M. Maestre (Universitat de València)

Overview of the Field

Let B_{c_0}, resp. B_{ℓ_∞} denote the open unit ball of c_0, the complex Banach space of null sequences (resp. ℓ_∞, the space of bounded sequences) both endowed with the supremum norm, and let D^∞ be a (somewhat careless) notation for the countable product of open unit discs $D \subset \mathbb{C}$ endowed with the product topology. Our interest focuses on certain Banach algebras of $\mathbb{C}-$valued holomorphic functions:

• $\mathcal{H}^\infty(B_{\ell_\infty})$, of bounded holomorphic functions $B_{\ell_\infty} \to \mathbb{C}$,
• $\mathcal{H}^\infty(D^\infty)$, of bounded holomorphic functions $D^\infty \to \mathbb{C}$,
• $\mathcal{A}(B_{c_0})$, of uniformly continuous holomorphic functions $B_{c_0} \to \mathbb{C}$, and finally
• $\mathcal{A}(D^\infty)$, of uniformly continuous holomorphic functions $D^\infty \to \mathbb{C}$.

By holomorphic, we mean complex Fréchet differentiable in the first and third cases, and continuous (with respect to the product topology) and separately holomorphic (i.e. holomorphic in each variable, with all other variables held fixed) in the second and fourth cases [5]. It is shown that

$$\mathcal{A}(B_{c_0}) \backsimeq \mathcal{A}(D^\infty) \subsetneq \mathcal{H}^\infty(D^\infty) \subsetneq \mathcal{H}^\infty(B_{c_0}) \subsetneq \mathcal{H}^\infty(B_{\ell_\infty}).$$

Scientific Progress Made, Open problems and Aftermath

In addition to such inclusion and equality relations, during our stay in Banff we have investigated several fundamental properties related to these algebras, such as their maximal ideal spaces and local structure of fibers in these maximal ideal spaces. Our attention was also directed at examining corona type, or cluster value, theorems in this context. In addition, in a second part of our studies, we investigated approximation of holomorphic functions by rational functions on infinite products of domains.

A primary motivating factor for our interest in these algebras is that they are natural analogues of the disc algebra $\mathcal{A}(D)$ and the standard $\mathcal{H}^\infty(D)$, and of $\mathcal{A}(D^n)$ and $\mathcal{H}^\infty(D^n)$, about whose maximal ideal spaces a considerable amount is known (although a considerable amount remains unknown as well). To be more specific, for a unital commutative Banach algebra $\mathcal{A}$ let $\mathcal{M}(\mathcal{A}) :\equiv \{\varphi : \mathcal{A} \to \mathbb{C} \mid \varphi \text{ is a non-trivial homomorphism } \}$. It is known that $\mathcal{M}(\mathcal{A}(B_{c_0})) \backsimeq \overline{B_{\ell_\infty}}$, via the composition of the canonical extension mapping with evaluation at $b \in \overline{B_{\ell_\infty}}$, given by

$$\tilde{\delta}_b : f \in \mathcal{A}(B_{c_0}) \to \tilde{f} \in \mathcal{A}(B_{\ell_\infty}) \to \tilde{f}(b).$$

The maximal ideal space of $\mathcal{H}^\infty(B_{\ell_\infty})$ is very complicated. Of course, this is to be expected by anyone who has studied the maximal ideal space of $\mathcal{H}^\infty(D)$ (see, e.g., [6]). On the other hand, the situation in infinite dimensions is much worse; for example, there is plenty of analytic structure in fibers over interior points of B_{ℓ_∞} (see, e.g., [3]). Now, $\mathcal{H}^\infty(D^\infty)$ is between $\mathcal{A}(B_{c_0})$ and $\mathcal{H}^\infty(B_{c_0})$, and the structure of its maximal ideal space is, so far, completely unknown.

To the non-specialist, a much less arcane topic is the study of fibers over boundary points in the ideal space of $\mathcal{H}^\infty(D^n)$, for some $n \in \mathbb{N}$. Here, the apparent lack of information regarding these fibers is astonishing. For instance, we have shown during our stay in Banff that there is an analytic isomorphism between $\mathcal{M}_1(\mathcal{H}^\infty(D))$ (which is, by definition, the subset of $\mathcal{M}(\mathcal{H}^\infty(D))$ consisting of those homomorphisms φ lying "over" the point 1, i.e. those φ such that $\varphi(z) = 1$) and $\mathcal{M}_{(1,b)}(H^\infty(D^2))$, for any fixed $b \in D$. On the other hand, it is unknown whether there is a homeomorphism of any kind between $\mathcal{M}_1(\mathcal{H}^\infty(D))$ and $\mathcal{M}_{(1,1)}(\mathcal{H}^\infty(D^2))$. Now, it is known that there is a holomorphic embedding of D into $\mathcal{M}_1(\mathcal{H}^\infty(D))$ and a similar argument shows that D^2 embeds holomorphically into $\mathcal{M}_{(1,1)}(\mathcal{H}^\infty(D^2))$. Surprisingly it is unknown whether D^2 embeds into $\mathcal{M}_1(\mathcal{H}^\infty(D))$. If our conjecture, that this does not occur, were proved to be correct then it would follow that $\mathcal{M}_1(\mathcal{H}^\infty(D))$ and $\mathcal{M}_{(1,1)}(\mathcal{H}^\infty(D^2))$ are different. Of course, one can go on to ask analogous questions for $\mathcal{M}(\mathcal{H}^\infty(D^n))$ for every $n \in \mathbb{N}$. The situation with an infinite product of discs introduces some new wrinkles into the problem. In the context of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, we are able to show the B_{ℓ_∞} can be embedded holomorphically into $\mathcal{M}_{(1,1,\ldots,1,\ldots)}(\mathcal{H}^\infty(B_{c_0}))$. In addition, we do not (yet) know the relation between the latter fiber and a fiber like $\mathcal{M}_{(1,0,1,0,\ldots,1,0,\ldots)}(\mathcal{H}^\infty(B_{c_0}))$. Actually, we cannot decide whether these two fibers are holomorphically homeomorphic using the technique that we have developed to prove that $\mathcal{M}_1(\mathcal{H}^\infty(D))$ is holomorphically homeomorphic to $\mathcal{M}_{(1,b)}(H^\infty(D^2))$. Moreover, we have yet to discover the "position" of fibers like $\mathcal{M}_{(\frac{1}{2},\frac{2}{3},\ldots,\frac{n}{n+1},\ldots)}(\mathcal{H}^\infty(B_{c_0}))$.

