23w5125: Extremal Graphs Arising from Designs and Configurations

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1 Objectives of the Workshop

The purpose of the workshop was to provide a forum for interaction among leading mathematicians in the areas of designs/configurations and extremal graphs. Recent work by some of the organizers has shown a significant potential of designs and configurations for providing solutions to Extremal Graph Theory problems. This served as the main motivation for proposing the gathering of experts from the two areas. Around the world, there are several groups in Combinatorics and Graph Theory working on these topics and the majority of these researchers have been meeting in the past at the IWONT gatherings: International Workshop on Optimal Networks Topologies, first held in Australia in 2005 (subsequent editions taking place in 2007, 2010, 2011, 2012 and 2016). Despite its name, IWONT has been more of a traditional conference than a workshop of the type usually supported by BIRS. The objective of the 23w5125 workshop was to gather active and new experts in these areas, along with graduate students and junior faculty, providing them with an appropriate setting for joint work and for successful production of relevant results. It was also the organizers' intention to give graduate students the opportunity to learn from the expert participants, to use the language, results and tools of each area and to interact with each other. To this purpose new working groups were established during the event, where the participation of students, tenure track researchers, tenured faculty researchers and emeritus professors was purposefully balanced.

Using designs and configurations, the main intention was to approach the following problems:

- Existence and construction of optimal bipartite biregular graphs with fixed girth or diameter.
- Optimization and construction of vertex transitive, arc transitive or edge-regular graphs.
- The study of the interaction between several constructions and operations applied to optimal graphs, and the study of the properties and symmetries of such graphs.
- Colorability problems in highly symmetric graphs.

Another objective was to maintain and improve the participation of female researchers in the proposed areas. This is why the organizers found it important to highlight the influence of female mathematicians in the involved research groups as well as those previously invited to present at the IWONT gatherings. One of the founders of the IWONT series, the late Mirka Miller, as well as Camino Balbuena, one of her most frequent co-authors, actively promoted the participation of young female researchers and also encouraged the growth in the number of female students. The list of participants invited to this meeting speaks for itself in this aspect.

2 Scientific Background and Motivation

In this workshop we propose to study the interaction of the combinatorial areas of Graph Theory and the Theory of Designs and Configurations. More specifically, we intend to use designs and configurations for solving classical problems in Extremal Graph Theory and problems related to graph colorings. These kinds of problems and approaches have been previously considered by different research groups throughout the world, however, we can venture to say that this is the first time anyone intends to bring together specialists from these areas to jointly work in combining the potentials of both subjects to obtain new results.

The academic relevance of our workshop can be highlighted by recalling that it includes an extensive list of topics related to Extremal Graph Theory, Finite and Discrete Geometry and optimal networks, but the main two topics are the Cage Problem and the Degree-Diameter Problem.

Both problems can be viewed as two-parameter optimization problems. For the Degree-Diameter Problem one seeks the largest possible graph with a given degree and diameter. For the Cage Problem, the smallest graph with a given degree and girth is sought. Both problems are central to Extremal Graph Theory and generalize the Moore graph problem in opposite directions. Moreover, work on these central queries has led to a number of interesting theoretical problems. There is an extensive bibliography list regarding these two topics published by the participants of this workshop but, for simplicity, we only include two dynamic surveys that show the relevance of these two topics [20, 27].

In particular, the Dynamic Cage Survey is regularly updated and includes most of the research work (even the most recent results) that has been done on this topic. To name more specifically the topics of the workshop besides the main ones just mentioned, they are: connectivity and reliability of networks, colourings, cycles and factors in graphs, construction techniques for large graphs and digraphs with prescribed degree and diameter or girth, structural properties of large graphs and digraphs of this type, spectral techniques in graph theory, network design problems related to communication. In addition, we intend to consider generalizations of the original problems such as extremal graphs of given degree and prescribed smallest even and odd cycle lengths, biregular graphs and bipartite biregular graphs, and edge-girth-regular graphs, which are graphs in which each edge is contained in the same number of girth cycles (and which are generalizations of edge-regular graphs).

