

Moduli, Motives and Bundles – New Trends in Algebraic Geometry (22w5187)

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The BIRS–CMO workshop “Moduli, Motives and Bundles – New Trends in Algebraic Geometry” was held at the Casa Matemática Oaxaca (CMO) in Oaxaca from September 18 to September 23, 2022, and was organized by Pedro Luis del Angel Rodriguez (Centro de Investigación en Matemáticas Guanajuato), Frank Neumann (University of Leicester), and Alexander Schmitt (Freie Universität Berlin). It focussed on new developments in algebraic geometry, in particular, in the areas of moduli problems of algebraic bundles and the theory of motives.

1 Overview of the Field

Classification problems are regarded as some of the most important problems in algebraic geometry and, in fact, in all of modern mathematics. Mathematical objects are normally classified with respect to a notion of symmetry, equivalence, or isomorphism. When those objects cannot be classified by discrete data, geometers try to endow the classifying set with a geometric structure as well. The resulting classifying or moduli spaces can then be studied themselves with methods from algebraic geometry, and their geometric properties encode important information about the classification problem. Nowadays, apart from the classical techniques to study these moduli spaces, such as geometric invariant theory, developed during the 19th and 20th century, new techniques are being employed, such as derived categories or motivic homotopy theory. A problem arises when the objects that shall be classified feature too many symmetries and no moduli space can be directly constructed. A modern way to deal with this problem is to use the technology of algebraic stacks. This allows to keep track intrinsically of all the symmetries and gives rise to the notion of a moduli stack, a more abstract, categorified version of a moduli space which is, nevertheless, tractable with methods from modern algebraic geometry. An example of a classification problem which is of fundamental importance to geometry, topology, arithmetic, and physics is the one of vector bundles and principal bundles over a fixed algebraic variety. The associated moduli spaces and stacks turn out to have a very rich geometric structure. The BIRS–CMO workshop “Moduli, Motives and Bundles - New Trends in Algebraic Geometry” offered fresh insights into current key ideas, techniques, and results related to the study of moduli spaces and stacks, including methods from geometric invariant theory (GIT), enumerative geometry, derived categories, derived geometry, and motivic homotopy theory.

2 Recent Developments and Open Problems

Some of the timely topics in the realm of classification problems which featured prominently at the BIRS–CMO workshop are:

- The use of quadratic invariants in the Grothendieck–Witt ring of a variety, when working over a non-algebraically closed field, for use in enumerative algebraic geometry. This is part of a recent larger

program for refining and extending such invariants to motivic cohomology theories on schemes over more general bases.

- Non-abelian Hodge theory, as initiated by Hitchin, Simpson, Donaldson, and Corlette. The fundamental result of the theory is the existence of a correspondence between the Betti and the Dolbeault moduli space of the variety. Simpson observed that the Betti side generalizes also to non-constant coefficients.
- Higher structures on moduli spaces, such as higher tangent spaces - which already appeared before when studying infinitesimal deformation theory - and obstruction theories as well as their derived versions.
- Refined height pairings. These were introduced in the 1980's by Bloch and Beilinson, who extended the Néron–Tate height pairing to cycles over smooth projective varieties on global fields, and further generalized by Rössler and Szamuely to smooth projective varieties over the function field of a smooth variety over an algebraically closed field.
- Applications of model theory to algebraic geometry.
- Techniques from algebraic topology and, in particular, homotopy theory. Though already successfully applied since the second half of the last century to study some aspects of algebraic geometry, mainly over the complex numbers, they have been extended to the general framework of motivic homotopy theory, following the work of Voevodsky and others. They were used, e.g., for a realisation of the theory of motives as first envisioned by Grothendieck and are now being prominently applied to arithmetic geometry.

3 Presentation Highlights

The workshop lectures and talks can be naturally divided into three interconnected groups. The talks of the first group were those of Florent Schaffhauser, H el ene Esnault, Ernesto Mistretta, C esar Lozano, Pedro Montero, Leonardo Roa, and Abel Castorena and were all related to the study of vector bundles and their applications. More precisely, the series of lectures by Florent Schaffhauser was devoted to the study of twisted local systems, twisted character varieties, and twisted Higgs bundles and offered an answer to Simpson's question about the existence of a generalization of the Dolbeault side of non-abelian Hodge theory to non-constant coefficients. In contrast, H el ene Esnault presented new ideas to approach conjectures by Simpson about the geometric nature of rigid local systems. Ernesto Mistretta's talk dealt with applications of vector bundle techniques to birational and biholomorphic characterizations of abelian varieties as well as insights into similar questions for complex tori and compact complex parallelizable manifolds. C esar Lozano, in turn, highlighted applications to the study of birational problems of certain moduli spaces of curves.