In the second part of our work during this week, we studied versions of Mergelyan's theorem that would be valid on products. Mergelyan's theorem is one of the classic results of complex analysis. Recently Gauthier and Nestoridis (who was originally part of our research team at BIRS but was unable to get funding to travel from Athens and, in addition, had health problems) proved a generalization of Mergelyan's theorem for holomorphic functions on products of planar domains, under the assumption that any two points of the domain can be joined by a path in the domain with length smaller than a fixed constant. We worked on an extension of this result, in which the complement of each of the planar domains is allowed to have a finite number of connected components. To be more specific, consider a finite or infinite number of planar domains $\{U_i\}_{i\in I}$ each of whose complements has a finite number of connected components, such that any two points of each domain can be joined by a path in the domain with length smaller than a fixed constant M_i.

Theorem [1] Let $\left(\prod_{i\in I} U_i\right)$ be as above. Then any function in $A\left(\prod_{i\in I} U_i\right)$ can be approximated by a rational function whose poles are fixed elements of the complement.

Finally, Gauthier reported on his recently submitted paper [4], in which he showed that the zero sets of most harmonic functions on open sets in $\mathbb{R}^N$ and most holomorphic functions on open sets in $\mathbb{C}^N$ are non-extendable. The fact that this is a generic phenomenon has "resonance" with a 35-year old work of Valdivia [7]. In fact, we discussed the possibility of investigating this phenomenon in infinite dimensional complex spaces and came to the conclusion that we could and should proceed in this direction.

Participants

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Chapter 46

Subconvexity bounds and simple zeros of modular L-functions (15rit201)

May 24 - 31, 2015

Organizer(s): Andrew Booker (University of Bristol), Micah Milinovich (University of Mississippi), Nathan Ng (University of Lethbridge)

Overview of the Field

This Research in Teams meeting focussed on questions concerning the behavior of L-functions in the critical strip, an old and fundamental problem in analytic number theory. Analytic number theory dates back to work of Riemann (1859) and Dirichlet (1837) who studied the behaviour or prime numbers via analytic methods using functions now known as L-functions. In particular, they defined what we know call the Riemann zeta-function, $\zeta(s)$, and Dirichlet L-functions, $L(s,\chi)$, which are certain meromorphic functions attached to group characters χ of $(\mathbb{Z}/q\mathbb{Z})^{\times}$. These are special cases of arithmetic L-functions which may be thought of as Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ where the sequence $\{a_n\}$ typically encodes arithmetic information. The Riemann zeta function and Dirichlet L-functions are the only examples of degree one L-functions. In the twentieth century, it was realized that there were many more examples of higher degree L-functions. Hecke and Maass discovered certain degree two L-functions which are attached to what are now known as holomorphic modular forms and to Maass forms. Godement and Jacquet constructed higher degree L-functions attached to irreducible cuspidal automorphic representation of $GL(n)$ over the rationals. From another point of view, Selberg [18] gave an axiomatic definition of L-functions. This set of L-functions became to be known as the Selberg class. Conjectures of Langlands suggest that Selberg class of L-functions coincides with the set of cuspidal automorphic L-functions.

L-functions are important since they encode information about many arithmetic problems. For instance, an exact formula for $\pi(x)$, the number of primes less than x, can be obtained by studying the Riemann zeta function and its zeros. The famous Prime Number Theorem, which asserts that $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$, may be deduced from the non-trivial fact that $\zeta(s)$ does not vanish on the line $\Re(s) = 1$. Dirichlet's theorem on the infinitude of primes in arithmetic progressions follows from the non-trivial fact that Dirichlet L-functions do not vanish at the point $s = 1$. Higher degree L-functions are related to other problems in arithmetic. For instance, the Birch and Swinnerton-Dyer conjecture (one of Clay Mathematics Institute's million dollar Millennium Prize Problems) describes a relationship between the group of rational points on an elliptic curve over $\mathbb{Q}$ to order of vanishing of a degree two L-function at a certain point in the complex plane.

