# Topological Complexity and Motion Planning (22w5182)

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## 1 Overview of the Field

Topological complexity is a numerical homotopy invariant of a space introduced by Farber [4] with the aim to providing a topological measure of the complexity of the motion planning problem in robotics. This is a recently created area of research located within the burgeoning field of applied algebraic topology. With an originally applied motivation but, at the same time, strong connections to classical subjects in algebraic topology such as the Lusternik-Schnirelmann category, this field has rapidly evolved with contributions from many theoretical topologists. More recently, applied flavored results have begun to re-emerge, and nowadays the subject is a vibrant, highly active, and fruitful ground for scientific activity, ready to face the current needs of our technological society.

## 2 Brief description of the workshop

This workshop was realized in an hybrid format, and as a the sequel to the online workshop 20w5194 "Topological Complexity and Motion Planning" which was held on September 17-20, 2020.

Due to the still existing pandemic situation and the corresponding restrictions of CMO capacity for in-person participants, we had 15 in-person participants and 71 confirmed online participants coming from very different time zones.

Workshop activities included 12 talks, a poster session, and extensive periods of time for discussion. In order to allow as much interaction as possible between all the participants, we included long intervals of discussion between the talks. We also scheduled the lunch time with the aim to make the earlier part of the afternoon activities more accessible to the online participants from far removed time zones.

## 3 Presentation Highlights and Scientific Progress Made

### 3.1 First day

The inaugural talk of the workshop, "Sequential Parametrized Motion Planning and its Complexity", was an online talk delivered by M. Farber. The notion of "parametrized topological complexity" of a fibration was introduced in 2020 by D. Cohen, M. Farber, and S. Weinberger as a variant of the original concept of

topological complexity taking the external conditions (such as positions of the obstacles) as parameters of the problem. Since its introduction and its presentation during the online workshop 20w5194, several works have been made on this new concept.

In this talk, Farber surveyed the various results obtained in collaboration with Cohen and Weinberger [2], [3] and presented in more details his recent results on the sequential parametrized topological complexity, coming from the joint work [7] with A. Kumar Paul. The adjective "sequential" (or "higher") refers to motion planning algorithms for robots/mechanical systems which are required to visit a prescribed sequence of states in a certain order. The authors of [7] have in particular obtained results on the sequential parametrized topological complexity of obstacle-avoiding collision-free motion of many robots in Euclidian space, which generalize some results of [2] and [3]. They also defined the TC-generating function of a fibration and propose some interesting general questions about the analytic properties of such a function.

In [2], the notion of parametrized topological complexity is defined for fibrations, with the Fadell-Neuwirth fibration as the main example. As maps which are not fibrations appears naturally in many problems, it is relevant to try to extend the notion for more general maps. In his talk "Formal aspects of parametrized topological complexity and its pointed version", which was the last talk of the first day, José Manuel García Calcines discussed and solved this question. Indeed making use of the notion of fibrewise sectional category, he showed how to extend the parametrized TC to fibrewise spaces which are not necessarily Hurewicz fibrations and presented the properties of this extended concept.

When considering the collision-free motion of several particles/robots on a network of wires, one is led to study configuration spaces of distinct points on a graph. At the time of our application for a CMO workshop, relatively little was known about the topological complexities of these spaces and one goal of the predecessor online workshop 20w5194 was to stimulate the investigation on this topic and the development of new techniques. Since then a great deal of progress have been made, and some results in this direction were reported on in the two remaining talks of the first day.

First, in her talk "Topological complexity of unordered configuration spaces of trees", Teresa Hoekstra Mendoza explained how to use discrete Morse theory to give bounds for the higher (sequential) topological complexity of configuration spaces of graphs without cycles. This new technique permits her to recover and generalize some results which were known for trees.

