New trends from Classical Theorems in Geometry, Combinatorics, and Topology

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1 Overview of the Field

Helly's theorem plays an important role in combinatorial geometry. It states the following:

Helly's Theorem [1913 [12]] Let $\mathcal{A}$ be a finite family of at least $d+1$ convex sets in R^d. If every $d+1$ members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.

Helly's theorem holds for infinite families of compact convex sets as well, and has stimulated numerous generalization and variants. Results of the type "if every m members of a family of objects have property $\mathcal{P}$ then the entire family has the property $\mathcal{P}$" are called Helly-type theorems. The minimum positive integer m that makes this theorem possible is called the Helly number. Helly-type theorems have been the object of active research, inspired by many of the problems posed in "Helly's Theorem and Its Relatives" [17].

Furthermore in the past decades, there has been a significant increase in research activity and productivity in the area. Notable, two key lines of research around Helly's theorem have gathered significant interest in the last few years. On one side we have quantitative Helly-type results. In these results, we seek to characterize finite families of convex sets whose intersection is not only non-empty, but also measurably large (such as having a large volume, for instance). State-of-the art results in this direction rely on analytic properties of convex sets, such as characterization of maximal-volume ellipsoids in a convex sets. The other active line of research is around topological Helly-type results. In these theorems, the condition of convexity is relaxed. A new line of Helly-type theorems has been found, in which instead of imposing conditions on contractibility of the sets, or vanishing of their homology groups (as was done prior), we just bound certain Betti numbers on their sets and their intersections.

There are many interesting connections between Helly's theorem and its relatives, the theorems of Radon, of Caratheodory and of Tverberg. In fact, one of the most beautiful theorems in combinatorial convexity is Tverberg's theorem, which is the r-partite version of Radon's theorem, and it is very closely connected with the multiplied, or colorful versions of the theorems of Helly, Hadwiger and Caratheodory. The first of these colorful versions was discovered by Barany and Lovasz and has many applications (see [3]).

For instance, Tverberg's theorem [28] still remains central and is one of the most beautiful and intriguing results of combinatorial geometry. It has been shown that there are many close relations between Tverberg's theorem and several important results in mathematics, such as: Rado's Centre Point Theorem on general measures, the Ham Sandwich Theorem and the Four Color Theorem, just to mention some examples. In the last two decades there has been an increasing amount of work that involve the use of new techniques in algebraic topology and other areas of mathematics. The topological techniques use range from simple homotopy arguments all the way to explicit computation of characteristic classes of associate fiber bundles.

2 Workshop Structure

We were able to have 31 in-person participants at CMO, together with some online participants. This turned it into a smaller workshop than originally anticipated, but it led to an environment where there was more time available for informal working in groups.

This workshop brought together senior and junior researchers in the area with the objective of interchanging ideas and assessing recent advances, of fostering awareness of the inter-disciplinary aspects of the field such as geometry, topology, combinatorics, and computer science, as well as mapping future directions of research.

The workshop combined and interesting mixture of talks, at least two problem sessions and many time for discussions in groups.

3 Recent Developments and Open Problems

During the week the academic interest was mainly centered around Helly type theorems, generalizations and variations of Tverberg's theorem and many other talks that had deep relationships with other areas of discrete and non-discrete mathematics, such as algebraic topology, algebraic geometry convexity and probability.

Among Helly-type results there were talks extending both the topological versions (bounding Betti numbers), as well as analytic versions, improving current bounds on several volumetric Helly-type results. Among the topological results, some of the talks extended the technical machinery needed to prove mass partitions results, such as those related to iterated partitions, mass assignments, or embeddability results. Other talks focused on simplifying the need for technical tools by finding relation to transveral results or using simpler methods.

3.1 Open Problems

There were two open problem sessions during the workshop, which gave the participants the chance to share and learn about interesting open questions in the area. The following open problems are a sample of some problems suggested during the workshop.