It is also important to note that applications of the results in this field are extensive and relevant for technological development. For example, small graphs of high girth can be used to construct the LDPC codes (Low Density Parity Check Codes). LDPC codes are linear error-correcting codes with sparse parity-check matrices. Long codes of this type can be constructed from shorter codes and sparse bipartite graphs of large girth (the so-called Tanner graphs). The efficiency of the decoding is directly related to the girth of the bipartite graph. Another example is that the Degree/Diameter Problem is related to the design of communication networks, recently used for constructing the so-called ”Bluetooth scatternets”. A Bluetooth scatternet is an ad-hoc computer network built from a number of small subnetworks, where the underlying network has degree at most 8.

3 Structure of the Workshop

The main focus of our workshop was on fostering collaboration between various groups of researchers. Thus, it has been decided that we would limit the number of presentations to a small number designed to outline the areas of possible research as envisioned by the organizers, and devote a significant part of the workshop to work in groups around the outlined research areas. The groups were not determined beforehand and were created based on participants' interest in a specific presentation. The presenters were chosen to cover the various possible work groups with each talk outlining possible direction of research in one of the groups.

The list of suggested groups was sent to the participants prior to the workshop together with lists of suggested article to be read before arriving at the workshop. The suggested groups together with the suggested reading lists were:

- Algebraic techniques in extremal graph theory [20, 15, 26, 21]
- Connections between extremal graph theory, designs and geometry [1, 2, 5, 12]
- Colorings in extremal graph theory [13, 25, 9, 8, 10]
- Extremal mixed graphs [29, 30, 18, 3, 4]
- Computational extremal graph theory [14, 16, 17, 19, 27, 28]
- General ideas on extremal graph theory and connections to graph theory and other areas of mathematics [6]

4 Recent Developments and Open Problems as Outlined in Presentations

Our workshop contained a small number of presentations designed to describe the current status of knowledge in the areas to be considered in the workshop, and to introduce specific topics for work in groups. Here, we include the list of titles for these talks together with the name of the presenter and their abstract.

4.1 Robert Jajcay: Advancing our understanding of cages through use of algebraic methods

A k-regular graph of girth g is is said to be a (k, g)-*cage* if its order $n(k, g)$ is the smallest among the orders of all k-regular graphs of girth g. The *Cage Problem* is the problem of finding cages and the corresponding values $n(k, g)$, for all parameter pairs k, g. Cages are only known for very limited sets of parameter pairs, and progress in Cage Problem is unfortunately quite slow. The aim of the presentation is to outline specific problems and questions concerning cages that appear to be particularly suitable for the use of algebraic methods. Notably, a significant proportion of the known cages exhibit a high level of symmetry; with many being vertex-transitive or even Cayley. This observation is the motivation behind restricting the general Cage Problem to the problem of finding smallest vertex-transitive graphs of given degree and girth. While focusing on vertex-transitive graphs may not necessarily directly lead to finding new cages, understanding the structure of small vertex-transitive graphs of given degree and girth adds to our understanding of the general Cage Problem and may even produce graphs that could possibly be altered to become cages (by giving up some of their symmetry properties). In our talk, we will discuss connections between general and vertex-transitive graphs of given degree and girth and the way progress in understanding either of these two classes may lead to progress in the other.

4.2 Gabriela Araujo: Bipartite biregular cages: block designs and generalized polygons

A *bipartite biregular* $(m, n; g)$-graph Γ is a bipartite graph of even girth g having the degree set $\{m, n\}$ and satisfying the additional property that the vertices in the same partite set have the same degree. The $(m, n; g)$-*bipartite biregular cages* were introduced in 2019 by Filipovski, Ramos-Rivera, and Jajcay. They are bipartite biregular $(m, n; g)$-graphs of minimum order. The authors calculate lower bounds on the orders of bipartite biregular $(m, n; g)$-graphs, and call the graphs that attain these bounds *bipartite biregular Moore cages*. We improve the lower bounds obtained in the above paper. Furthermore, in parallel with the well-known classical results relating the existence of k-regular Moore graphs of even girths $g = 6, 8$ and 12, to the existence of projective planes, generalized quadrangles, and generalized hexagons, we prove that the existence of an $S(2, k, v)$-Steiner system yields the existence of a bipartite biregular $(k, v - 1k - 1; 6)$-cage, and, vice versa, the existence of a bipartite biregular $(k, n; 6)$-cage whose order is equal to one of our lower bounds yields the existence of an $S(2, k, 1 + n(k - 1))$-Steiner system. Moreover, in the special case of Steiner triple systems, we completely solve the problem of determining the orders of $(3, n; 6)$-bipartite biregular cages for all integers $n \geq 4$. Considering girths higher than 6, we relate the existence of generalized polygons (quadrangles, hexagons and octagons) to the existence of $(n+1, n^2+1; 8)$-, $(n^2+1, n^3+1; 8)$-, $(n, n+2; 8)$-, $(n+1, n^3+1; 12)$- and $(n+1, n^2+1; 16)$-bipartite biregular cages, respectively. Using this connection, we also derive improved upper bounds for the orders of other classes of bipartite biregular cages of girths 8, 12, and 14.

