Interactions between Algebraic Topology and Geometric Group Theory 23w5034

Javier Aramayona (Instituto de Ciencias Matemáticas, España),
Kenneth Bromberg (University of Utah),
Daniel Juan Pineda (Centro de Ciencias Matemáticas, UNAM)
Conchita Martínez-Pérez (Universidad de Zaragoza),
Rachel Skipper (The Ohio State University)

May 29th, 2023-June 2nd 2023

1 Overview of the Field

Discrete groups are often studied through their actions on topological or geometric spaces. Traditionally, actions on topological spaces have been the focus of algebraic topology, while geometric group theory revolves around actions on spaces with a rich geometric structure. However, there has been a recent trend in applying ideas and techniques coming from geometric group theory to solve well-known questions in algebraic topology.

Through influential work of Bartels, Lück, and others, the Farrell-Jones Conjecture is known to be true for many classes of groups, such as hyperbolic groups, relatively hyperbolic groups, $\mathrm{CAT}(0)$ groups, and lattices in connected Lie groups. In a more recent breakthrough, Bartels-Bestvina have established the conjecture for mapping class groups. In all these proofs there has been beautiful interplay between of geometry, dynamics, geometric group theory, and algebraic topology.

Hyperbolic groups, relatively hyperbolic groups and $\mathrm{CAT}(0)$ groups, by their very definition, act on spaces of negative or non-positive curvature and for this reason have been of central interest to geometric group theorists. There is much current work trying to understand what kind of actions more general groups have on negative curved spaces and how this can be used to prove theorems about the groups. For example Bestvina-Bromberg-Fujiwara developed an axiomatic setup to construct group actions on quasi-trees (or more generally metric spaces of quasi-trees) and this plays a key role in the Bartels-Bestvina resolution of the Farrell-Jones conjecture for the mapping class group. In fact Bartels-Bestvina added additional *flow axioms* to the original axioms of Bestvina-Bromberg-Fujiwara giving a general framework for showing that a group satisfies the conjecture. Of course, it is natural to ask if this framework can be applied to other groups with $Out(F_n)$ be the most obvious candidate. Another natural class to examine is *acylindrically hyperbolic groups* of which there has been much recent progress by Osin and others. At present this class is probably too broad to hope for such a general result, although there may be some extra structure that could be added to get a new class of groups satisfying the conjecture.

1.1 Classifying spaces of optimal dimension

Classifying spaces constitute a powerful tool for studying algebraic properties of discrete groups. These are contractible spaces on which the group act subject to suitable topological conditions on the fixed-point sets of preferred families of subgroups. An important feature of classifying spaces is that they are unique up to equivariant homotopy, and thus in many occassions it suffices to find a good model for such a space. For instance, the cohomological dimension of a torsion-free group, defined in terms of projective resolutions of group modules, coincides with the ordinary cohomological dimension of any of its classifying spaces.

By a celebrated theorem of Eilenberg-Ganea, if a (discrete) torsion-free group G has cohomological dimension $n \geq 2$ then it admits a classifying space $\mathrm{E}G$ of dimension n, which is moreover the minimal dimension that a classifying space for G may have. In this light, the Eilenberg-Ganea theorem says that torsion-free groups *admit classifying spaces of optimal dimension*.

On the other hand, if G has torsion then it does not admit a finite-dimensional $\mathrm{E}G$. For this reason, it is more natural to consider a *classifying space for proper actions of* G, denoted EG, where we require that finite (resp. infinite) subgroups of G act with contractible (resp. empty) fixed-point sets. As further motivation, the classifying space EG appears on the topological side of the *Baum-Connes Conjecture*, also of central importance in algebraic topology.

An old question of Brown, motivated by the Eilenberg-Ganea theorem, asked if the minimal dimension of an EG coincides with the *virtual* cohomological dimension of G. The answer is known to be negative through the work of Leary-Nucinkis, although equality is known to hold for many classes of natural groups, such as Coxeter groups, lattices in (classical) simple Lie groups, mapping class groups, and outer automorphism groups of free groups.

In all of these cases, there is a beautiful interplay of algebraic-topological and geometric ideas. Important ingredients of the former are Lück's theorem equating the minimal dimension of an EG and the Bredon cohomological dimension of G, and a result of Degrijse-Martínez-Pérez interpreting Bredon cohomological dimension in terms of the cohomology of EG relative to the singular set. On the geometric side, one often works with a well-known model of an EG (e.g. a symmetric space) and have good control over the possible dimension of fixed-point sets, or find a deformation retract of this space of the correct dimension and on which G acts correctly.