1. *Transversal complex lines to real lines* by Arocha, Brancho y Montejano: Given a family $\mathcal{F}$ of real lines en $\mathbb{C}^2$. Assume that every $T(6)$ real lines $L_1, L_2, \dots L_6$ admit a transversal complex line through the origin. Is there a transversal complex line through the origin that intersects every line in $\mathcal{F}$?

2. *A decidibility problem* by Boris Bukh:

Let $C_1, C_2, \dots C_n$ convex sets in $\mathbb{R}^d$ and $p \in \mathbb{R}^d$ define a pattern $pat_C(p) = \{i : p \in C_i\} \in 2^[n]$. Given $int(C) = \{pat_C(p) : p \in \mathbb{R}^d\}$. Given $\mathcal{F} \in 2^{2^{[n]}}$ Is there $C = (C_1, C_2, \dots C_n)$ such that $int(C) = \mathcal{F}$? Is this problem decidable for $d = 3$?

3. *Iterated Partitions* By Pablo Soberón: We say that a partition of $\mathbb{R}^d$ into n partss is an iterated hyperplane partition if it can be obtained by the following recursive process. First, for $n = 1$, it is simply $\mathbb{R}^d$. Otherwise, find an iterated hyperplane partition of $\mathbb{R}^d$ into $n - 1$ parts, pick a piece, and split only that piece into two using a hyperplane.

Given two finite measures in $\mathbb{R}^2$, each absolutely continuous with respect to the lebesgue measure, and an even number n, is it possible to split $\mathbb{R}^2$ into n parts using an iterated hyperplane partition so that each piece in the partition has the same size under each measure? This is easy when n is a power of two, and the requirement of n being even if $n > 1$ is necessary. All other cases are open. If $n = 6$, the problem has a positive solution if additional conditions on the measures are imposed.

4. *Extended Colorful fractional Helly theorem* By Martin Tencer:

Theorem[Colorful fractional Helly theorem] For every $\alpha \in (0, 1]$ and $d \geq 1$ there exists $\beta(\alpha, d) \in (0, 1]$ with the following property: Let $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \dots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ colorful $(d+1)$-tuples have nonempty intersection, then there is $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least βn_i with a nonempty intersection. (By a colorful $(d+1)$-tuple we mean a collection of $d+1$ sets $F_1 \in \mathcal{F}_1, \dots, F_{d+1} \in \mathcal{F}_{d+1}$, not necessarily distinct.)

For the theorem above, it is even known that the optimal $\beta = 1 - (1 - \alpha)^{1/(d-1)}$.

What is the optimal β for a topological version? (I.e. when $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{d+1}$ forms a *good cover*; that is a collection of open sets such that the intersection of each subcollection is either contracible or empty.) In particular, is the topological version true with $\beta = 1 - (1 - \alpha)^{1/(d-1)}$?

5. *Realizable orders* by Edgardo Roldan Pensado:

In the sense of Leonardo Martínez Sandoval's definition regarding representing orders and pre-orders as orderings of pairwise Euclidean distances:

Is a total order $K_{4,4}$ realizable in $\mathbb{R}^3$, Is $K_{2,2}$ representable in $\mathbb{R}$ how about $K_{3,3}$ in representable in $\mathbb{R}^2$? Find a non computer assisted proof.

6. *Helly type theorem for transversals* by Gergely Ambrus:

Let $\{K_1, K_2, \dots K_n\}$ a finite family of convex sets in $\mathbb{R}^3$ if every 3 sets intersect in order there is a line that intersects all of them. What happens if every two intersects? Can we say show, the existence of a constant $c > 0$ such that there exist a line l that intersects at least c of the K_i? This question was asked by Martínez-Sandoval, Roldán-Pensado, and Rubin. Gergely Ambrus wanted to bring attention to this open problem. There is a positive answer for certain families of convex sets, proven by Imre Bárány.

7. *Separation problem* by Ricardo Strausz

There exists 8 convex sets in some dimension d, $3 \leq d \leq 7$ such that for every four of them there is a 2-transversal plane and for every 3-coloring of the 8 two colors separate?