Due to this connection between arithmetic and L-functions, researchers are interested in studying the analytic properties of L-functions. Our Research in Teams project involves studying tthe *size* of certain degree two L-functions and the properties of their *zeros*. The Lindelöf hypothesis asserts that any L-function $L(s)$ satisfies the bound $|L(\frac{1}{2}+it)| = O(|t|^{\varepsilon})$. The convexity bound for a degree d L-function, a consequence of the Phragmén–Lindelöf convexity principle, states that $|L(\frac{1}{2}+it)| = O(|t|^{\frac{d}{4}+\varepsilon})$. A *subconvexity bound* is an improvement of the convexity bound for a degree d L-function of the form $|L(\frac{1}{2}+it)| = O(|t|^a)$ where $a < \frac{d}{4}$. For any given L-function it is desirable to prove such subconvexity bounds as many arithmetic consequences can be derived from such bounds.

It is also important to have knowledge of the zeros of an L-function. It is widely believed that all non-trivial zeros of an L-function lie on the line $\Re(s) = 1/2$. This is known as the "Generalized Riemann Hypothesis," and it is another of Clay Mathematics Institute's million dollar Millennium Prize Problems. This is perhaps the most famous open problem in number theory. Another important conjecture is that all non-real, non-trivial zeros of an L-function are simple. As a first step toward establishing this conjecture, it is an open and difficult question to show that an L-function has (quantifiably) many simple zeros. Showing that certain degree two L-functions have many simple zeros is one of they key topics we focussed on in this Research in Teams project.

Recent Developments and Open Problems

In recent years, establishing a subconvexity result for an L-function has been a very popular research subject. In a series of influential articles Duke, Friedlander, and Iwaniec [6], [7], [8], [9] established some of the first results for degree two L-functions. Many of their arguments were generalized considerably in the following years (see, for instance, the article by Michel and Venkatesh [16] and the references contained within). Attached to a modular L-function are various parameters, such as the level and the weight. In some of the above articles, the authors were interested in subconvexity results with respect to the level or weight. We are interested in a subconvexity result with respect to the parameter t, the height of the L-function above the real axis. In this case, we would like to establish there exists $\delta \in (0, \frac{1}{2})$ such that for any $\varepsilon > 0$

$$|L_f(\tfrac{1}{2}+it)| \ll |t|^{\delta+\varepsilon} \tag{46.0.1}$$

where $L_f(s)$ is the L-function attached to a modular form f. In the special case where f is a modular form on the full modular group, Good [5] proved that $\delta = \frac{1}{3}$ is admissible. A different proof of Good's result was later provided by Jutila [12]. Recently, Wu [20] established (46.0.1) with $\delta = \frac{1}{2} - \frac{25}{256}$ for all newforms f and this was slightly improved by Kuan [14] to $\delta = \frac{25}{178}$ in some (but not all) cases. Despite these advances, it remained an open problem to establish the analogue of Good's subconvexity estimate for modular form L-functions of all weights and levels with the value $\delta = \frac{1}{3}$.

The question of showing that a fixed L-function has infinitely many simple zeros began with work of Levinsion [15] for the Riemann zeta function. He showed at that least $1/3$ of the zeros of the Riemann zeta function are simple and lie on the critical line. His work was extended to Dirichlet L-functions by Bauer [1]. For primitive degree two L-functions this first result is due to Conrey and Ghosh [5], established in 1989. They considered Ramanujan's delta function $\Delta(z)$, a level one, weight 12 modular form. They showed that the $L_\Delta(s)$, the L-function attached to Δ, has infinitely many simple zeros. More precisely, let $N_\Delta^s(T)$ denote the number of simple zeros of $L_\Delta(s)$ to height T in the critical strip. They showed the following:

$$\text{For any } \varepsilon > 0 \text{ there are arbitrarily large values of } T \text{ such that } N_\Delta^s(T) \geq T^{\frac{1}{6}-\varepsilon}. \tag{46.0.2}$$

For L-functions attached to both arbitrary modular forms or to Maass form f, we would like to show that

$$\text{for any } \varepsilon > 0 \text{ there are arbitrarily large values of } T \text{ such that } N_f^s(T) \geq T^{\frac{1}{6}-\varepsilon} \tag{46.0.3}$$

where $N_f^s(T)$ denote the number of simple zeros of $L_f(s)$ up to height T in the critical strip. Since Conrey and Ghosh's work [5] it has been desirable to obtain lower bounds for $N_f^s(T)$ for other modular forms or Maass forms f. Cho [4] established the analogue of (46.0.2) for a few Maass forms f. Furthermore, Milinovich and Ng [17] have shown conditionally on the generalized Riemann Hypothesis for $L_f(s)$ where f is *any* holomorphic newform that for any $\varepsilon > 0$, $N_f^s(T) \geq T(\log T)^{-\varepsilon}$, for T sufficiently large. Recently, in a major breakthrough, Booker [3] proved the qualitative estimate that $N_f^s(T)$ is unbounded for all holomorphic newforms f. Despite these results, it is still currently unknown how to prove (46.0.3) or any other quantitative estimate for the number of simple zeros of $L_f(s)$ for all holomorphic newforms f.

Scientific Progress Made

The main goals of this meeting were to establish a general subconvexity result for modular form L-functions $L_f(s)$ and to provide lower bounds for $N_f^s(T)$, the number of simple zeros of the L-function $L_f(s)$ whose imaginary parts lie in $[0, T]$. During this meeting we established the following results.