In light of this, a natural question is to extend these techniques and ideas to other interesting classes of groups, such as automorphism groups of right-angled Artin groups, (certain) acylindrically hyperbolic groups, lattices in more general Lie groups, etc.

1.2 Dimension bounds on classifying spaces

As mentioned above, another well-studied type of classifying space associated to a group G is the classifying space $\mathrm{E}_{vc}G$ with respect to the family of virtually-cyclic (VC) subgroups of G. This is a contractible space on which G acts, subject to the condition that every VC subgroup has a contractible fixed-point set, while for other subgroups these are empty. Finiteness properties of $\mathrm{E}_{vc}G$ are important in light of the Farell-Jones conjecture, and thus a natural question is whether a given class of groups admits a finite-dimensional $\mathrm{E}_{vc}G$. This is known to be the case for hyperbolic groups, relatively hyperbolic groups, virtually polycyclic groups, many amenable groups, braid groups, mapping class groups, etc. In most of these cases, an explicit model is constructed by attaching appropriate equivariant pieces to an explicit model of an EG. The construction of the model of EG is closely related to the geometric structure of G, which again displays a strong interplay between the geometric group theory and algebraic topology sides. It would be interesting to know if these results may be extended to other classes of widely studied groups, such as outer automorphism groups of free groups, automorphism groups of right-angled Artin groups, etc.

As a byproduct of the arguments used for proving the existence of a finite-dimensional $\mathrm{E}_{vc}G$, one obtains an upper bound for its dimension. Therefore, an obvious problem is to determine the minimal dimension of an $\mathrm{E}_{vc}G$. Lück proved that this dimension is bounded above by one plus the minimal dimension of an EG, whenever G acts geometrically on a proper complete $\mathrm{CAT}(0)$ space. In stark contrast, Degrijse and Petrosyan have recently proved that the gap between the two quantities can be arbitrarily large. However, as was the case with Brown's question above, the above inequality does hold for certain families of groups, such as planar braid groups, or mapping class groups of "low-complexity" surfaces. It would be interesting to find other classes of "geometrically-defined" groups for which this holds.

2 Recent Developments and Open Problems

M. Bestvina presented a decomposition of the curve complex C for a surface S by giving a tower of $Mod(S)$ complexes

$$C = C_N \to C_{N-1} \to \cdots \to C_1 \to C_0,$$

where C_0 is a quasi-tree, all maps are Lipschitz, equivariant and coarsely onto and the coarse fibers are quasi-trees. This decomposition may give new insights in various geometric aspects of the curve complex.

Rita Jiménez presented an upper bound of the virtually abelian dimension of $Mod(S)$, this is in relation to the various geometric dimensions for families of subgroups of a given group. This notion is strongly related to the well-known Farrell-Jones conjecture for a group, proved by Bartels-Bestvina in the case of mapping class groups.

In a related topic Kevin Li gave a computation of the complexity for families of subgroups of a given group, this is via the study of suitable covers for the corresponding classifying spaces and relates these with the different notions of dimension, cohomological or virtual cohomological dimension of a group.

Luis Jorge Sánchez presented a method to study the homotopy type of the space of n-commuting elements in 3-manifold group. This is via Thurston's geometrization , the results is that this space has the homotopy type of a wedge of circles, possible infinitely many.

Carolyn Abbott presented how to study the a subspace of the visual boundary of a hyperbolic group to overcome the problem that the usual visual boundary is not a quasi-isometric invariant.

Kai-Uwe Bux presented finiteness properties of the so-called Houghton groups and some subgroups of spacial big mapping class groups, these are called B_n, he proved that these have similar finiteness properties as Houghton groups, namely they are F_{n-1} but not F_n.

Claudio Llosa, using complex geometry, gave a collection of examples of subgroup of hyperbolic groups that are of type F_{n-1} but not of type F_n.

Ferrán Valdez gave a relation between model theory and the study of big mapping class groups to study properties such as automatic continuity.

Xiao Lei gave a computation of the homology of the big mapping class group for the specific case of a surface with at least boundary component on end or the disc minus a cantor set.

Macarena Arenas explores the problem of finding a good model for the classifying space of a hyperbolic group and the small cancellation property.

Sahana Balasubramanya explores the problem of understanding all hyperbolic and cobounded isometric actions on a hyperbolic space. She gives a characterization for the case of an acylindrically hyperbolic group.