4 Presentation Highlights

4.1 Jorge Ramirez Alfonsín: Around the vertices of projective polytopes

A projective transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ is a function such that $T(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$, where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$, such that at least one of $c = 0$ or $\delta = 0$. T is said to be *permissible* for a set $X \subset \mathbb{R}^d$ if and only if $\langle c, x + \delta\rangle = 0$ for all $x \in X$.

In (1971) P. McMullen's Propose the following problem that is closely connected with different notions/problems such as minimal Radon partitions, tolerance of finite sets and arrangements of hyperplanes, that say the following: Determine the largest integer $n(d)$ such that given any set of points $X \subset \mathbb{R}^d$ in general position with $|X| = n(d)$ there is a permissible projective transformation mapping this points onto vertices of a convex polytope.

In (1972) D. Larman proved that $2d + 1 \leq n(d) \leq (d + 1)^2$ for $d \geq 4$ and conjeture that $n(d) = 2d + 1$ proving in particular that $n(2) = 5$ and $n(3) = 7$. Later on Las Vergnas improved the upper bound to $n(d) \leq \frac{d(d+1)}{2}$ for any

$d \geq 2$. During this talk a more general question was raised: Find the maximum number $n(d,t)$, such that for any set with n points in general position, there is a permissible projective transformation were some tolerance t is given, that is, by allowing some vertices to be at the interior of the convex hull, $(n(d,0) = n(d))$. During this talk Ramirez Alfonsín use among other thing very interesting combinatorial methods of Oriented Matroids. This is a joint work with N. Garcia-Colin and L.P. Montejano.

4.2 Gergely Ambrus: Quantitative Helly-type theorems via sparse approximation.

As we mention before, Helly's theorem, [12] is a classical and corner stone theorems in convex and combinatorial geometry states (in its finite) version that the intersection of a finite family of convex sets in $\mathbb{R}^d$ is empty if and only if there exists a subfamily of $d+1$ sets such that its intersection is empty. In 1982 Bárány, Katchalski and Pach [14] proved the following quantitative versions of Helly's theorem; one of them according to the Quantitative Volume Theorem: if the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of size at most $2d$ is of volume at most $v(d)$, a constant depending only on d. The Quantitative Diameter Theorem states that if the intersection of a family of convex sets in $\mathbb{R}^d$ is of diameter one, then the intersection of some subfamily of size at most $2d$ is of diameter at most $\delta(d)$, a constant depending only on d. In [14], the authors established an upper bound of roughly d^{d^2} and conjectured that $v(d) \leq (d)^{cd}$ for a constant $c > 0$. Naszódi confirmed this conjecture in 2016 [21] using contact points of the John ellipsoid of the intersection of the family of convex sets. The current best bound $v(d) \leq (cd)^{\frac{3d}{2}}$, is due to Brazitikos (2017) [15].

For the diameter question, Brazitikos [16] proved the first polynomial bound on $\delta(d)$ by showing that $\delta(d) \leq (cd)^{\frac{11}{2}}$ for some $c > 0$. In 2021, Ivanov and Naszódi [18] found the best known upper bound, $\delta(d) \leq (2d)^3$, and also proved that $\delta(d) \geq (cd)^1/2$. Thus, the value conjectured in [14] for $\delta(d)$ would be asymptotically sharp. In this talk Gergely show the following sparse approximation result for polytopes. Assume that $Q \subset \mathbb{R}^d$ is a polytope in John's position. Then there exist at most $2d$ vertices of Q whose convex hull Q' satisfies $Q \subseteq (-2)d^2Q'$. Obtaining as a consequence, the best bound for the quantitative Helly-type result for the volume, achieved by Brazitikos, and improve on the strongest bound for the quantitative Helly-type theorem for the diameter, shown by Ivanov and Naszódi in [18]:

4.3 Leonardo Ignacio Martínez Sandoval: Representing orders and pre-orders as orderings of pairwise Euclidean distances

The talk focused on the study of distances among points in Euclidean space topic in discrete geometry that has stimulated a lot of research and has interconnected several areas of mathematics. Given set of n points $\mathcal{P}$ in $\mathbb{R}^d$ in general position, by measuring the distance between all possible pairs of points it is cleat that it is possible to generate some ordering, that is:

Let $D_n = \binom{[n]}{2} = \{(i,j) : 1 \leq i < j \leq n\}$, where $[n] = \{1, 2, \dots, n\}$ given by $(i_1, i_2) \leq (j_1, j_2)$ if and only if $\|p_{i_1} - p_{i_2}\| \leq \|p_{j_1} - p_{j_2}\|$.