Theorem 46.0.1 *Let $f \in S_k(\Gamma_1(N))^{new}$ be a normalized Hecke eigenform. Then*

$$|L_f(\tfrac{1}{2}+it)| \ll_{f,\varepsilon} (1+|t|)^{\frac{1}{3}+\varepsilon},$$

with polynomial dependence on N and k.

For level 1 modular forms, this was previously established by Good [5]. Our proof of Theorem 46.0.1 follows an approach of Jutila [12] which was later refined by Huxley [11]. Their argument makes a very clever use of Farey fractions and the Voronoi summation formula. In addition exponential sum techniques and a large sieve inequality are required. A key difficulty in establishing the generalization given in Theorem 46.0.1 is that we needed to prove a new version of Voronoi's summation formula that has a number of new restrictions on the set of "admissible" Farey fractions. These restrictions do not appear in Jutila and Huxley's argument, making the proof of Theorem 46.0.1 quite a bit more involved. We were able to overcome these restrictions and succeeded in making the Jutila-Huxley argument work in this more general setting.

Using the subconvexity estimate in Theorem 46.0.1 along with ideas from the important papers of Conrey and Ghosh [5] and Booker [2, 3], we were able to establish the following quantitative result on the number of simple zero zeros of a Dirichlet twist of a modular L-function.

Theorem 46.0.2 *Let $f \in S_k(\Gamma_1(N))^{new}$ be a normalized Hecke eigenform. There is a primitive character χ such that, for any $\varepsilon > 0$, there exists a sufficiently large T with $N^s_{f\times\chi}(T) \geq T^{\frac{1}{6}-\varepsilon}$.*

In other words, given a modular form f with L-function $L_f(s)$ there exists a 'twisted' L-function which has many simple zeros. Furthermore, we have been able to show that if quasi Riemann hypothesis holds for $L_f(s)$ then it is possible to obtain a result like (46.0.2) with an exponent which depends on the width of the zero-free region.

Outcome of the Meeting and Future directions

In this meeting we succeeded in proving a subconvexity bound in the "t-aspect" for any modular form L-functions. Our result is of the same quality as Hardy and Littlewood's classical bound for the the Riemann zeta function, namely $|\zeta(\frac{1}{2}+it)| = O(|t|^{1/6+\varepsilon})$, for any $\varepsilon > 0$. This generalized an argument of Jutila [12] and Huxley [11] to all modular form L-functions. One future direction would be extend this result to all Maass form L-functions. If this were possible, then it would follows that all degree two L-functions $L_f(s)$ should satisfy a subconvexity bound $|L_f(\frac{1}{2}+it)| = O(|t|^{1/3+\varepsilon})$. We were interested in subconvexity bounds since they have applications to the number of simple zeros of an L-function.

A key open question that still remains is to prove a good quantitative lower bound for the number of simple zeros of any degree two L-function. We were unable to show a good quantitative lower bound for the number of simple zeros of any fixed modular form L-function $L_f(s)$. However, we could show that some twist of $L_f(s)$ has many simple zeros. It is still desirable to establish (46.0.2) for any holomorphic newform f.

Participants

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Chapter 47

Analysis and Computation of Vector Functionalized Cahn Hilliard Equations and Application to Amphiphilic Materials (15rit196)

July 12 - 19, 2015

Organizer(s): Brian Wetton (UBC)

Overview of the Field

Patterns are ubiquitous in nature [3, 4, 5] and have been systematically studied for over a century [19]. This BIRS research in teams group focussed on mathematical models for the diffusion driven self-assembled morphologies of amphiphilies: molecules containing both hydrophilic and hydrophobic components. This class of materials ranges in size from lipids and surfactants, with molecular weight around 500, to block copolymers and graft copolymers which can be 10-1000 times larger [1]. Despite the size disparity, all amphiphiles are driven to form similar nanoscaled, phased separated network patterns and their defects. There are a wide variety of structures to be understood, such as bilayers, pores, micelles, pearled structures, end-caps and Y-junctions [11, 18, 9, 13, 6]. Each of these patterns are abundant in biological structures including cell membranes, and act as indispensable functional elements with technical applications to pharmaceutics, emulsion stabilization, detergent production and energy conversion devices [1, 2, 14, 17, 15].

The team of Arjen Doelman (Mathematics, Leiden), Keith Promislow (Mathematics, MSU), Brian Wetton (Mathematics, UBC) and Qiliang Wu (Mathematics, MSU) came together to discuss how the gradient flow of the functionalized Cahn-Hilliard (FCH) free energy could describe the structures appearing in these materials. The group has expertise in dynamical systems techniques, combined with differential geometry, using computational simulation tools to guide the analysis and confirm results. The emphasis of the proposed work was on the existence of solutions representing patterns in amphiphilic morphology, their defects and their interfaces, together with their qualitative properties, such as linear and nonlinear stability, instabilities, bifurcations and continuations.

Given the ubiquity in biological structures and wide-range technical applications of amphiphiles, the importance of revealing the pattern-forming mechanism behind the rich amphiphilic morphology is self-evident. Due to the lack of such a mechanism, it is always a big challenge to control and tune the mixture and its external and internal conditions to achieve the desired nano scale self-assembled morphologies. Unfortunately, even the finest coarse-grained particle based simulation is on the order of nanoseconds, while the time scale for the amphiphilic morphology is in minutes, hours and even days [33, 36]. The functionalized Cahn-Hilliard free energy is the first analytical continuum model, admitting stable pattern solutions which resemble those in amphiphilic selfassemblies. As such, it provides a prototype, a platform for further and deeper understanding about amphiphilic morphology.