Priyam Patel describes how to approach Thurston's geometrization theorem in a more combinatorial way.

Bena Tishiku gave an overview and new results on the Nielsen realization problem, namely given a manifold anf a group $G \subset Out(\Gamma)$ where $\Gamma = \pi_1(M)$, Can we realize G by an action on M.

Noé Bárcenas presented some aspects on the Zimmer program for 3-dimensional manifolds and Alexandrov spaces.

3 Presentation Highlights

The workshop was diverse in attendance, presenters and topics. We have an attendance of 28 in-site people. There were presentations that highlighted the objectives of the workshop, namely: show interactions between algebraic topology and geometric group theory. There were 14 talks, 7 of these delivered by women, presenters range from senior to posdoctoral fellows. The topics pointed all aspect expected according to the theme of the workshop. The problem session also included a wide variety of topics and presenters. We point out the constant discussions around the facility, these went into late night on many occasions. We also have a good participation of graduate students, posdoctoral fellows and local people from the local mathematics community.

4 Scientific Progress Made

Many collaborations were started or continued during this conference based on the topics and the talks and significant progress was made on these collaborations. This includes, for instance, a joint project between Abbott, Patel and Skipper, but there were numerous other active discussions which occurred during the breaks between talks.

5 Outcome of the Meeting

An important outcome of the workshop is a list of problems suggested by the participants, these are directly related to the topics of the workshop and show some of the advances in the different topics as well as new directions for future research.

1. (Kenneth Bromberg) Do curve complexes of finite-type surfaces quasi-isometrically embed equivariantly in a finite product of quasi-trees?

   A positive question to this answer would give lower bounds on the asymptotic dimension of the associated curve complexes.

   Let X be a geodesic metric space. If X is non-hyperbolic, then the usual visual boundary $\partial_\infty X$ consisting of equivalence classes of geodesic rays of X (issuing from a base point) may not be a quasi-isometric invariant. One way to fix this is to restrict our attention to the **Morse geodesic rays** which, roughly speaking, are the rays pointing in the *hyperbolic directions* of X. Considering the remainder of the visual boundary, we could ask if it makes sense to look at the *non-hyperbolic* directions of X to define other pieces at infinity with good metric properties.

2. (Daniel Juan-Pineda) Is it possible to define from the non-Morse geodesic rays a quasi-isometric invariant metrizable space with good metric properties?.

3. (Kai-Uwe Bux) Is the group $\mathrm{SL}_2(\mathbb{Z}[t, t^{-1}])$ finitely generated?.

   Using the abelianization morphism from the free group F_n to the abelian free group $\mathbb{Z}^n$, we induce a morphism $Out(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. In a similar way, we have the symplectic representation $\mathrm{Map}(S_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$ in the context of mapping class groups. Considering that mapping class groups of punctured-spheres with free fundamental group of rank n embed in $\mathrm{Out}(F_n)$ in the same way as that $\mathrm{Map}(S_g)$ embeds in $\mathrm{Out}(\pi_1(S_g))$; we could think of the previous morphisms as special cases of homomorphisms from groups related to mapping class groups to *Chevalley groups* (reductive algebraic groups) over $\mathbb{Z}$.

4. (Kai-Uwe Bux) Can the previous analogy be generalized to every Chevalley group over the integers?. In a more concrete way, we could ask whether or not there are covers of Chevalley groups with non-arithmetic covers.

5. (Carolyn Abbott) Let G be a group acting acylindrically on two hyperbolic spaces X and Y. Given a loxodromic element g for the action of G on X and a loxodromic element h for the action of G on Y, does there exist an acylindrical action of G on another hyperbolic space Z for which both elements $g, h \in G$ act loxodromically?.

6. (Macarena Arenas) Is the geometric dimension a quasi-isometric invariant for torsion-free groups?.

   Let Γ be a finite simple graph, i.e. a finite graph without loops or multiple edges between vertices. We define the *right-angled Artin group* (RAAG) A_Γ as the group generated by the vertices of Γ with all the relations of the form $vw = wv$ whenever v and w are joined by an edge.

7. (Ric Wade)

   (a) Let G be a group and suppose that A_Γ is a finite-index RAAG subgroup of G. Does G have a cocompact $\underline{E}G$ whose dimension equals the cohomological dimension of $A_\Gamma \leq G$?.

(b) Does $Out(A_\Gamma)$ have a cocompact $\underline{E}Out(A_\Gamma)$ whose dimension equals the virtual cohomological dimension of $Out(A_\Gamma)$?.