Under these assumptions, the point collection $\mathcal{P}$ induces a total pre-order on the family of pairs. When $\mathcal{P}$ induces pairwise distinct distances, this pre-order is also antisymmetric and thus it is a total order on D_n.

The aim of this talk was to study weather or not, for given predetermined given ordering, is it possible to find a suitable set of points $\mathcal{P}$ in $\mathbb{R}^d$ for some d such that $\mathcal{P}$ is a "geometric realization" of the given ordering. During this talk it was shown that every given total preorder or total order is achievable if and only if d is large enough with respect to n. And presented some optimal bound on the minimal dimension required for this to happen.

In particular they show that when $n \geq 3$, the minimal dimension into which any total order on D_n can be induced by the pairwise distances of a point collection in $\mathbb{R}^d$ is $d = n-2$. And the minimal dimension into which any total pre-order on D_n can be induced by the pairwise distances of a point collection in $\mathbb{R}^d$ is $d = n-1$.

4.4 Nikola Sadovek: Iterated convex partitions

In 2006, Nandakumar and Ramana-Rao conjectured that every convex polygon P in the plane can be partitioned into any prescribed number n of convex pieces that have equal area and equal perimeter. Later the same authors [22] gave an answer in case $n = 2$ and proposed a proof for other cases. In [4] Bárány, Blagojević and Szűcs gave the positive answer for the case $n = 3$ and Soberón [24] solved a similar question. Naturally, a higher dimensional extension of the conjecture was soon formulated by two groups of authors; Karasev, Hubard, Aronov in [20] and by Blagojević and Ziegler in [8], and they have shown that the regular convex partitions of a Euclidean space into n parts yield a solution to the generalised Nandakumar and Ramana-Rao conjecture when n is a prime power. This was obtained by parametraising the space of regular equipartitions of a given convex body with the classical configuration space. During his talk, Sadovek show how by a repetition of the process of regular convex partitions many times, first partitioning the Euclidean space into n_1 parts, and then each part into n_2 parts, and so on, it is possible to obtain an iterated equipartions of a given convex body into $n = n_1 ... n_k$ parts. Then he show how to parametrise such iterated partitions by the (wreath) product of classical configuration spaces, and develop a new scheme for solving the generalised Nandakumar and Ramana Rao conjecture. The new scheme yields a solution to the conjecture if and only if all the n_i's are powers of the same prime. In particular, for the failure of the scheme outside prime power case he gave three different proofs. (This lecture was based on the joint work with Pavle V. M. Blagojević.)

4.5 Zuzana Patáková: On Radon, fractional Helly and (p,q)-type theorems

As we mention before, one of the fundamental statements of combinatorial convexity is Radon's lemma [23] which says that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two parts whose convex hulls intersect. This property was extended to partitions in to k parts, by the celebrated theorem of Tverberg [28], stating that any set of $(d+1)(k-1)+1$ points in $\mathbb{R}^d$ can be partitioned in to k parts whose convex hulls have a point in common. One other very interesting generalization of Helly's theorem is the famous (p, q) theorem due to Alon and Kleitman [2], whose proof combined a large number of sophisticated tools and results that had been developed over the years since Helly's original theorem.