Second, Ben Knudsen, in his talk "Around Farber's conjecture" explained how Farber's argument to prove that the topological complexity of the configuration space $\mathrm{Conf}(\Gamma, k)$ of $k$ ordered points on a tree $\Gamma$ is 2 times the number $m(\Gamma)$ of essential vertices (at least for $k$ sufficiently large) can be adapted to the case of a planar graph but not to the case of a nonplanar case. In particular new ideas would be necessary to prove Farber's conjecture asserting that, as soon as $k$ is suficiently large, $\mathrm{TC}(\mathrm{Conf}(\Gamma, k)) = 2m(\Gamma)$ for any graph $\Gamma$. As an highly satisfactory fact we mention that, during the workshop, B. Knudsen developed such new ideas and shortly after the workshop was able to present a proof of Farber's conjecture in [8].

## 3.2 Second day

The second day opened with an online talk, "Relative and equivariant cohomological dimensions" by Mark Grant. The importance of the notion of cohomological dimension in the field comes from the fundamental work of Eilenberg and Ganea [21] asserting that the Lusternik-Schnirelmann category of the classifying space $BG$ of a group $G$ agrees with the cohomological dimension of $G$. A compelling problem, put forward by Farber, is to characterize the topological complexity of $BG$, denoted by $\mathrm{TC}(G)$, solely in terms of the algebraic properties of the group $G$. Much progress has been made in the past few years. In particular, a new approach using Bredon cohomology and the associated cohomological dimension was developed by Farber, Grant, Lupton and Oprea in [6]. More recently, the use of Adamson cohomology (presented on the third day in the poster of A. Espinosa, see below) motivates the investigation on the relationship between $\mathrm{TC}(G)$ and the Adamson cohomological dimension of $G \times G$ relative to the diagonal subgroup. In his talk, M. Grant discussed the several possible definitions of relative cohomological dimension associated to the notions of relative cohomology respectively due to Takasu, Adamson and Bredon giving comparison results as well as

interesting open questions. This also leads to three possible versions of equivariant cohomological dimension whose relationships between them and equivariant LS-category are discussed.

The following two talks reported on new developments on symmetric forms of topological complexity. Considering motion planners for which the path from $B$ to $A$ is the reverse of the path from $A$ to $B$ has motivated the introduction of the symmetric topological complexity, $\mathrm{TC}^S$, in [5] and of the symmetrized topological complexity, $\mathrm{TC}^\Sigma$, in [1].

In his talk "Bidirectional sequential motion planning", Enrique Torres-Giese defined a new invariant $\mathrm{TC}_n^\beta$ which is a sequential version of $\mathrm{TC}^\Sigma$ corresponding to a natural condition of symmetry imposed to the motion planning problem and which appears to be a lower bound of the invariant $\mathrm{TC}_n^\Sigma$ defined in [1]. Properties and examples of this new invariant, called bidirectional sequential TC are discussed. This also open a new perspective to study $\mathrm{TC}_n^\Sigma$ as the final term of a sequence of invariants starting with the ordinary and bidirectional sequential TC.

Next, Marzieh Bayeh, in her online talk "Motion planning of symmetric navigator robots" considers the situation where there is a symmetry on the mechanical system or its configuration space given by a group action. She reviewed the different equivariant approach to the topological complexity which have been developed by several authors and proposed a new approach using the concept of transversal, that is a closed subspace representing the orbit space inside the configuration space. This leads to a new invariant, which is suitable for measuring the complexity of motion planning algorithms for symmetric navigator/walker robots and which is a lower bound for the different equivariant versions of TC.

Finally, the last talk of the day, "Borsuk-Ulam property via sectional category" was presented by Cesar Ipanaque on a joint work with Daciberg Gonçalves. For a topological space $X$, a free involution $\tau$ on $X$ and a Hausdorff space $Y$, the triple $(X, \tau, Y)$ satisfies the Borsuk-Ulam property if for any map $f : X \to Y$ there exists $x \in X$ such that $f(x) = f(\tau x)$. In this work, the authors characterized this property in terms of the sectional category of related maps involving in particular the configuration space $\mathrm{Conf}(Y, 2)$ of 2 distinct points in $Y$. As an application, they obtained a new lower bound for the index in terms of sectional category.