(c) Is there a poly-cyclic subgroup of $Out(A_\Gamma)$ of Hirsch length equal to the virtual cohomological dimension of $Out(A_\Gamma)$?.

Let $S_{g,1}$ be the orientable surface of genus g with one boundary component and let $\mathcal{C}$ be a copy of the Cantor set inside $S_{g,1}$.

8. (Xiaolei Wu) Does the inclusion of the infinite-type surface $S_{g,1} \setminus \mathcal{C}$ into $S_{g,1}$ induces an isomorphism on homology in every degree?. By [2, Theorem 2.3] we know that the answer is affirmative in degree one for all surfaces of finite-type.

Consider an infinite-type surface S with a distinguished point p. Following Bavard and Walker ([1]), we can define the short-ray graph $\mathcal{R}_s(S;p)$ of S as the graph whose vertices are isotopy classes of proper rays starting at p by isotopies fixing p for all times, and adjacency is given by disjointness of representatives. The quotient group $\mathrm{Map}(S;p)$ of $\mathrm{Homeo}^+(S;p)$ by isotopies which fix p for all times acts naturally on the short-ray graph $\mathcal{R}_s(S;p)$ by the morphism

$$\begin{aligned} \mathrm{Map}(S;p) &\longrightarrow \mathrm{Aut}\big(\mathcal{R}_s(S;p)\big) \\ [f] &\longmapsto ([\gamma] \longmapsto [f \circ \gamma]). \end{aligned}$$

9. (Ferrán Valdez) Does the previous action admit a parabolic element?.

Recall that the **amenable category** of a group G can be given as the minimun $n \in \mathbb{N}$ for which the natural map $EG \to E_{\mathrm{AME}}(G)$, where AME is the family of amenable subgroups of G, factors through the n-skeleton $X^{(n)} \hookrightarrow E_{\mathrm{AME}}(G)$.

10. (Kevin Li) What is the amenable category of a Coxeter group?.

11. (Claudio Llosa Isenrich) How *wild* subgroups of hyperbolic groups can be?.

Putting this question in perspective, we could ask how *exotic* the geometry of subgroups of hyperbolic groups can be.

12. (Claudio Llosa Isenrich)

(a) Is there a characterization of groups that embed in some hyperbolic group?.

(b) Does every type F group with no subgroups isomorphic to a Baumslag-Solitar group embed in a hyperbolic group?.

Consider the n-fold direct product of the free group F_2. For each factor of the direct product let us denote its generators by a_i and b_i for $i = 1, \ldots, n$. Define the homomorphism

$$\begin{aligned} \phi : F_2 \times F_2 \times \ldots \times F_2 &\to \mathbb{Z} \times \mathbb{Z} \\ a_i &\mapsto a \\ b_i &\mapsto b \end{aligned}$$

where $\mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle$.

13. (Claudio Llosa Isenrich) What is the kernel of ϕ and what is the behavior of its growth function $\delta_{\ker(\phi)}$?.

Let $\mathcal{N}_g = \mathrm{Map}(N_g)$ be the mapping class group of the non-orientable surface of genus g. For genus 4 and 5, Hidber-Saldaña-Trujillo in [3] proved the following inequalities:

$$\begin{aligned} 3 = vcd(\mathcal{N}_4) \leq \underline{cd}(\mathcal{N}_4) = \underline{gd}(\mathcal{N}_4) \leq 6, \\ 5 = vcd(\mathcal{N}_5) \leq \underline{cd}(\mathcal{N}_5) = \underline{gd}(\mathcal{N}_5) \leq 6. \end{aligned}$$

14. (Luis Jorge Sánchez Saldaña) For the non-orientable surfaces of genus 4 and 5, what is exactly the proper geometric dimension of the respective mapping class groups?.

The list has been compiled by Néstor Colín, Porfirio L. León and Carlos Pérez Estrada.

References

[1] Bavard, Juliette and Walker, Alden, Two simultaneous actions of big mapping class groups. Preprint at arXiv: 1806.10272 [math.GT], 201

[2] Calegari, Danny and Chen, Lvzhou, *Trans. Amer. Math. Soc. Ser. B*, **9** 2022, 957–976,

[3] Hidber, Cristhian E. and Sánchez Saldaña, Luis Jorge and Trujillo-Negrete, Alejandra, On the dimensions of mapping class groups of non-orientable surfaces, *Homology Homotopy Appl.*, **24** 2022, 347–372