This talk was based on joint work with Xavier Goaoc and Andreas Holmsen and it was concerned with one particular (and important) generalization of Helly's theorem due to Katchalski and Liu [19] known as the fractional Helly theorem. It states the following. Let F be a family of $n \geq d+1$ convex sets in $\mathbb{R}^d$, and suppose the number of $(d+1)$-tuples of F with non-empty intersection is at least $\alpha\binom{n}{d+1}$, for some constant $\alpha > 0$. Then there are at least βn members of F whose intersection is non-empty, where $\beta > 0$ is a constant which depends only on α and d. The fractional Helly theorem plays a crucial role in the proof of the (p, q) theorem, as well as in various fractional Helly theorems are known (see for example [1, 6, 10]). In 2019 it was proved by Holmsen and Lee see [13] that Radon's theorem implies fractional Helly Theorem. By standard techniques this consequently yields an existence of weak epsilon nets and a (p, q)-theorem. During this talk Zuzana Patákoá show that it is possible to obtain these results even without assuming convexity, by replacing it with very weak topological conditions. More precisely, given an intersection-closed family F of subsets of $\mathbb{R}^d$, and measure the complexity of F by the supremum of the first $d/2$ Betti numbers over all elements of F. And show that constant complexity of F guarantees versions of all the results mentioned above, most notably that fractional Helly number of such families is $d + 1$.

4.6 Pavle Blagojević: Many partitions of mass assignments

Problems of the existence of mass partitions by affine hyperplanes in a Euclidean space have a long and exciting history since the 1930' sham-sandwich theorem of Hugo Steinhaus and Karol Borsuk. The ham-sandwich theorem claims the existence of a hyperplane which equiparts d given masses in a d-dimensional Euclidean space. A mass in a Euclidean space V is assumed to be a finite Borel measure on V which vanishes on every affine hyperplane. A more general problem for masses asks for the minimal dimension $d = \Delta(j, k)$ of a Euclidean space V in which every collection M of j masses can be equiparted by an arrangement of k affine hyperplanes. In this talk P. Blogojević presented a general framework for treating problems of partitions of mass assignments with prescribed hyperplane arrangements on Euclidean vector bundles. Developing a new configuration test map scheme, as well as an alternative topological toolkit will allow us to reprove known results, extend them to arbitrary bundles as well as to put various types of constraints on the solutions. Moreover, the developed topological methods allow us to give new proofs and to extend results of Guth & Katz, Schnider, Soberón and Takahashi, and Blagojević, Calles Loperena, Crabb and Dimitrijević-Blagojević. In this way all these results will be under one "roof". (This lecture is based on the joint work with Michael C. Crabb.)

4.7 Florian Frick: Hyperplane partitions and transversals

The classical Ham Sandwich theorem conjectured by Steinhaus and proved by Banach (see [7]), is by no doubts the origin of topological methods in discrete geometry.

Theorem Let $\{X_1, \dots X_d\}$ be finite subsets of $\mathbb{R}^d$. Then there exists an affine hyperplane H such that both of the corresponding open half spaces H^+ and H^- contain no more than half the points of each X_i.

The hyperplane H given by the Theorem simultaneously bisects each set X_i. There are two generalizations of the Ham Sandwich theorem. First, Dolnikov gave a combinatorial criterion for not necessarily balanced hyperplane partitions of point sets. This recovers the Ham Sandwich theorem in the balanced case. Second, generalizations for equipartitions by multiple hyperplanes have been proven. In the same way that Dolnikov's result extends the classical Ham Sandwich theorem.

In his talk, Frick established the connection between three families of problems. First, mass partition results using hyperplanes. These are problem in which we aim to cut measures or families of point in $\mathbb{R}^d$ into pieces of prescribred size simultaneously with families of hyperplanes. The second are hyperplane transversal theorems. In these problems, we aim to determine if families of convex sets with certain intersection structure can be pierced with a small number of hyperplanes. The third kind of problem are non-embedability problems, in which we are interested in determining if there exist equivariant maps between two topological spaces. The connection between the first two kind of problems is done via Gale duality. The connection with the third family is done with a variation of the test map scheme. With this, Frick was able to prove extensions of the ham sandwich theorem and Dolnikov's theorem. Moreover, this framework leads to natural conjectures in hyperplane transversal theory using open problems in mass partitions as a starting point.

