A unified view of quasi-Einstein manifolds

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23–28 April 2023

1 Overview of the Field

Einstein metrics continue to be an active area of study in mathematics and theoretical physics. Driven by advances in several disparate areas of mathematics, there has been recent interest in generalizations of these metrics which arose independently, especially with regard to their classification. Given the explosive growth of the literature in each of these fields, the time is right to unite experts to share techniques and develop new insights that will enrich our combined understanding.

The Einstein equation relates the Ricci tensor to a multiple of the metric. The equations of interest here, quasi-Einstein equations, relate the Ricci tensor, the Lie derivative of the metric with respect to a vector field, and the tensor product of vector field with itself to a multiple of the metric. In the gradient case, the vector field is replaced by the gradient of a potential function.

Some of the applications of quasi-Einstein equations were outlined in our workshop proposal. We will not repeat them here, except to say that some of the primary areas of investigation for this workshop are

- the investigation of special cases of the quasi-Einstein equation, such as the Ricci soliton equation and the near horizon geometry equation for extreme (i.e., degenerate) black holes,
- the generalization of Ricci curvature bounds to nonsmooth settings such as RCD spaces and Lorentzian (spacetime) analogues, and
- developments in the problem of determining the fundamental gap between the first two eigenvalues of Laplace operators on domains in $\mathbb{R}^n$ and on manifolds.

2 Recent Developments and Open Problems

The classification of near horizon geometries is an important special case of the problem of solving the quasi-Einstein equations

$$(R_X^m)_{ij} := R_{ij} + \frac{1}{2}\left(\nabla_i X_j + \nabla_j X_i\right) - \frac{1}{m} X_i X_j = \lambda g_{ij} \tag{1}$$

on closed manifolds. Here g_{ij} is a Riemannian metric on the closed manifold M, X_i is a 1-form, and λ and m are constants. This classification informs the uniqueness (and non-uniqueness) of extreme black hole horizons. A small but important part of this problem is to find all quasi-Einstein manifolds with $m = 2$ (and more generally for any m) on a 2-sphere. An open problem is to show that the only $m = 2$ examples, other than the round sphere metric with $X \equiv 0$, are the $U(1)$-symmetric Kerr (for $\lambda = 0$), Kerr-de Sitter (for $\lambda > 0$), and Kerr-anti de Sitter (for $\lambda < 0$) near horizon metrics.

There has been very important recent progress in the generalization of Ricci curvature bounds and the Einstein and quasi-Einstein equations to the non-smooth setting. This goes back to work over the past 20 years by Sturn, Lott, Villani, Gigli, McCann, Mondino and others generalizing Ricci curvature bounds for Riemannian metrics. There are formulations of the strong energy condition (or, more correctly, the timelike convergence condition) by McCann and by Mondino and Cavaletti. The latter have also proved a non-smooth version of the Hawking singularity theorem. In the workshop McCann and Ketterer described their formulations of the null energy condition. This is a rapidly developing area.

In the setting of homogeneous manifolds, Christoph Böhm and R Lafuente very recently showed that a homogeneous Einstein manifold with negative scalar curvature is diffeomorphic to Euclidean space, confirming the Alexseevskii conjecture from 1975. This has several important consequences. First combining with their earlier result that homogeneous Einstein metrics on Euclidean spaces are Einstein solvmanifolds and the work of Heber and Lauret, this reduces the classification of non-compact homogeneous Einstein spaces to that of nilsolitons. Second, combinied with the work of Jablonski, this gives that any compact, locally homogeneous Einstein manifold with negative scalar curvature is locally symmetric. Third, any homogeneous expanding Ricci soliton is isometric to a solvsoliton. Important open problems are the classification of nilsolitons, and the classification of compact homogeneous spaces, which remains wide open despite several structure results concerning existence and non-existence. Another question is whether the Einstein condition of homogeneous spaces is a strong or a weak condition. And are there more general existence theorems like the simplicial simplex theorem?

After the seminal work of Andrews-Clutterbuck resolving the fundamental gap conjecture for convex domains of $\mathbb{R}^n$, recently great progress has been made on the fundamental gap estimates of convex domains of $\mathbb{S}^n, \mathbb{H}^n$ by Wei and coauthors. Question still remains for Einstein manifolds, even $\mathbb{C}P^n$.

3 Presentation Highlights

Monday was used for overview talks that tried to capture the general state of the research landscape in each of five main areas. First up was Christoph Böhm, who reported on recent progress concerning the classification of homogeneous Einstein manifolds. In joint work with R Lafuente he showed that a homogeneous Einstein manifold with negative scalar curvature is diffeomorphic to Euclidean space, confirming the Alexseevskii conjecture from 1975. A Ricci flat homogeneous space is flat. The classification of homogeneous Einstein manifolds with positive scalar curvature is wide open, even though there exist several structure results concerning existence and non-existence.