Recent Developments and Open Problems

Some analytic analytic results have been obtained for the FCH relevant to the underlying materials science questions. One version of the scalar FCH equation is

$$u_t = \Delta \left((\varepsilon^2 \Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2 \Delta u - W'(u)) + \varepsilon\eta_d W'(u) \right), \tag{47.0.1}$$

see [10] for details. It is the H^{-1} gradient flow of the FCH energy. It is expected that stable patterns are solutions of

$$(\varepsilon^2 \Delta - W''(u) + \varepsilon\eta_1)(\varepsilon^2 \Delta u - W'(u)) + \varepsilon\eta_d W'(u) = \varepsilon\gamma. \tag{47.0.2}$$

The constant γ can be thought of as a Lagrange multiplier arising from the mass conservation. The fundamental differences to the classical Cahn-Hillard equation are that interface area can be promoted (depending on the sign of the η parameters); in addition, the distinguished limit of interest does not have a constant ratio of material in the phases. Rather, there is a dominant bulk phase (solvent in some cases) and a thin interface region of the second phase (single component amphiphilic material). Some results on the scalar system are known [8, 10, 12, 16]. These are of the type where local structures are found considering asymptotic versions of (47.0.2) and found to be stable (or have long time evolution of known character) under perturbation of lower order terms. There is mounting evidence from this work that the FCH model can describe the observed structure, and changes in the behaviour in structure in changing conditions, of these materials.

Numerical computation of (47.0.1) is challenging due to the wide variations in space and time scales that need to be captured. The high PDE order makes the time evolution of spatial discretizations very stiff. It should also be said that the computational problem is not a fixed target: numerical methods must be easily adapted to new terms and new parameter scaling as researchers explore the models for regimes in which mathematically interesting and physically relevant phenomena occur. Some progress has been made on such a computational framework [7] on scalar computational architecture.

A major extension to this model is to a vector version, capable of describing three phase problems. Of early interest are materials with a solvent and two different amphiphilic materials. This type of model could describe lipid bilayers, in which the lipids on either side of the bilayer are different, leading to an intrinsic curvature. Vector models naturally allow a non-conservative perturbation to the energy, and so to the asymptotic interface profile problem, that leads to mathematical complications. The first extension of the numerical framework to the vector FCH (unpublished) is relatively crude. The scalar case was already a difficult computational problem, and the vector case has yet another source of stiffness in sharp transitions in the vector potentials of interest. Our purpose at BIRS was the preliminary investigation of vector FCH models and their relevance to materials science applications.

Scientific Progress Made

It is clear in the scalar case that the shape of the potential has an impact on the type of structures that emerge. In the scalar case, this impact can be described by only a few parameters. In the vector case, the shape of the potential (a function of two variables) has a much richer variety. We identified two cases that we believed were amenable to analysis and would lead to interesting structures: "billiard wells" that would allow bilayer structures, and "radial wells" in which at highest order the two amphiphilic materials could co-exist without energy penalty. Having these two particular cases in mind focussed the discussion of the analysis and gave concrete goals for the computations.

Computations on the vector problem done at the meeting revealed that there is a much more complicated interaction of interfacial structures with the background state than in the scalar case. This was then understood analytically.

At a high level, a general framework for the analysis of the interfacial structure (a high dimensional ODE system) was realized. We understood how instabilities of three types (meander, phase separation and pearling) would enter. While a great deal of technical analysis remains to put these ideas on rigorous footing, there is now a clear path forward into this complicated system.

Outcome of the Meeting

We have planned a sequence of papers coming from the discussion at BIRS:

- Analysis paper on the vector FCH equation with mollified billiard potentials, showing the persistence of interface profiles with mollification and other perturbations, and the existence of a sharp interface limit.
- A more general paper looking at the non-symmetric local vector interface profile problem and the three types of instability we identified. Numerical examples using the billiard and radial potentials will be used to show the structure. This paper is intended to be an invitation to analysts, highlighting the phenomena that could be amenable to rigorous study.
- More rigorous analysis of some of the cases from the previous general paper. We believe the phase separation due to morphology in the radial potential is amenable.
- An application paper, targeted to the experimental lipid researchers, showing the correspondence of phenomena they observe to parameter values in the model. Optimistically, this paper would include a direct correspondence of material properties they can measure to FCH model parameters.
- A technical paper on the analysis of a particularly interesting edge bifurcation that occurs in the study of the vector interface problem.
- Computational study validating the convergence of the FCH with the billiard potential to the sharp interface limit predicted in paper #1.

Thanks

Three of us have administrative roles at our home institutions. We are especially thankful to BIRS for giving us this chance to get together and work on these problems uninterrupted by the daily routine.

Participants

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Chapter 48

Low-lying zeros of quadratic Dirichlet L-functions (15rit197)

August 9 - 16, 2015

Organizer(s): Daniel Fiorilli (University of Ottawa), James Parks (University of Lethbridge), Anders Södergren (University of Copenhagen)

Background

Prime numbers form the building blocks of the natural numbers. As such, in number theory our goal is to improve our understanding of their behavior. One of the most important unresolved questions in number theory is the Riemann Hypothesis, first conjectured by Riemann in 1859. In simple terms the Riemann Hypothesis tells us that the distribution of prime numbers amongst the natural numbers is as nice as possible. The original statement was given in terms of the zeros of a function called the Riemann zeta function, but this conjecture has since been generalized to similar statements for many other so-called L-functions.