4.8 Andrew Newman: Linear embeddings of random complexes

In his talk, Newman described new results about embedabbility of random simplicial complexes into $\mathbb{R}^d$, improving earlier results on planarity of Ếrdos–Renyi graphs. The simplicial complex model that Newman studied is when we are given a vector $(\alpha_1, \dots, \alpha_d)$ of numbers in $[0, 1]$, and we construct our simplex with n points inductively. First, each possible edge is included with probability α_1, independently. Then, if the $(i-1)$-dimensional skeleton has been constructed, for each set of i vertices that such that all possible $(i-1)$-dimensional faces spanned by them are included in the complex, we include their i-dimensional face with probability α_i.

Newman gave tight conditions on such vectors of probabilities that determine whether the induced simplicial complex is linearly embeddable in $\mathbb{R}^d$ or not (both with high probability).

4.9 Oleg Musin: Quantitative Sperner type lemmas

Sperner's lemma is a discrete analog of the Brouwer fixed point theorem. This lemma states that Every Sperner $(n + 1)$–coloring of a triangulation T of an n–dimensional simplex Δ_n contains an n–simplex in T colored with a complete set of colors, (a simplex of the triangulation whose vertices all have different colors) [25]. Sperner's lemma is a key result in combinatorial topology and there are several interesting generalizations of it. It is widely used in proving the existence of mass partition results, and the existence of envy-free partitions. Musin provided new results that describe under which conditions we can guarantee several fully colored simplices. The key parameter for Musin's lower bound is the degree of an associated piecewise linear map between high-dimensional spheres. Musin extended his results to Sperner coloring of polytopes and to framed cobordisms.

4.10 Edgardo Roldán-Pensado: Transversals to colored sets

Roldán-Pensado proved several results regarding the existence of transversals to colorful families of convex sets. In particular, he showed how, in collaboration with Martínez-Sandoval, one can improve earlier results related to a conjecture of Jerónimo–Castro, Magazinov, and Soberón regarding piercing of translates of a convex body in the plane. They showed that for n families of translates of a convex body of constant width in the plane, if every two translates of different families intersect, then there are $n-2$ families whose union can be pierced with four points (the general conjecture being three points). The tools involved an earlier dichotomy between piercing numbers of families and the possibility of piercing a family with only two lines, which involves the use of the KKM theorem as its topological backbone.

4.11 János Pach: Geometric Applications of the Linear Algebra Method

Pach discussed an extension of a math competition problem. In the setup, we are given n grasshoppers in the plane, and at any moment we allow any grasshopper to jump over another, landing at the same distance to the jumped grasshopper as it was at the beginning. The goal is to see if it is possible to end up with a scaled isometric copy of the original configuration. For four grasshoppers forming a square, this is a nice math olympiad style question. For other configurations, Pach showed (in collaboration with Tardos) how one can reduce the problem to the study of certain transformations of matrices, ultimately solving the problem in full generality.

4.12 Marton Naszodi: Quantitative Steinitz theorems

The classical result of E.Steinitz say that If the origin is at [26]. the convex hull of a set $S \subset \mathbb{R}^d$. Then there are at most $2d$ points of S whose convex hull contains the origin in the interior. The first quantitative version of this result was obtained in [5], where the following statement was proven.

Theorem (Quantitative Steinitz theorem). There exists a constant $r = r(d) > 0$ such that for any subset Q of $\mathbb{R}^d$ whose convex hull contains the Euclidean unit ball B^d,there exists a subset F of Q of size at most $2d$ whose convex hull contains the ball rB^d, for $r(d) > d^{-2d}$.

The topic of Naszodi's talk was about several quantitative variations of Helly's theorem. In one problem, the goal is to guarantee that, given a finite family of convex sets in $\mathbb{R}^d$ such that the intersection of every $2d$ or fewer contains a ball of radius 1, then the intersection of the entire family contains a ball of some positive radius $r(d)$. Naszodi presented the first polynomial lower bound on $r(d)$, which is $r(d) = O(1/d^2)$. The method to prove this builds on earlier ideas of Almendra et al, concerning quantitative Helly-type theorems for containing homotetic copies of a convex set, and duality argument.