Ernani Ribeiro Jr gave an overview of four-dimensional gradient shrinking Ricci solitons. He spoke about the geometry of four-dimensional gradient shrinking Ricci solitons. He presented some examples and classical results on gradient Ricci solitons in order to justify special attention to dimension four, and presented some recent classification results for four-dimensional gradient shrinking Ricci solitons. He discussed some open problems.

Xuan Hien Nguyen spoke on the fundamental gap in Euclidean, spherical, and hyperbolic spaces. The fundamental gap is the difference between the first two eigenvalues of the Dirichlet problem for the Laplace operator. She gave a brief history of the problem, stated the main conjecture, and gave a survey of techniques and recent results for the subject.

James Lucietti surveyed near-horizon geometries. Extremal black holes possess a well defined notion of a near-horizon geometry which describes the intrinsic geometry of the horizon. The spacetime Einstein equations imply that such near-horizon geometries correspond to a class of Riemannian quasi-Einstein metrics. As such, near-horizon geometries can be studied and classified independently of any parent black hole spacetime, and hence provide a useful arena to study the geometry and topology of black holes in a purely Riemannian setting. Lucietti gave an overview of the current state of research on near-horizon geometries in a variety of dimensions and theories, focusing on general constraints on their topology and symmetry, their explicit classification, and highlight open problems in this area. He described a number of recent applications of near-horizon geometries to the classification programme for extremal black holes.

Robert McCann presented a synthetic definition of the null energy condition. While Einstein's theory of gravity is formulated in a smooth setting, the celebrated singularity theorems of Hawking and Penrose describe many physical situations in which this smoothness must eventually breakdown. In positive-definite signature, there is a highly successful theory of metric and metric-measure geometry which includes Riemannian manifolds as a special case, but permits the extraction of nonsmooth limits under dimension and curvature bounds analogous to the energy conditions in relativity: here sectional curvature is reformulated through triangle comparison, while and Ricci curvature is reformulated using entropic convexity along geodesics of probability measures. McCann explored recent progress in the development of an analogous theory in Lorentzian signature, whose ultimate goal is to provide a nonsmooth theory of gravity. He began with a simplified approach to Kunzinger and Saemann's theory of (globally hyperboloid, regularly localizable) Lorentzian length spaces in which the time-separation function takes center stage. He showed compatibility of two different notions of timelike geodesic used in the literature. He then proposed a synthetic (i.e. nonsmooth) reformulation of the null energy condition by relating to the timelike curvature-dimension conditions of Cavalletti and Mondino (and Braun), and discussed its consistency and stability properties.

The remaining days were dedicated to focused research talks. On Tuesday, Zhiqin Lu spoke on joint work of N Charalambous on the L^p spectrum of complete Riemannian manifold, computing the L^p spectrum of Laplacians on k-forms on hyperbolic spaces. Lu proved the L^p boundedness of certain resolvent of Laplacians by assuming the Ricci lower bound and manifold volume growth. This generalized a result of M Taylor, in which bounded geometry of the manifold is assumed.

Christian Ketterer then spoke on a characterization of the null energy via displacement convexity of entropy. He presented a characterization of the null energy condition for an $(n+1)$-dimensional, time-oriented Lorentzian manifold in terms of convexity of the relative $(n-1)$-Renyi entropy along null displacement interpolations on null hypersurfaces. More generally, Ketterer considered a Lorentzian manifold with a weight function and introduced the Bakry-Emery N-null energy condition that we characterize in terms of null displacement convexity of the relative N-Renyi entropy. He then proved a version of Hawking's area monotonicity theorem for a black hole horizon and a Penrose singularity theorem in the context of weighted Lorentzian manifolds.

Raquel Perales spoke on the maximal volume entropy rigidity for RCD spaces. For n-dimensional Riemannian manifolds with Ricci curvature bounded below by $-(n-1)$, the volume entropy is bounded above by $n-1$. If M is compact, it is known that the equality holds if and only if M is hyperbolic. She presented a similar result for RCD spaces. in oint work with C Connell, X Dai, J Nunez-Zimbrón, P Suárez-Serrato, and G Wei.