4.3 Gyorgy Kiss: Girth-(bi)regular graphs and finite geometries

Two new classes of graph regularities have been introduced recently. Let Γ denote a simple, connected, finite graph. For an edge e of Γ let $n(e)$denote the number of girth cycles containing e. For a vertex v of Γ let $\{e_1, e_2, \dots, e_k\}$ be the set of edges incident to v ordered such that $n(e_1) \leq n(e_2) \leq \dots \leq n(e_k)$. Then $(n(e_1), n(e_2), \dots, n(e_k))$ is called the *signature* of v. The graph Γ is said to be *girth-(bi)regular* if (it is bipartite, and) all of its vertices (belonging to the same bipartition) have the same signature. We show that girth-(bi)regular graphs are related to (biregular) cages, finite projective and affine spaces and generalized polygons. We also provide a short, concise introduction to these geometric objects.

4.4 Christian Rubio-Montiel: On the harmonious chromatic number of Levi graphs

The *harmonious chromatic number* of a graph G is the minimum number of colors that can be assigned to the vertices of G in a proper way such that any two distinct edges have different color pairs. Therefore, if G has order v and diameter at most two, then $h(G) = v$. Therefore, if G is the Levi graph of a linear space $\mathcal{S}$ of n points, then $h(G) \geq n$, since every two points in $\mathcal{S}$ determine a line. In [11] is proved that:

1. If G is the Levi graph of the finite projective plane of odd order q then
$$q^2 + q + 1 \leq h(G) \leq q^2 + q + 2.$$
2. If G is the Levi graph of the complete graph K_n, then
$$\frac{3}{2}n \leq h(G) \leq \frac{5}{3}n + \theta(1).$$

A possible problem is to estimate the harmonious chromatic number of the $(q+1, 8)$-cages (the Levi graphs of generalized quadrangles).

4.5 Mika Olsen: Using geometric properties to color Levi graphs of generalized quadrangles $Q(4, q)$

Geometric properties can be used to determine chromatic classes of coloring and chromatic classes can determine geometric properties. Packing colorings and $L(h, k)$ colorings are somehow related. The packing chromatic number, $\chi_\rho(G)$ of a graph G is the minimum number of colors (positive integer) of a vertex coloring of G such that vertices of color i are at distance at least $i + 1$, for all i. The $L(h, k)$ chromatic number, $\lambda_{h,k}(G)$, of a graph G is the minimum number of colors (non-negative integers) of a vertex coloring of G such that the colors of adjacent vertices must differ in at least h and the colors of vertices of distance 2 must differ at least k. In [23] it was proved that:

1. Let q be an odd prime power. If G is the Levi graph of a $Q(4, q)$, then
$$(q^2 + 1)(q - 1) + 3 \leq \chi_\rho(G) \leq (q^2 + 1)(q - 1) + 4.$$
2. A $Q(4, q)$ contains two disjoint ovoids or two disjoint spreads if and only if
$$\chi_\rho(G) = (q^2 + 1)(q - 1) + 3,$$

where G is its Levi graph.

In [24] it was proved that:

1. The points of a $Q(4, q)$ have a partition into a daisy structure and the lines of $Q(4, q)$ have a partition into a sunflower structure.

2. If G is the Levi graph of a $Q(4, q)$ and $q \geq 5$ is a prime number then,

$$(2q + 2)k \leq \lambda_{h,k}(G) \leq (q + 1)2k/2 + h.$$

A possible problem is to extend the results for Levy graphs of $Q(4, q)$ to Levi graphs of generalized quadrangles which are not $Q(4, q)$, another possible problem es improve the bounds for the packing chromatic number of $(q + 1, 12)$-cages or improve the bounds of $L(h, k)$ chromatic number of Levi graphs of $Q(4, q)$.