The second group includes the talks by Barbara Fantechi, Andrei Yafaev, Leticia Brambila Paz, Juan V azquez, and Montserrat Vite. It consisted of presentations concentrating on the study of moduli spaces in a broad sense. Barbara Fantechi introduced the audience to the circle of ideas behind obstruction theory and higher derived geometry, focussing on the deformation theory of Quot schemes and moduli spaces of coherent sheaves. Andrei Yafaev's talk provided an overview on the celebrated Andr e–Oort conjecture, concerning irreducible components of the Zariski closure of the set of special points of a Shimura variety. Leticia Brambila Paz spoke about moduli spaces of unstable vector bundles with fixed endomorphism rings.

The talks of the third group were delivered by Marc Levine, Bruno Kahn, Donu Arapura, Andrew Kobin, and Tuan Ngo Dac. These talks were mostly concerned with algebraic cycles and arithmetic questions. Marc Levine outlined quadratic enumerative geometry whose goal it is to produce invariants with values in the Grothendieck–Witt ring of non-degenerate quadratic forms over a field k . Bruno Kahn presented a generalization of work by R ossler and Szamuely on height pairings on cycles which, in turn, generalized earlier height pairings constructed by Beilinson. Donu Arapura's talk was about the algebraicity of Hodge and Tate cycles on varieties fibered over lower dimensional varieties. Andrew Kobin discussed categorification and generalizations of zeta and L -functions within a homotopical framework. Last but not least, Tuan Ngo Dac used Langton's semistable reduction method to construct compactifications of moduli stacks of certain *shtukas*, a kind of bundles with extra structures composed of Frobenius twists and Hecke modifications.

The workshop was opened by Marc Levine. In his first lecture, Levine explained quadratic Euler characteristics, using methods from motivic homotopy theory, and provided important examples of calculations in enumerative algebraic geometry. In the next talk, Florent Schaffhauser introduced the concept of a twisted local system which plays the Betti part in the non-abelian Hodge correspondence for twisted coefficients. Afterwards, César Lozano presented recent results on problems from birational algebraic geometry based on higher dimensional Lagrange interpolation. Donu Arapura spoke about new results on Hodge and Tate cycles and gave criteria for the Hodge and Tate conjectures to hold under the assumption that the maps induced by the Leray filtration are surjective in certain regions. Pedro Montero gave a talk about joint work with V. Benedetti, Y. Prieto, and S. Troncoso on the Ulrich property of tangent bundles of projective manifolds. The final talk of the first day was delivered by Leonardo Roa who presented joint work with H. Torres and A.G. Zamora on H -stable vector bundles of rank two with fixed Chern classes and the Segre invariant.

The second day of the workshop began with Barbara Fantechi's first lecture on higher structures on moduli spaces. The lecture was about deformation and obstruction theory with a focus on Hilbert schemes. Florent Schaffhauser continued his series of lectures with a discussion of twisted character varieties. He explored stability in the sense of geometric invariant theory and proved a Riemann–Hilbert correspondence between twisted local systems and twisted torsors with integrable connection. Leticia Brambila Paz presented her recent work with R. Ríos Sierra about moduli spaces of unstable vector bundles having a Harder–Narashiman filtration of length 2 and a fixed endomorphism ring. In the following talk, Juan Vázquez spoke about joint work with C. Reynoso on computations of intersection Betti numbers of the moduli space of plane quartic curves and the moduli space of holomorphic foliations of degree two in the complex plane which were performed with the help of the Kirwan stratification. A. Castorena reported on joint work with H. Torres and with E. Mistretta and H. Torres on the equivalence of linear stability of a linear system and slope-stability of the associated syzygy bundle for general curves and discussed examples of smooth plane curves of degree 7 where this equivalence fails. In the last talk of the second day, Bruno Kahn explained the construction of a new height pairing for algebraic cycles with values in $\text{Pic}(B) \otimes \mathbb{Q}$, on a smooth projective variety X defined over the function field $K(B)$ of a smooth variety B over an arbitrary field. This construction generalizes the height pairings of Rössler and Szamuely and of Beilinson.