In 1973, Montgomery [9] noticed that certain statistics of the zeros of the Riemann zeta function bear a striking similarity to statistics coming from random unitary matrices in $U(N)$ in the large N limit. In recent years, such similarities were also seen to be present for certain families of L-functions. Depending on the family, one might have to replace $U(N)$ with one of the groups $O(N), SO(2N+1), SO(2N), Sp(N)$; this is what Katz and Sarnak coined the symmetry type of the family.

The 1-level density is a central statistic, which analyzes the low-lying zeros of members of a family. It has the advantage of allowing one to isolate the symmetry type, while being quite versatile and tractable under certain restrictions on the involved test function. It should be noted that low-lying zeros of L-functions are of central importance in many number-theoretical problems, and their thorough understanding could lead to the solution of several longstanding problems.

The family that we will discuss in this report is that of Dirichlet L-functions of real primitive Dirichlet characters of modulus $8d$, i.e. the family of L-functions attached to the quadratic characters

$$\chi_{8d}(\cdot) := \left(\frac{8d}{\cdot}\right).$$

More precisely, we consider

$$\mathcal{F}(X) := \left\{L(s, \chi_{8d}) : 1 \leq |d| \leq X, d \text{ is odd and square-free}\right\}.$$

This family is known to have significant advantages over that of all real characters [12, 5], and will allow us to obtain precise results. We introduce an even Schwartz test function ϕ, which is assumed to be real-valued. Given a positive number X, the 1-level density for the single L-function $L(s,\chi_{8d})$ is defined as the sum

$$D_X(\chi_{8d};\phi) := \sum_{\gamma_{8d}} \phi\left(\gamma_{8d}\frac{L}{2\pi}\right),$$

with $\gamma_{8d} := -i(\rho_{8d}-\frac{1}{2})$, where ρ_{8d} runs over the nontrivial zeros of $L(s,\chi_{8d})$. Here,

$$L := \log\left(\frac{X}{2\pi e}\right). \tag{48.0.1}$$

We introduce the following weighted 1-level density

$$\mathcal{D}^*(\phi;X) := \frac{1}{W(X)}\sideset{}{^*}\sum_{d \text{ odd}} w\left(\frac{d}{X}\right) D_X(\chi_{8d};\phi),$$

where $\sum^*$ indicates that the summation is restricted to square-free numbers, $w(t)$ is an even nonnegative Schwartz test function which is not identically zero (which essentially restricts the sum to $d \ll X$) and

$$W(X) := \sideset{}{^*}\sum_{d \text{ odd}} w\left(\frac{d}{X}\right).$$

Note that in the literature the weight w is often taken to be a sharp cutoff. However, it is more convenient to use a smooth function, especially when computing lower order terms.

After a detailed analysis of a corresponding function field family, Katz and Sarnak made the following prediction:

$$\lim_{X\to\infty} \mathcal{D}^*(\phi;X) = \widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\widehat{\phi}(x)dx. \tag{48.0.2}$$

In fact they predict that this family is symplectic. Note that there is a transition when the supremum of the support of $\widehat{\phi}$ reaches 1. It is therefore particularly interesting to obtain results for test functions having larger support and to describe this transition also with respect to lower order terms.

Katz and Sarnak have shown [6] that their prediction holds under the restriction $\sup(\text{supp}(\widehat{\phi})) < 1$, and under the additional assumption of GRH (the Generalized Riemann Hypothesis) they were able to relax this condition to $\sup(\text{supp}(\widehat{\phi})) < 2$. Rubinstein [10] has extended their unconditional result to the n-level density ($n \geq 1$) for test functions with suitably restricted support. As for the GRH result, it was extended by Gao [5] to the n-level density; again with the admissible support doubled. Note that for several years it was not known how to match Gao's asymptotic with the random matrix theory predictions. However, this was recently established for $n \leq 7$ by Levinson and Miller [7], and for all remaining n by Entin, Roditty-Gershon and Rudnick [2].

Recent progress and results

While the symmetry type of a family is expected to predict an asymptotic for the 1-level density of any reasonable family, it can not predict lower order terms. Such terms of order $1/\log X$ have been found in many families so far (see, e.g., [15, 4] in the case of families of elliptic curves), and depend on the arithmetic properties of the family under consideration. In the family we are considering, such terms have been isolated by Miller [8] under the restriction $\sup(\text{supp}(\widehat{\phi})) < 1$. As noted above, this restriction hides the transition at 1, and therefore one might believe that the global picture is quite different.