4.6 Grahame Erskine: Totally regular mixed graphs in the diameter, girth and geodecity problems

Mixed graphs can be viewed as generalisations of both undirected graphs and digraphs. The problems of the largest (totally regular) mixed graph of given degree and diameter and the smallest such graph of given degree and girth (or geodecity) share some features in common with those for undirected and directed graphs. However, the mixed graph problems have certain unique aspects. We will outline some results from the last few years, and note some open problems and possible lines of attack.

4.7 Geoffrey Exoo: Negative Results on Cages and Degree Diameter Graphs

Some problems on cages and degree diameter graphs are surveyed. The focus is on problems where a computational approach to the problem has fallen short, and for which a little non-computer based analysis might be enough to make the computations feasible.

4.8 Nacho López: Measuring the closeness to mixed Moore graphs

Moore graphs are extremal graphs that appear as solutions of a combinatorial problem known as the degree/diameter problem. This problem is key to the design of topologies for interconnection networks and other questions related to data structures and cryptographic protocols. Besides, the relationship between vertices or nodes in communication networks can be undirected or directed depending on whether the communication between nodes is two-way or only one-way. Mixed graphs arise in this case and it is therefore natural to consider network topologies based on mixed graphs, and investigate the corresponding degree/diameter problem and their solutions (mixed Moore graphs). In this talk we present mixed radial Moore graphs as an approximation of mixed Moore graphs and we describe how to measure they closeness to the properties that a mixed Moore graph should have.

Related references can be found in [14, 16, 17, 19, 27, 28].

4.9 Linda Lesniak: On the existence of (r, g, χ)-cages

For integers $r \geq 2$, and $g \geq 3$, an (r, g)-*graph* is an r-regular graph with girth g, and an (r, g)-*cage* is an (r, g)-graph of minimum order. It is conjectured that all (r, g)-cages with even g are bipartite, that is, have chromatic number 2. Here we introduce the idea of an (r, g, χ)-*graph*, an r-regular graph with girth g and chromatic number χ. We investigate the existence of such graphs and study in detail the $(r, 3, 3)$-graphs of minimum order. We also consider (r, g, χ)-graphs for which there is a χ-coloring where the color classes differ by at most 1.

4.10 Marco Buratti: An overview of my dearest differences

I will try to give an overview of the main constructions for classic designs and graph decompositions that my collaborators and I obtained over the years with the so-called *method of differences* and its many variants.

4.11 Cristina Dalfó, Miguel Àngel Fiol, and Mónica Reyes: On the spectra of token graphs of a cycle

The k-token graph $F_k(G)$ of a graph G is the graph whose vertices are the k-subsets of vertices from G, two of which being adjacent whenever their symmetric difference is a pair of adjacent vertices in G. In this talk, through lift graphs and voltage assignments, we first show the spectrum of a k-token graph of a cycle C_n for some n. Since this technique does not give the results for all possible n, we introduce a new method that we called over-lifts. This method allows us to present the spectra of any k-token graph of any cycle C_n.

This talk, presented as series of three consecutive talks by the above presenters, was also broadcast as a part of the *Mirka Miller Webinar* whose organizers were all participants of the workshop.

4.12 Dimitri Leemans: Edge-girth-regular graphs arising from biaffine planes and Suzuki groups

An edge-girth-regular graph $egr(v, k, g, \lambda)$ is a k-regular graph of order v, girth g, and with the property that each of its edges is contained in exactly λ distinct g-cycles. An $egr(v, k, g, \lambda)$ is called extremal for the triple (k, g, λ) if v is the smallest order of any $egr(v, k, g, \lambda)$. In this talk, I will show two families of edge-girth-regular graphs. The first one is a family of extremal $egr(2q^2, q, 6, (q-1)^2(q-2))$ for any prime power $q \geq 3$. The second one is a family of $egr(q(q^2+1), q, 5, \lambda)$ for $\lambda \geq q-1$ and $q \geq 8$ an odd power of 2. In particular, if $q = 8$ we have that $\lambda = q - 1$.