The Wednesday session started with the second lecture of the series of Marc Levine. Levine introduced Milnor–Witt K -sheaves and sheaves of Witt groups in order to construct quadratic refinements of the Chow ring and of virtual fundamental classes in Gromov–Witten theory. The session closed with the second lecture of Barbara Fantechi's series. Fantechi worked out the higher obstruction theory of Quot schemes.

Barbara Fantechi opened the Thursday session with her final lecture. She spoke about derived geometry, first concentrating on the affine case for differentiable graded algebras and then illustrating the general machinery by Illusie's fundamental work on simplicial resolutions and cotangent complexes. The second talk on Thursday was Florent Schaffhauser's last lecture. Schaffhauser introduced the concept of Higgs bundles for non-constant structure groups and explained the non-abelian Hodge correspondence between twisted torsors with integrable connections and completely reducible monodromy and polystable twisted Higgs bundles with vanishing Chern classes. Andrei Yafaev spoke then about the André–Oort conjecture and his proof assuming the generalized Riemann hypothesis, as well as about Pila's ideas to use model theory in order to prove some general unconditional instances of the conjecture. Andrew Kobin discussed new homotopical realizations of classical zeta functions as elements of appropriate incidence algebras and gave some ideas on how to extend this construction to motivic zeta and L -functions. Tuan Ngo Dac revisited Langton's semistable reduction method for vector bundles to construct compactifications of moduli stacks of $\text{GL}(r)$ -shtukas with arbitrary modifications. The closing talk of the day, by Montserrat Vite, was a report on her thesis on the geometry of some families of curves in projective 3-space, motivated by liaison theory.

Friday's session started with the third and final talk by Marc Levine who explained some tools for computing quadratic invariants, such as characteristic classes of vector bundles with values in Witt sheaf cohomology and quadratic Euler characteristics of smooth hypersurfaces in projective space. Hélène Esnault talked about joint work with M. Groechenig and with J. de Jong on Simpson's conjecture on the geometric origin of rigid local systems. The closing talk of the workshop was presented by Ernesto Mistretta. Mistretta gave birational and biholomorphic characterizations of abelian varieties, using semi-ampleness of vector bundles. He also indicated how this circle of ideas could be adapted to complex tori and compact complex parallelizable manifolds.

4 Scientific Progress Made

The BIRS–CMO workshop allowed for many open talks and informal discussions among the participants. The stimulating atmosphere at CMO was very useful and instrumental for initiating future research and new collaborations. In particular, the workshop helped launching and supporting joint research initiatives with the Mexican research community. Several Mexican postdoctoral researchers and final year PhD students contributed talks to the workshop and could, in this way, present their work to the world wide research community in algebraic geometry. Important scientific progress was presented at the workshop, too. Some collaborations were initiated or extended as a direct result of the workshop, for example, between A. Castorena and F. Neumann [1, 2] and between P.L. del Angel Rodriguez and F. Neumann [5]. The organizers are planning to edit and publish proceedings of the workshop in the prestigious LMS Lecture Note Series of the London Mathematical Society (LMS) published by Cambridge University Press [4]. This proceedings volume will feature lecture notes from the series of talks by Fantechi, Levine, and Schaffhauser as well as invited contributions from speakers and various other participants. Some articles for the proceedings are already in preparation ([3], [6], [7], [8],[9], [10], [11]). These articles will provide overviews and introductions to the various themes of the workshop as well as feature novel research contributions.

5 Outcome of the Meeting

The workshop consisted of many highly relevant talks and scientific discussions and was very successful. The participants from different research areas of algebraic geometry found many important points of common interest. Conditions provided by CMO Oaxaca were excellent and created a stimulating research environment. The workshop also helped to support young researchers and postdoctoral students from Mexico by connecting them to the wider international research community and inspiring them to expand their own work to new research areas. The meeting was granted additional funding by the Clay Mathematics Institute (CMI). This is an indication of the relevance and importance of the research themes that were discussed. As already mentioned, proceedings of the workshop are in preparation for publication with the LMS and Cambridge University Press.

References

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