An analogous question over function fields was studied by Rudnick [11]. He studied the symplectic family of zeta functions of hyperelliptic curves $y^2 = Q(X)$ defined over $\mathbb{F}_q[x]$, where $Q(x)$ is a monic square-free polynomial of degree $2g+1$. The hyperelliptic curve has genus g, and Rudnick considered the associated 1-level density when q is fixed and $g\to\infty$ (this is a more direct analogue to number fields than the $q\to\infty$ limit). His result is the following estimate for this quantity, under the restriction $\sup(\text{supp}(\widehat{\phi})) < 2$:

$$\int_{\text{USp}(2g)} Z_\phi(U)\,dU + \frac{1}{g}\left[\widehat{\phi}(0)\sum_{P \text{ monic irr.}}\frac{\deg P}{q^{2\deg P}-1} - \frac{\widehat{\phi}(1)}{q-1}\right] + o\left(\frac{1}{g}\right). \tag{48.0.3}$$

Here, $Z_\phi(U)$ is the 1-level density of small eigenvalues of the matrix U (and hence it is a purely random matrix theoretical object). Note that the term involving $\widehat{\phi}(1)$ is only present when the supremum of the support of $\widehat{\phi}$ reaches one, and hence this confirms the existence of a transition at this point.

The result we obtained during our stay in Banff is an asymptotic expression for $\mathcal{D}^*(\phi; X)$ in descending powers of $\log X$, under the assumption of GRH and the restriction $\sigma := \sup(\text{supp}(\widehat{\phi})) < 2$. As with Rudnick's result (48.0.3), our result uncovers the transition at 1, and therefore is expected to hold regardless of the support of $\widehat{\phi}$. In fact, the first lower order term we obtain has a striking similarity to that obtained by Rudnick, since it involves both $\widehat{\phi}(0)$ and $\widehat{\phi}(1)$. We now give a more precise statement of our result.

Theorem 48.0.1 *Fix $\varepsilon > 0$. Let w be a nonnegative Schwartz function on $\mathbb{R}$ which is not identically zero and let ϕ be an even Schwartz function on $\mathbb{R}$ whose Fourier transform satisfies σ =sup(supp($\widehat{\phi}$)) < 2. Then, assuming GRH, the 1-level density for the zeros of the family of L-functions attached to real primitive Dirichlet characters of modulus $8d$, where d is odd and square-free, is given by*

$$\mathcal{D}^*(\phi; X) = \widehat{\phi}(0) - \frac{1}{2}\int_{-1}^{1} \widehat{\phi}(u)\, du + \frac{1}{\log X}\left[C_{w,1}\widehat{\phi}(0) + C_{w,2}\widehat{\phi}(1)\right] + O\left(\frac{1}{(\log X)^2}\right),$$

where $C_{w,1}$ and $C_{w,2}$ can be given explicitly in terms of integrals involving the weight function w.

Using similar techniques, we can also study the low-lying zeros in the related family

$$\mathcal{F}_1(X) := \left\{L(s, \chi_d) : 1 \le |d| \le X\right\}.$$

In this case we observe that the family contains an abundance of repetitions, that is, the L-functions in $\mathcal{F}_1(X)$ appear with certain multiplicities. There are several known examples in the literature where allowing repetitions in a family lead to a more manageable analysis and also to more precise results (cf. [16, 3, 4]). This turns out to be true also in the present case, where we can prove a result of the same flavor as Theorem 48.0.1, but with better control of the involved error terms. The improved quality of the error terms is particularly interesting in comparison with the predictions made for the 1-level density by the powerful L-functions Ratios Conjecture of Conrey, Farmer and Zirnbauer [1].

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Chapter 49

Uniqueness results in geometric tomography (15rit189)

August 16 - 23, 2015

Organizer(s): Alexander Koldobsky (University of Missouri), Dmitry Ryabogin (Kent State University), Vladyslav Yaskin (University of Alberta), Artem Zvavitch (Kent State University)

Overview of the Field

The research of the group was focused on problems in geometric tomography. The latter area is concerned with the study of geometric properties of convex bodies based on information about sections or projections of these bodies. Geometric tomography has important applications to many areas of mathematics and science, in general. The book "Geometric Tomography" [4] by Gardner gives an excellent account of various problems and techniques that arise in geometric tomography. Of paramount importance are questions about unique determination of convex bodies from the size of their sections or projections. For many years the dominating tools for proving uniqueness were those involving spherical harmonics and direct geometric methods. In recent years, we have seen a rapid development of new methods, based on Fourier analysis, which allowed to solve many open problems in convex geometry. The general idea is to express geometric characteristics of a body in terms of the Fourier transform and then use methods of harmonic analysis to solve geometric problems. This approach has led to several results including Fourier analytic solutions of the Busemann-Petty and Shephard problems, characterizations of intersection and projection bodies, extremal sections and projections of certain classes of bodies. These developments are described in the books "Fourier Analysis in Convex Geometry" [8] by Koldobsky and "The Interface between Convex Geometry and Harmonic Analysis" [9] by Koldobsky and Yaskin. The most recent results include solutions of several longstanding uniqueness problems, and the discovery of stability in volume comparison problems and its connection to hyperplane inequalities.

Recent Developments and Open Problems

One of the classical results in the area (attributed to Funk and Minkowski; see [4]) is that an origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by $(n-1)$-dimensional volumes of its central hyperplane sections. More precisely, if K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$|K \cap \xi^\perp| = |L \cap \xi^\perp|$$

for every $\xi \in S^{n-1}$, then $K = L$. Here and below, $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$.

Note that this result is false without the symmetry assumption. It is natural to ask what information is needed to uniquely determine convex bodies that are not necessarily origin symmetric. One of the results in this direction was obtained by Falconer [2] and Gardner [3], who proved that in order to determine a non-symmetric convex body one needs to know volumes of hyperplane sections through two interior points. However, the following problem is still open, even in $\mathbb{R}^2$.