5 Scientific Progress Made

It was a shared impression among all participants that the interactions and collaboration at the workshop were quite successful. The discussions in the groups were lively and productive and the exchange of ideas was useful. Many of the participants praised the informal format and the ample time devoted to active research in groups. By the end of the workshop, the groups prepared and presented summaries of their results, which included among others the following reports.

1. The *mixed bipartite Moore graphs group* reported some of the results known already before the workshop

   - There are no mixed bipartite Moore graphs with diameter $k \geq 4$.
   - The diameter 3 bipartite bound is $M_B(r, z, 3) = 2((r+z)^2 - r + 1)$.
   - There are Moore graphs with $k = 3, r = 1$ and any z.

   and compared these with results obtained during the workshop

   - There are two mixed bipartite Moore graphs with parameters $r = 2, z = 1, k = 3, n = 16$ shown in Figure 1.

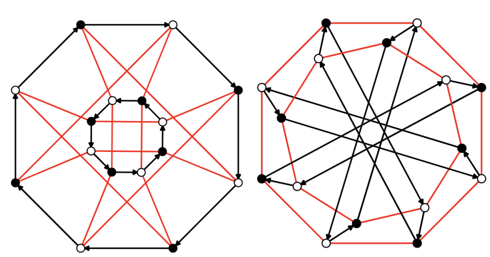

Figure 1: Mixed bipartite Moore graphs with parameters $r = 2, z = 1, k = 3, n = 16$

- With the big remaining question: Are there graphs which attain the bipartite Moore bound for other values of r, z?.

With regard to large bipartite mixed graphs with $r = z = 1, k \geq 4$ they discovered

- A selection of largest graphs for $k = 4, 5, 6$.
- A construction for $(1, 1, k)$ bipartite graphs for even $k \geq 6$ which is asymptotically similar to a recent construction for $r = z = 1$ in the general case as shown in Figure 2.

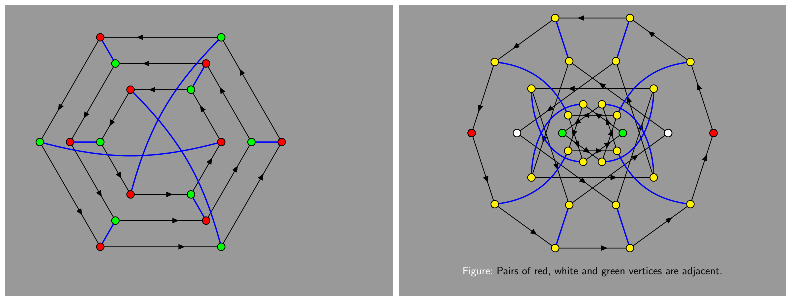

Figure 2: $(1, 1, k)$ mixed graphs

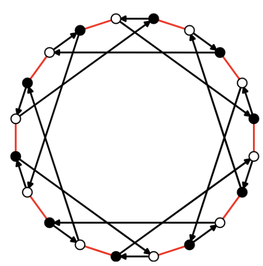

Figure 3: Bipartite mixed graphs with $z = r = 1$ and order $n \sim \sqrt{2}^k$

As for the large general mixed graphs, it has been known

- A construction for general mixed graphs given r, z combining cycle prefix digraphs and pancake graphs.
- Proved diameters for this construction for $r = 1, 2, 3$ and all z.
- Conjectured diameters for all larger r and z.

which was extended during the discussions to

- Proved diameters for this construction for $r = 4, 5, 6, 7$ and all z.
- The conjecture seems to be "nearly correct"...
- There may be a small discrepancy due to the complications of calculating the diameters of pancake graphs (a long-standing open problem).
- In any event, we now have a way to build tables of 'good' mixed graphs for all r up to 7 and many values of z and k.

The group also outlined for ongoing work to

- Check for other mixed bipartite Moore graphs at $k = 3$
- Find best bipartite graphs at $r = z = 1$ and extend to other values

- Build tables of best known general mixed graphs in the diameter problem for small values of r, z, k
- Update CombinatoricsWiki and the degree-diameter survey paper
- Subproblems to consider:
  - A revision to the "nearly correct" conjecture
  - Bounding the minimum defect in $(1, 1, k)$ bipartite graphs as k increases (extending existing results for the general case)
  - Circulant and abelian Cayley mixed graphs in the diameter problem
  - Results for "balanced" mixed graphs where $r = z$
  - Modifications to an existing construction for "sequence mixed graphs" to give regular mixed graphs

2. The summary of the group working on the *colorings in extremal graph theory* consisted of the following observations and results.