## 3.3 Third day

The third day was dedicated to a poster session. Posters submitted by participants had been made available in a pdf format to all participants since the very beginning of the workshop. The poster session consequently consisted of a short oral presentation of each poster, which was projected on the main screen, and followed by a discussion with the other participants. The three first posters were presented by online participants while the two last posters were presented by in-person participants. All the presenters were very young researchers, PhD students or post-docs.

The first poster, "On the sectional category of subgroups inclusions and Adamson cohomology theory", was presented by Arturo Espinosa. In this work, in collaboration with Z. Blaszczyk and J. Carrasquel, the authors extend to the context of sectional category of subgroup inclusions the characterization of TC of an aspherical space given in [6] as well as the notion of Berstein class. They also develop an approach using Adamson cohomology, which introduces the notion of Adamson cohomological dimension in the discussion of the topological complexity of aspherical spaces.

In the second poster, David Mosquera Lois reported on his joint work with E. Macías-Virgós and M. J. Pereira-Sáez on the "Homotopic distance and Generalized motion planning". The notion of homotopic distance $D(f, g)$ between two maps has been defined in [9] and recovers both the Lusternik-Schnirelmann category and the topological complexity as special cases. In this work, given a manifold $M$ equipped with a Morse-Bott function $\Phi$ and two maps $f$ and $g$ defined on $M$, the authors give a relationship between the homotopic distance $D(f, g)$ and the homotopic distances between the restriction of $f$ and $g$ to the different critical sets of $\Phi$. This generalizes some similar results for the category and the topological complexity.

José Luis León Medina presented the third poster, "(Higher) Topological Complexity of non-$k$-Equal Spaces", on a work in collaboration with J. González. The notion of non-$k$-equal space is a natural generalization of the concept of configuration space of distinct ordered points in the $d$-dimensional euclidian space. In this work, the authors obtained results on the LS-category and (higher) topological complexity of these spaces by making use, in particular, of a combinatorial description of their cohomology in terms of "string pre-order" when $d = 1$ and "$k$ forests" when $d > 1$.

The fourth poster, "An algorithmic discrete gradient field", was presented by Emilio González Rivero on a joint work with J. González. In this work the authors develop an algorithm which constructs a discrete gradient field on a finite simplicial complex given with an ordered vertex set and study its properties. The algorithm is applied to the Munkres' discrete model of the configuration space of 2 distinct points on a complete graph with the aim to describe the cohomology ring of this space.

Finally, Said Hamoun presented the last poster "On the rational topological complexity of elliptic coformal space" based on a work in collaboration with Y. Rami and L. Vandembroucq. Using Sullivan models, the authors study the rational topological complexity ($\mathrm{TC}_0$) of a pure elliptic space, that is, a simply-connected space admitting a Sullivan model of a certain form whose rational homotopy and homology are both finite dimensional. The results presented include, in particular, an expression of $\mathrm{TC}_0$ in terms of the rational LS-category and the homotopy characteristic in the formal case and a new lower bound in the coformal case (that is, when the differential of the model is quadratic).

### 3.4 Fourth day

The fourth day opened with the talk "Free cyclic actions on surfaces and the Borsuk-Ulam theorem", delivered by Daciberg Gonçalves, work in collaboration with John Guaschi (Normandie Univ, UNICAEN, CNRS, LMNO, France) and Vinicius Casteluber Laass (Federal University of Bahia, Brazil). The talk reported on results concerning generalized versions of the Borsuk-Ulam property for maps from spheres with antipodal actions to Euclidean spaces. The lecture also included a very informative survey of the results obtained when spheres and Eucliden spaces are replaced by closed surfaces, stressing the role of configuration spaces and braid groups in their approach. In particular the authors use such technology in order to generalize the Borsuk-Ulam property for more general actions. Thus, in their main main result, a classification is obtained for quadruples $(M, \mathbb{Z}_n, \tau, \mathbb{R}^2)$ satisfying the Borsuk-Ulam property for maps $M \to \mathbb{R}^2$. Here $M$ is a compact surface without boundary, and the action $\tau\colon \mathbb{Z}_n \times M \to M$ is free.