Problem 1. *Let K and L be convex bodies in $\mathbb{R}^n$ containing a sphere of radius t in their interiors. Suppose that for every hyperplane H tangent to the sphere we have $|K \cap H| = |L \cap H|$. Does this mean that $K = L$?*

The problem in this form was stated by Barker and Larman [1], though a similar question on the sphere was considered earlier by Santaló [14]. Let us give an overview of partial results pertaining to this problem. In their paper Barker and Larman have shown that if a planar body has chords of constant length at distance t from the origin, then the body is a Euclidean disk. It is shown by Yaskin [16] that the answer to the problem is affirmative in the class of convex polytopes in $\mathbb{R}^n$. Recently, the following modification of Problem 1 was considered by Yaskin and N. Zhang; see [17].

Let K and L be convex bodies in $\mathbb{R}^2$ and let D_1 and D_2 be two disks in the interior of $K \cap L$. Assume that neither of D_1 or D_2 is contained in the other. If the chords $K \cap H$ and $L \cap H$ have equal length for all H supporting either D_1 or D_2, then $K = L$.

In this context let us also recall several results related to maximal sections. Let K be a convex body in $\mathbb{R}^n$. The inner section function m_K is defined by

$$m_K(\xi) = \max_{t \in \mathbb{R}} |K \cap (\xi^\perp + t\xi)|,$$

for $\xi \in S^{n-1}$.

An old question of Klee dating back to 1969 asks whether a convex body is uniquely determined (up to translation and reflection in the origin) by its inner section function. Recently, this question was answered in the negative by Gardner, Ryabogin, Yaskin, and Zvavitch [6]. Klee also asked whether a convex body in $\mathbb{R}^n$, $n \geq 3$, whose inner section function is constant, must be a ball. Nazarov, Ryabogin, and Zvavitch disproved this conjecture; see [10] and [11]. They also constructed a counterexample (in even dimensions) to a much older question of Bonnesen asking whether a convex body is determined by its inner section function and its brightness function. However, the odd-dimensional case is still unresolved.

Problem 2. *Let K and L be convex bodies in $\mathbb{R}^n$, n is odd, such that $m_K(\xi) = m_L(\xi)$ and the projections $K|\xi^\perp$ and $L|\xi^\perp$ have equal $(n-1)$-dimensional volumes. Does this information guarantee that $K = L$ (up to translation and reflection)?*

Another version of this question is open in all dimensions.

Problem 3. *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, such that $m_K(\xi)$ =const and all projections $K|\xi^\perp$ have the same $(n-1)$-dimensional volume. Is K necessarily a Euclidean ball?*

Instead of areas of sections and projections, one can also consider other intrinsic volumes. For example, the following problem from [4] has attracted a lot of interest recently.

Problem 4. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^3$ such that the sections $K \cap \xi^\perp$ and $L \cap \xi^\perp$ have equal perimeters for every $\xi \in S^2$. Does it follow that $K = L$?*

Of course, one can also consider higher dimensional versions of this problem (instead of perimeters it is natural to take the surface area of sections). There is still very little progress done toward the solution of this problem. In particular, it is not known whether the uniqueness holds if one of the bodies is the Euclidean ball. That is, if K is an origin-symmetric convex body in $\mathbb{R}^n$ such that the surface area of $K \cap \xi^\perp$ is independent of ξ, does this mean that K is a ball? It was shown by Howard, Nazarov, Ryabogin and Zvavitch [7] the latter problem has an affirmative answer in the class of C^1 star bodies of revolution. Rusu [13] settled an infinitesimal version of this problem for one-parameter analytic deformations of the ball. Yaskin [15] solved Problem 1 in the class of origin-symmetric convex polytopes.

Discrete tomography is a related area, where instead of convex bodies one deals with lattice sets. A finite subset A of $\mathbb{Z}^n$ is called a *convex lattice set* if $A = (\text{conv}A) \cap \mathbb{Z}^n$. For such sets one can ask questions similar to those in geometric tomography. For example, Gardner, Gronchi and Zong [5] studied an analogue of Alexandrov's projection theorem in discrete settings.

Problem 5. *Let A and B be origin-symmetric convex lattice sets in $\mathbb{Z}^n$ such that for each $u \in \mathbb{Z}^n$ the projections $A|u^\perp$ and $B|u^\perp$ have the same number of points. Is then A necessarily the same as B?*

Gardner, Gronchi and Zong have shown that the answer is negative if $n = 2$. However, the counterexample constructed does not provide a complete understanding of the 2-dimensional case. Are there any other counterexamples? Is it possible to impose a mild condition that would make the answer positive for $n = 2$? In the 3-dimensional case the problem is completely open.

Some progress in this direction has been recently obtained by Ryabogin, Yaskin and N. Zhang [12].

Outcome of the Meeting

During the meeting we discussed the problems described in the previous section, as well as other related questions. In particular, we tried to attack Problem 4 using recent advances in Fourier analytic methods. A. Koldobsky gave an overview of his results concerning stability and separation in problems on sections and projections. We identified possible new problems and directions. There are many questions similar in spirit to Problem 5 and it is worth looking at possible applications of Fourier methods to such discrete problems. In fact, it looks like a good idea to try to organize a BIRS workshop with the goal of bringing together researchers with analytic background and discrete background. This would help to facilitate collaboration between these two communities of convex geometers, and hopefully to lead to a better understanding of such problems.

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