   An (r, g, χ)-graph is defined to be an r-regular graph of girth g with chromatic number χ. Such graphs of minimum order are called (r, g, χ)-cages. Our group worked primarily on questions of existence.

   Among other things we were able to show that:

   - The $(r, g, 3)$-cages exist for all $r \geq 2$ and $g \geq 3$. This improves the result that assumes that all (r, g)-cages of even girth are bipartite.
   - Using a construction due to Ch. Rubio Montiel we know that $(r, 3, \chi)$-cages exist for all $r \geq 2$ and $\chi \geq 3$.
   - Given a graph of girth 4, chromatic number χ and maximum degree Δ, we can construct a $(\Delta, 4, \chi)$-graph.
   - Given a graph of girth g, order n and chromatic number χ and an (n, g')-cage with $g' \geq g/2$, we can construct an $(r + 1, g, \chi*)$-graph with $\chi* \geq \chi$. The goal here is to show that $\chi = \chi*$.

3. Miklós Bóna had productive discussions with Marston Conder and Miguel Angel Fiol about log concavity in vertex transitive graphs. These discussions improve the chances to prove log-concavity results *in a uniform way* for the many graphs that appear in the combinatorics of genome rearrangements, something that Bóna has been trying to accomplish for 12 years.

4. The group working on *harmonious colorings for Levi graphs of designs* reported that it is easy to note that the harmonious chromatic number of the Levi graph of a 2-(v, k, λ)-design is bounded below by v, the number of points.

   Most of what the group has done is obtain examples and constructions of designs having harmonious chromatic number of the Levi graph exactly v. We may call a design whose Levi graph realizes the lower bound v a *Banff Design*.

   In [11] it was shown that the harmonious chromatic number of the Levi graph of $PG(2, q)$, q a prime power, is either $v = q^2 + q + 1$, or $v + 1$; using difference methods we showed that it must be v, and thus that $PG(2, q)$ is a Banff design.

   We also proved, using also results on cyclotomic numbers, an asymptotic existence results for Steiner 2-designs having a prime power number of points: when $q \equiv 1 \pmod{k^2 - k}$ is a big enough prime power, there exists a $2 - (q, k, 1)$ Banff design.

   We then considered existence results for Banff-Steiner triple systems, i.e. 2-$(v, 3, 1)$-designs: this case points to an interesting relationship between harmonious coloring problems and some classical problems regarding nestings of triple systems, which made us realize that some constructions of Banff-STSs can be already found in the existing literature. We are still investigating this connection, which seems to give rise to some new nesting notions and problems.

5. The colleagues working on *The harmonious chromatic number of the Levi graph of the complete graph* reported the following.

   A harmonious colouring of a graph G is a proper vertex colouring such that every pair of colours appears on the end vertices of at most one edge of G. The harmonious chromatic number, denoted by $h(G)$, is the smallest number of colours required for a harmonious colouring of G. The Levi graph of a graph G is the incidence graph of G. Thus, a harmonious colouring of the Levi graph of G is a proper total colouring of G with the additional property that, if we write the colours occurring on a vertex and each edge incident to it as unordered pairs of colours, then each pair would occur at most once. Hence, we can refer to the harmonious chromatic number of the Levi graph of a graph G as the harmonious total chromatic number of G, denoted by $h_t(G)$.

   In [11], the authors give a lower bound and an upper bound for the harmonious chromatic number of the Levi graph of the complete graph K_n. In our notation, they prove that $\frac{3}{2}n \leq h_t(K_n) \leq \frac{5}{3}n + \theta(1)$. During the workshop, we considered this problem with the aim of closing the gap between the two bounds.

   We divided the cases according to $n \bmod 4$ and, using an appropriate construction, we obtained the following results:

   (a) if $n \equiv 0 \bmod 4$, then $6k \leq h_t(K_{4k}) \leq 6k + 1$;
   (b) if $n \equiv 1 \bmod 4$, then $h_t(K_{4k+1}) = 6k + 2$;
   (c) if $n \equiv 2 \bmod 4$, then $h_t(K_{4k+2}) = 6k + 3$;
   (d) if $n \equiv 3 \bmod 4$, then $6k + 5 \leq h_t(K_{4k+3}) \leq 6k + 6$.