The second kind of results were fractional and colorful results for Helly-theorems for the volume. The key novelty in these results is that the number of color classes, or subfamilies one has to check in Naszodi's result is linear in the dimension, as opposed to quadratic in earlier results. To achieve this, Naszodi and Jung introduced a family of filtered simplicial complexes related to intersection volume of families of convex sets. This allowed the reduction of the problem to combinatorial statements regarding filtrations of simplicial complexes.

4.13 Martin Tancer: Weak saturation of multipartite hypergraphs

M. Tancer gave a talk on weak saturation of multipartite graphs (joint work with D. Bulavka and M. Tyomkyn). Given q-uniform hypergraphs (q-graphs) F, G and H, where G is a spanning subgraph of F, G is called weakly H-saturated in F if the edges in $E(F) \setminus E(G)$ admit an ordering $e_1, \dots, e_k$ so that for all $i \in [k]$ the hypergraph $G \cup \{e_1, \dots, e_i\}$ contains an isomorphic copy of H which in turn contains the edge e_i. The weak saturation number of H in F is the smallest size of an H-weakly saturated subgraph of F. The weak saturation number is known in some special cases, for example when H and F are complete q-uniform hypergraphs.

During the talk, M. Tancer sketched how to obtain the weak saturation number of multipartite hypergraphs in so-called directed setting. The proof of this result combines tools of combinatorial, topological and algebraic nature.

In particular, one of the key steps is to find a suitable boundary operator which can be found with an aid of exterior algebra. (This ideas of this type have been pioneered by Kalai but we also need new ingredients.)

4.14 Antonio de Jesús Torres: From word representable graphs to Tverberg type results

As we mention before, Tverberg's theorem describes guarantees the existence of partitions of sufficiently large sets of points in $\mathbb{R}^d$ into r sets whose convex hulls all intersect. In other words, the nerve complex of said convex hulls is an $(r-1)$-dimensional simplex. A natural question is whether we can guarantee that other simplicial complexes can always be obtained this way (see [9]). In his talk, Torres showed positive results from a collaboration with De Loera, Hogan, Oliveros, and Yang for several families of 1-skeletons of simplicial complexes .

All the geometric results that were presented in this talk are motivated from Ramsey theory (see [11]) where one studies how every sufficiently large system must contain a large well-organized subsystem. Here "sufficiently large" is governed by Ramsey numbers. In geometry a classical example of a Ramsey-type theorem is the Erdős-Szekeres theorem that says for every sufficiently large point set in the plane must contain a sub-configuration forming a convex k-gon (in this case where the constant is hard to compute, (see [27] and references there).

In his talk, Torres observe an engaging connection between general k-representable graphs and Ramsey-Tverberg–type results, where nerve structures are shown to arise once we have sufficiently many points. By finding an interesting relationship between this problem and one of the most recent findings in the area of graph theory, the notion of word-representable graphs, which is a common generalization of several well-studied classes of graphs such as circle graphs, comparability graphs, 3-colorable graphs and graphs of degree at most 3 (also known as subcubic graphs). These include triangle-free graphs, circle graphs, bipartite graphs, and outerplanar graphs.

This approach to Tverberg-type problems opens a completely new line of research, which was very welcome in this meeting.

4.15 Efren Morales Amaya: On Baker-Larman Conjecture relative to a convex body with central symmetric sections

Morales Amaya presented new results related to characterizations of ellipsoids by properties of its sections. In the talk, two possible conditions on the hyperplane sections of a convex body were discussed. The first is that each section is centrally symmetric. The second is that each section is a body of constant width. Morales Amaya proved that the first condition implies that the body is an ellipsoid, and the second implies that the body is a ball. These represent significant improvement over earlier similar characterizations of ellipsoids (for example, with conditions such as every section being an ellipsoid). The proofs relied on geometric arguments related to the projection of the center of the original body to each section.