Julie Rowlett discussed joint work with Nelia Charalambous on the L^p-Laplace spectrum of conformally compact manifolds. Conformally compact manifolds are a class of non-compact manifolds with variable curvature that were introduced by Fefferman and Graham to study conformal invariants. They are a broad class and include many examples, including manifolds with Poincaré-Einstein metrics. Motivation to study the Laplace operator acting on L^p for $p \neq 2$ comes from physics. For example, the most natural space on which to study heat diffusion is L^1. Rowlett shows that for general values of p, the L^p Laplace spectrum contains a certain parabolic region and is contained in a certain parabolic region. These regions are determined by the geometry of the conformally compact manifold.

The structure of manifolds with positive spectrum was the subject of Ovidiu Munteanu's talk. A well-known sharp estimate of Cheng compares the bottom spectrum of a complete Riemannian manifold with Ricci curvature bounded from below by a negative constant with the counterpart bottom spectrum on a space form of constant negative curvature. The equality case in this estimate is understood in some cases, although not that much in general. Munteanu generalized this theory for manifolds admitting smooth densities, in particular for Ricci solitons, and in dimension three for manifolds with scalar curvature bounded from below.

The final talk of the day was by Malik Tuerkoen on fundamental gap estimates on positively curved surfaces. For the Dirichlet boundary condition the log-concavity estimate of the first eigenfunction plays a crucial role, which was established for convex domains in the Euclidean space and the round sphere. Joint with G Khan, H Nguyen, and G Wei, Malik obtained log-concavity estimates of the first eigenfunction for convex domains in surfaces of positive curvature and consequently establish fundamental gap estimates. In subsequent work with G Kahn and G Wei, he improved the log-concavity estimates and obtained stronger gap estimates which recover known results on round spheres.

On Wednesday morning Christine Guenther presented new stability results for the Ricci flow on manifolds of bounded geometry. She presented simple conditions which ensure that an elliptic operator L generates an analytic semigroup on any complete manifold of bounded geometry. As applications she proved well-posedness of the Bach flow, and long-time continuous dependence of the Ricci flow, on manifolds of bounded geometry.

Gunhee Cho discussed his work with Jihun Yum on stochastic Bergman geometry. He defined the embedding from the bounded domain Ω in C^n into the collection of probability measures on Ω as a statistical manifold so that Fisher information metric restricted to Ω becomes the Bergman metric.

Shin-ichi Ohta spoke on the geometry of weighted Finsler spacetimes. A Lorentz-Finsler manifold is a generalization of a Lorentzian manifold in the same way that a Finsler manifold generalizes a Riemannian manifold. One can further equip a Lorentz-Finsler manifold with a time orientation as well as a weight, then we have a weighted Finsler spacetime. In this general framework, one can successfully develop the theory of Ricci curvature (singularity theorems, various comparison theorems, etc). This was joint work (partly in progress) of Ohta with Mathias Braun (Toronto), Yufeng Lu (Hong Kong), Ettore Minguzzi (Firenze).

On Thursday morning we focused on asymptotically hyperbolic manifolds. Klaus Kroönke introduced a new mass-type invariant for asymptotically hyperbolic manifolds. He considered a particular linear combination of the renormalized volume and the boundary integral for the usual ADM mass, called the volume-renormalized mass. It is well-defined and diffeomorphism invariant under weaker falloff conditions for the metric at infinity than one needs to define the renormalized volume and the hyperbolic ADM mass separately. He used the volume-renormalized mass to define a variant of the expander entropy on asymptotically hyperbolic manifolds which is monotonically increasing under the Ricci flow. He then used the expander entropy to prove a local positive mass theorem for Poincaré-Einstein metrics. This was joint work of Krönke with Mattias Dahl and Stephen McCormick.

Xianzhe Dai then presented a positive mass theorem for manifolds with fibred Euclidean ends. The positive mass theorem of Schoen-Yau (and Witten) states that an asymptotically Euclidean manifold with nonnegative scalar curvature must have nonnegative ADM mass. Moreover, one has the rigidity statement that the mass is zero if and only if the manifold is the Euclidean space. Recently there has been a lot of interest and activity in positive mass theorems for spaces which are asymptotically approaching an Euclidean space times a compact manifold, or more generally a fibration. Dai discussed some of the work in this direction going back to his earlier work as well as some recent work and related work by Chen-Liu-Shi-Zhu and others. In particular there is a variant of the ADM mass introduced by Minerbe which plays an essential role in proving the rigidity statement here.

Jie Qing spoke about recent work on regularity, compactness and uniqueness of asymptotically hyperbolic Einstein manifolds.