   The general idea behind the constructions is to first colour each of the vertices using a different colour and decompose K_n into factors. The edges of one of the factors are intelligently coloured using the colours of the vertices, and then the remaining edges are coloured using the unused colours.

   We note that, in the case $n \in \{4k + 1, 4k + 2\}$, we have equality. In the case when $n = 4k$, we proved that $h_t(K_4) = 7$ and that $h_t(K_8) = 12$. We are thus still pondering whether we need to split the case when $n = 4k$ into two further subcases (depending on the cases $n = 0 \bmod 8$ and $n = 4 \bmod 8$), or whether the case $n = 4$ is too small and $h_t(K_{4k}) = 6k + 1$ for all $k \geq 2$. Further investigation is also required to decide the case K_{4k+3}.

6. The following table shows possible numbers of circuits of length g traversing through a particular edge in a $(3, g)$-cage. The results are computer assisted.

girth	cage #	# of girth circuits for an edge
5	1	4
6	1	8
7	1	6, 8
8	1	16
9	1	6-12
	2	8-12
	3	10, 12
	4	6-11, 13
	5	6-11, 13
	6	6-12, 14
	7	7-13
	8	8-12
	9	8-13
	10	5-11, 13

girth	cage #	# of girth circuits for an edge
9	11	5-12
	12	8-12
	13	7-12
	14	8-12
	15	4, 8-12
	16	5-12
	17	5-10
	18	6, 8-11
10	1	24, 26
	2	24,26
	3	24,27
11	1	14, 16, 20, 32
12	1	64

   A problem to consider in the future:

Does there exist a function ψ linear in g such that each edge of each cubic cage of girth g lies in at least $\psi(g)$ circuits of length g?

7. Progress summary: *Girth-diameter graphs arising from finite geometries and designs.*

   We are interested in the problem of finding the smallest regular graphs with given degree $k \geq 3$, girth 5, and diameter $d \geq 3$.

   During the workshop managed to identify the smallest example for $k = 3$, $g = 5$ and $d = 4$, which exhibits cyclic connectivity, and attains the natural *Moore bound* of $1 + k + k(k-1) + k + 1 = k^2 + k + 2 = 14$ vertices. See Figure 4.

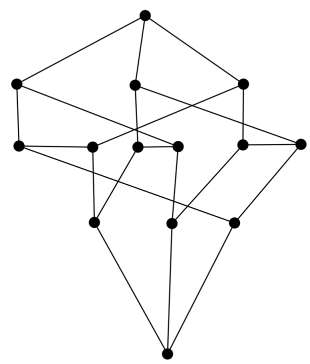

Figure 4: Crossing number graph 3H (aka the Banffrican graph)

   We then focused on finding a minimal graph with $k = 4$, $g = 5$ and $d = 4$. Initially, we discovered a vertex-transitive example with 24 vertices. Subsequently we proceeded to generate though a computer search four distinct examples of order 22 that are pairwise non-isomorphic and minimal, as the Moore bound for this case is precisely 22. These examples are shown in Figure 5, and we have verified that they are the only minimal examples (up to isomorphism).

   In 2022, Araujo and Leemans constructed two graphs with 32 vertices, both with degree 5, girth 5 and diameter 4, thus attaining the Moore bound for these parameters [7]. One of the two graphs is vertex-transitive, while the other is not. Also we found five more (up to isomorphism) using a systematic approach for considering the subgraph induced on vertices at distance 2 from a given vertex, when the Moore bound is attained.

   In work shortly after the workshop we found a second (non-isomorphic) version of the 'Banffrican graph' in Figure 4, and constructed a graph of order 44 with degree $k = 6$, girth 5 and diameter 4, again attaining the Moore bound. Currently, we are working on a generic approach for constructing graphs that attain the Moore bound of $k^2 + k + 2$ vertices for girth 5, diameter 4, and given degree $k > 5$.