4.16 Boris Bukh: Enumeration of interval graphs and d-representable complexes.

Studying intersection patterns of convex sets was a core topic of this meeting. Given a family of convex sets, their nerve complex is one of the most commonly used tools to understand the intersection of its sets. A simplicial complex is called d-representable if it is the nerve of a tuple of convex sets in $\mathbb{R}^d$. Such a tuple is called a d-representation. One of the closest combinatorial analog of d-representability is d-collapsability. A free face in a simplicial complex is a face that is contained in a unique facet. A d-collapse is the operation of deleting a free face with dimension $d-1$ or less (and all faces that contain it). Finally, a simplicial complex Δ is called d-collapsible if there is a sequence of d-collapses from Δ to the empty complex. Bukh and Jeffs sudied the representability of complexes as the nerve complex of a given family. More precisely, how many simplicial complexes with n vertices are the nerve complex of a family of convex sets in $\mathbb{R}^d$? During his talk Bukh showed upper and lower bounds for this number for general dimension, with more precise bounds on dimension 1. In particular, the results of this talk show that the number of d-collapsible complexes is much larger than the number of d-representable complexes.

4.17 Pablo Soberón: The central transversal theorem revisited

The central transversal theorem is a classic result proven independently by Živaljević and Vrećica, and by Dolnikov. It guarantees the existence of affine k-dimensional spaces of $\mathbb{R}^d$ that are "very deep" within any $d-k+1$ measures. The known proofs often relies on advanced topological tools. Soberón presented a relatively simple proof of this result using a Borsuk–Ulam type theorem for Stiefel manifolds, which in turn can be proven using only homotopy arguments and parity counting of orbits of zeros of certain equivariant maps.

The main tool is a recent theorem of Asada et al. that claim that every $(\mathbb{Z}_2)^k$-equivariant continuous map $f : V_k(\mathbb{R}^d) \to \mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^{d-k}$ has at least one zero. The methods presented also imply new generalizations of Yao–Yao partitions in discrete geometry. These are partitions of one measure into convex pieces of the same size such that no hyperplane intersects all of them.

One example of a new result that can be proven with this method is the following.

Theorem: Let μ_1, μ_2 be two probability measures in $\mathbb{R}^3$, each absolutely continuous with respect to the Lebesgue measure. Then, there exist two planes H_1, H_2, such that they split μ_1 into four parts of size $1/4$, and the line $\ell = H_1 \cap H_2$ satsifies that every half-space that contains ℓ has μ_2-measure greater than or equal to $1/3$.

4.18 Ricardo Strausz: How do 9 points look like in $\mathbb{E}^3$?

Strausz showed a simple proof of a result of Montejano related to transverals to families of convex sets in $\mathbb{R}^3$. The result is that, given two families of convex sets in $\mathbb{R}^3$, each with three sets, if every pair of sets of different families have a non-empty intersection, then one family has a hyperplane transversal. The proof consists on a reduction to the non-embedability of the complete bipartite graph $K_{3,3}$ on S^2. Strausz also established the conections of the proof to separoids.

4.19 Pavel Patak: Shellability is hard even for Balls

Shellability is an important property of simplicial complexes. Intuitively, it means that the complex can be constructed by gluing one facet at a time with some simple properties. In his talk, Pavel Patak showed that the problem of determining whether certain classes of complexes are shellable is NP complete. This includes d-balls (for $d \geq 3$) and 2-complexes embedded in $\mathbb{R}^3$. The same methods also proved that determining whether a 3-complex embedded in $\mathbb{R}^3$ is collapsible is also NP. The proof method consists of a reduction to 3-SAT, a known NP complete problem. Patak showed how, given an instance of 3-SAT, one could construct complexes in $\mathbb{R}^3$ whose shellability or collapsility would imply a solution to the 3-SAT problem.

5 Outcome of the Meeting

This workshop was valuable for the discrete geometry community. The diversity of topics and techniques was a highlight of the meeting. This meant that experts on one kind of results were able to learn about others, while also hearing about (or giving a talk) about the state of the art in their field. We expect several collaborations to start as a consequence of this workshop.

The time allotted between talks and the two problem sessions were used actively to form collaborations or continue ongoing work. The graduate student participants were able to meet and discuss results with senior researchers.

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