In the second talk of the day "Homotopic invariants for small categories", Enrique Macías-Virgós introduced the notion of homotopic distance in a purely categorical context. He additionally reported on a Varadarajan-type theorem for homotopic distance of Grothendieck bi-fibrations between small categories.

The third talk of the day "On the zero-divisor cup-length of real Grassmann manifolds" was given by Marco Radovanovic, who reported on his results on the zero-divisor cup-length, and its higher analogs, for real Grassmannians and some related manifolds.

The final talk of the day "Fibrewise TC of a map" was delivered by Petar Pavesic, who addressed the fibrewise topological complexity of maps, and examined several variants of this concept with respect to various definitions of topological complexity of a map. The lecture included detailed descriptions on how these ideas can be used to model problems in robot manipulation planning.

## 4 Outcome of the Meeting

The workshop was highly formative, with discussions of state-of-the-art problems in the subject, and with the participation of many of the leading experts in the area and many young researchers. Important aspects of the topological complexity theory were discussed and new perspectives identified.

As already mentioned, the schedule was organized in order to provide both in-person and online participants with time and opportunities to interact. Of course, the in-person participants had in addition the possibility to profit from the excellent physical facilities and environment provided by CMO. After the difficult time of lockdown, the importance of having the ability to discuss mathematics face to face / in front of a blackboard was repeatedly recognized and appreciated.

As reported in the participant testimonials and through other means, the activities of both BIRS-CMO workshops 20w5194 and 22w5182 had a very positive impact on the TC-community and especially on young researchers. The activities of the first online workshop led to a number of new collaborations and new works. And now the workshop 22w5182 has shown to have a similar impact. As a first example and as already mentioned, the paper [8] was produced directly after the present workshop. The acknowledgements of this paper read (in part) as follows:

> **Acknowledgements.** The seeds of this paper were sown when Andrea Bianchi noticed an early error in [Knu21], prompting the author to reconsider the role of cohomology in the non-planar setting. The ideas came during the 2022 workshop "Topological Complexity and Motion Planning," held at CMO BIRS in Oaxaca, and the author thanks Dan Cohen, Jesús González, and Lucile Vandembroucq for their skillful organization. The author benefited from fruit-full conversations with Daciberg Gonçalves and Teresa Hoekstra Mendoza, and he thanks Andrea Bianchi, Jesús González, and the anonymous referee for comments on an earlier draft.

We also note that a forthcoming collaboration between Jesús González and Daciberg Goncalves has recently emerged in context of the Borsuk-Ulam property for maps between graphs.

It is also worth mentioning that after the workshop 20w5194 and in response to the extensive interest in the subject demonstrated at this occasion, two of the organizers (DC and JG), in collaboration with J. Oprea (Cleveland State University), launched an ongoing online *Topological Complexity Seminar*, supported by the Applied Algebraic Topology Research Network (AATRN)

`https://sites.google.com/view/aatrn-tc-seminar/home`

This has become a very active monthly meeting. In fact, the combination of the monthly meeting and of the two TC workshops supported by BIRS-CMO has already had and will continue having a strong impact on our community. As an illustration of this impact, we include a portion of an email sent to the organizers by a virtual participant following our most recent CMO workshop:

> Dear Lucile, dear Dan, dear Jesus,
>
> I just want to thank you for organising the workshops at CMO Oaxaca in 2020 and 2022. I really benefited from this opportunity to at least virtually meet many people from the TC community and I very much appreciate the AATRN-TC seminar which would probably not exist without the workshop that you organized...

Finally, we note that of the 17 communications which have been presented (considering talks and posters), 3 have been delivered by very young mexican researchers (post-doc and phD-students based in Mexico city). We hope that the impact on these young scientists will be an important positive consequence of the activities of the Topological Complexity and Motion Planning workshop 22w5182.

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