In the afternoon, the focus shifted to near horizon geometries and the quasi-Einstein equations. Hari Kunduri spoke about special solutions of the quasi-Einstein equations. The quasi-Einstein equations with $m = 2$ arise naturally within the study of the horizon geometries of 'degenerate' black holes in general relativity. There has been considerable progress in constructing explicit solutions in the presence of symmetries. Kunduri reviewed these constructions and discussed the prospects of extending them for general m.

Sharmila Gunasekaran then spoke about the Kerr-AdS near horizon geometry. She studied uniqueness of the near horizon geometry that arises from degenerate AdS Kerr black holes in a neighbourhood of nearby near horizon geometries. The result is relevant in answering whether the only solutions to the near horizon geometry equations on a 2-sphere with negative cosmological constant arise from extremal AdS Kerr black holes.

The afternoon ended with a talk by Will Wylie on quasi-Einstein metrics and related equations on homogeneous spaces. He surveyed some rigidity results for gradient solitons to geometric flows and the Quasi Einstein equation on homogenous spaces. The approach is to characterize when the Hessian of a function can solve two general classes of equations involving a two-tensor that is invariant under a transitive group of isometries. This was joint work of Wylie with Peter Petersen of UCLA.

On the final morning of the workshop, Eric Chen discussed his work with Richard Bamler on the existence of expanding Ricci solitons asymptotic to cones with nonnegative scalar curvature. In dimensions four and higher, the Ricci flow may encounter singularities modelled on cones with nonnegative scalar curvature. It may be possible to resolve such singularities and continue the flow using expanding Ricci solitons asymptotic to these cones, if they exist. Bamler and Chen have developed a degree theory for four-dimensional asymptotically conical expanding Ricci solitons, which in particular implies the existence of expanders asymptotic to a large class of cones.

Giovanni Catino described two recent results concerning the rigidity of complete, immersed, orientable, stable minimal hypersurfaces: they are hyperplane in R^4 while they do not exist in some positively curved closed Riemannian $(n+1)$-manifold when $n \geq 5$. The first result was recently proved also by Chodosh and Li, and the second is a consequence of a more general result concerning minimal surfaces with finite index. Both theorems rely on a conformal method, inspired by a classical paper of Fischer-Colbrie. He also presented an application of these techniques to the study of critical metrics of a quadratic curvature functional. This was joint work of Catino with P Mastrolia (Università degli Studi di Milano) and A Roncoroni (Politecnico di Milano).

The final talk of the workshop was delivered remotely via Zoom by Yohei Sakurai. The weighted Ricci curvature is one of the central objects in the study of smooth metric measure spaces. Recently, Lu-Minguzzi-Ohta have suggested a new approach that enables us to investigate the so-called curvature-dimension condition and the Wylie-Yeroshkin curvature condition for weighted affine connections in a unified way. Sakurai presented various comparison geometric results under their curvature bound, and a characterization result via optimal transport. His talk was based on joint work with Kazuhiro Kuwae (Fukuoka University).

One unique feature of our meeting was a special evening session on Monday night dedicated to the introduction of junior participants. The organizers asked each non-tenured participant who was present in-person at the workshop site to prepare a short slide presentation introducing themselves and discussing their general research interests and presenting a slide with their best work. This proved to be a high-energy session that helped foster inclusion and discussions with these participants throughout the week. Presentations in this special session were given by Eric Chen, Gunhee Cho, Davide Dameno, Erin Griffin, Sharmila Gunasekaran, Christian Ketterer, Jihye Lee, Fabio Ricci, Shoo Seto, and Malik Tuerkoen.

4 Progress/Outcome of the Meeting

It is unusual to be able to measure progress in mathematics on short time scales. However, an exception is the progress reported at the meeting on formulating synthetic definitions of the energy conditions of general relativity, especially the work on the null energy condition reported by McCann and by Ketterer. This is a very rapidly developing area.

The Thursday morning session on asymptotically hyperbolic manifolds also produces surprises. The topic pertains more specifically to Einstein manifolds than to the more general quasi-Einstein manifolds, but the study of the mass, renormalized volume, and mass-like invariants of these manifolds still leads to surprises.

One of the most important outcomes of the meeting is that it opened a conversation between physicists interested in general relativity (particularly black holes), Riemannian geometers for whom Einstein and quasi-Einstein manifolds are familiar objects, and analysts interested in geometry of objects that are not smooth enough to allow for the structure of a differentiable manifold.

In summary, it was great seeing quasi-Einstein metrics play a role in so many different subjects, even in "Bernstein problem" in the recent work of Cattino-Mastrolia-Roncoron. Many participants liked the idea of Monday overview talks, and Monday night short talks by graduate students and postdocs. The presentations are all amazingly good.