8. The *algebraic graph theory* group worked mostly on refining their understanding of the (k, g)-graphs constructed via a series of arithmetic conditions by Lazebnik and Ustimenko. They also adjusted these originally undirected graphs by introducing direction to some of their edges which allowed them to construct mixed graphs which, they feel, might be of interest to their fellow participants working on mixed graphs.

   In addition, they addressed (and partially solved) a problem suggested during the workshop by G. Erskine who asked for constructions of infinite families of k-regular graphs of girth $g = 2r + 1$ and diameter $r + 1$. The proposed construction is based on removing vertices and edges from the incidence graphs of generalized polygons and works only for limited parameter sets. It turned our during the workshop that a similar solution had been found by the 'cages-and-designs' group who were not aware of Grahame's problem. It should be pointed out that the existence of infinite families of girth $g = 2r+1$ and diameter $r + 1$ for other degrees k than those associated with projective planes would most likely lead to record (k, g)-graphs of orders about twice as big as the corresponding Moore bounds.

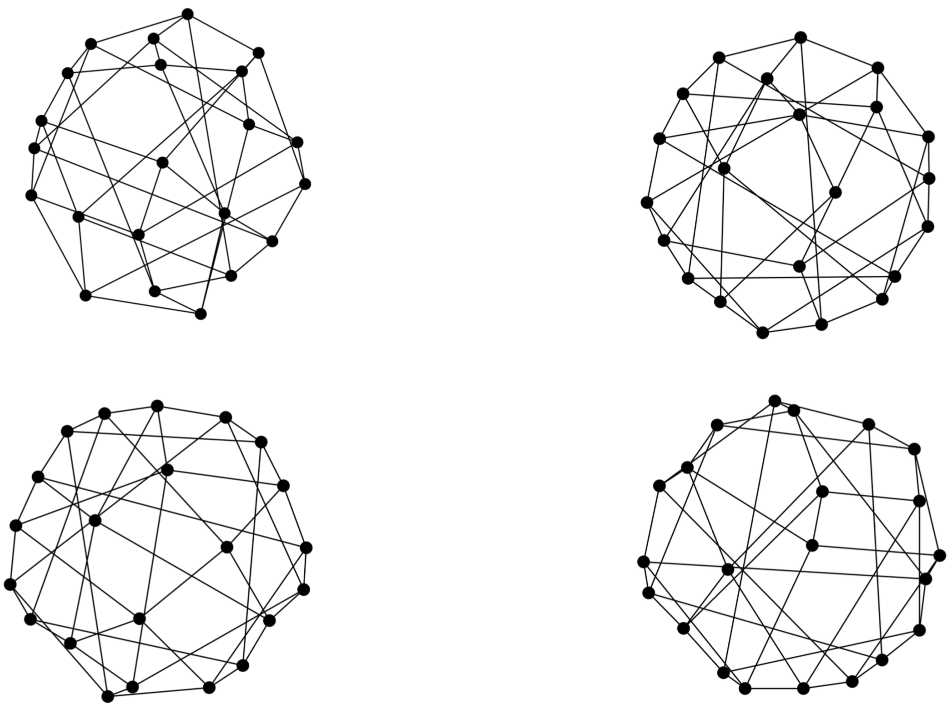

Figure 5: The four non-isomorphic graphs of order 22 having degree 4, diameter 4 and girth 5

6 Outcome of the Meeting

During the workshop, several of the groups produced research papers at various levels of completion. A special issue of *Art of Discrete and Applied Mathematics* `https://adam-journal.eu/index.php/ADAM` has been arranged for to accommodate the publication of articles conceived or started at the workshop in a single place.

The participants expect to further continue developing the ideas considered in Banff. The forthcoming *International Workshop on Optimal Network Topologies* `https://www.icms.org.uk/workshops/2023/international-workshop-optimal-network-topologies` that covers similar concepts and ideas will take place in July 2023 in Edinburgh, UK. A significant number of the participants of *23w5125* will also participate in this workshop which will allow them to pursue some of their ideas within a relatively short period following the end of *23w5125*. The majority of the participants also regularly attend the monthly *Mirka Miller Webinar* `http://combinatoricswiki.org/wiki/Mirka_Miller%27s_Combinatorics_Webinar_Series`.

Both survey papers [20] and [27] are due for an update, and their authors expect the participants of the Banff workshop to contribute to these updates.

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