Moduli and Invariants

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1 Introduction and overview

This workshop focused on the newest of results in the theory of moduli spaces and applications to enumerative invariants in algebraic geometry. The workshop showed new advances in computations of Gromov-Witten, Donaldson-Thomas, quasi-maps invariants through an array of different and new techniques and perspectives, form classical localization, cosection techniques as well as incoming techniques from derived deformation theory.

The workshop brought several experts from different groups working in different enumerative invariants to interact and share new insights, techniques and ideas, in hopes to foster new collaborations. The topics covered the following.

1.1 Gromov-Witten and Donaldson-Thomas invariants.

Gromov-Witten theory was motivated by the work of physicists, specially string theorists, who were interested in counting curves embedded in a fixed target variety, such as the three complex dimensional manifolds known as the Calabi-Yau threefolds. In order to systematically study the so-called curve counting problem, one needs to study the curves with certain fixed numerical data (such as their genus or the homology class) in a family, i.e in the moduli spaces of algebraic curves which are embedded in the ambient variety. The moduli spaces of curves have frequently appeared in many areas of mathematics, such as algebraic geometry, differential geometry and low dimensional topology and during the past 30 years their properties have been intensively studied from numerous points of view. These spaces are non-compact and admit important compactifcations. In 1992, Kontsevich [26] provided the moduli space of stable maps as one example of such compactifications. The stable maps are defined as maps from the domain curves to the ambient target variety such that they satisfy certain stability conditions. This breakthrough, later enabled mathematicians to define the Gromov-Witten invariants as intersection numbers associated to the moduli space of stable maps. Donaldson-Thomas theory was the result of rather another compactification of the moduli space of curves, namely the Hilbert scheme parameterizing them as one dimensional subschemes. Such invariants were first defined by Richard-Thomas [44] for stable sheaves (representing the subschemes) on Calabi-Yau threefolds and played a central role in the Gromov-Witten/Donaldson-Thomas correspondence formulated by Maulik, Nekrasov, Okounkov and Pandharipande [32, 33]. Indeed, the Donaldson-Thomas invariants can more generally be defined as virtual enumerative invariants for Bridgeland stable objects in the derived categories of Calabi-Yau threefolds. During the recent years however, the machinery of Donaldson-Thomas theory has been further developed, which allows one to define them for more general geometries, such as some cases where the base variety is given as a non-Calabi-Yau threefold or when it is given as an algebraic surface. One of the objectives of the workshop was to discuss new developments in Donaldson-Thomas theory for general geometries, such as 4-folds.

Further key developments in the field are the stable pairs theory of Pandharipande and Thomas [40], which is the theory of studying subschemes with, roughly speaking, certain controlled deformations in the ambient variety, the motivic Donaldson-Thomas theory of Kontsevich and Soibelman [27] which studies the components of the moduli space of such subschemes as elements in certain motivic rings, as well as the theory of generalized Donaldson-Thomas invariants of Joyce and Song [23] which also takes a motivic approach to such moduli spaces. In particular the Kontsevich-Soibleman and Joyce-Song theories enable one to derive a motivic wall-crossing formula for such invariants which has been the focus of very intense activities in the past years. We discussed with some of the participants new applications of motivic wall-crossing.

From a physics point of view, Donaldson-Thomas invariants are supersymmetric indices counting D-brane bound states in string compactifications. Their physical interpretation has been the origin of several important results in the field including the topological vertex [1] and its refined generalization [22]. More recently, a mathematical construction of the algebra of BPS states of Harvey and Moore was developed by Kontsevich and Soibelman in [28]. This construction yields new connections between enumerative invariants and representation theory, which are yet to be fully understood. The interaction between physics and geometry in this context is expected to produce further exciting results. We discussed the connection between enumerative invariants and representation theory.

1.2 Donaldson-Thomas and Gromov-Witten theories and dualities in physics.

The existence of certain dualities which have roots in physics, such as the strong weak electric-magnetic duality have been the source to spectacular predictions about enumerative geometry of moduli spaces of sheaves and curves on fixed target varieties. As an example one can mention the extensive research activity carried out during the past years to prove the modularity properties of Gromov-Witten or Donaldson-Thomas invariants. Some of the most recent developments in this particular direction include; Pandharipande-Thomas proof of KKV conjecture for K3 surfaces [41], Maulik-Pandharipande proof of modularity of Gromov-Witten invariants for K3 fibered threefolds [35], Gholampour-Sheshmani proof of modularity of Donaldson-Thomas invariants of stable sheaves with two dimensional support on similar K3 fibrations [20] (as well as the generalization of the latter to the case of strictly semi-stable sheaves [2]) the study of modular property of count of curves on surfaces deforming freely inside ambient Calabi-Yau threefolds [13], and finally numerous exciting results on modularity of Gromov-Witten invariants on threefolds given by elliptic fibrations [38], [3] and [21]. We Discussed the latest advances on modularity of Gromov-Witten invariants.

1.3 Equivariant Methods and Gromov-Witten gauged theory

Given a smooth projective variety X equipped with an action of a reductive group G (or a Hamiltonian symplectic manifold) one can associate several equivariant extensions to the usual GW invariants of X. Using symmetries to aid the computation of the invariants has proven to be an important tool, Kontsevich introduced localization to compute GW invariants in special cases [25, 30]. Givental defined [15] equivariant GW invariants by using equivariant integration of equivariant cohomology classes over the space of stable maps, which were used to prove his mirror symmetry theorem [14]. Recently, researchers have focused in a natural generalization of GW invariants, which is given by taking integration of equivariant cohomology classes over larger moduli spaces, such as the moduli of quasi-maps [5], the moduli of gauged maps [11, 24] or the moduli of symplectic vortices in the symplectic category [6, 36, 37]. In the last few years, important progress has been made towards the understanding of the relation of these gauged versions of GW invariants to the GW invariants of the GIT quotient $X/\!\!/G$, via adiabatic limits and wall-crossing [47]. This relation gives a systematic understanding of the (stacky) quantum cohomology of the quotients, as the quotient varies. As a result one obtains a novel interpretation of the original mirror symmetry transformations proposed by Givental, as well as a version of the crepant conjectures [7, 8]. One expects that these results could generalise to other theories such as FJRW [10], or the theory with p-fields [29]. Recent progress in the field [43, 42, 11, 19, 18] expects that the theory to be extended for a full degeneration of the curve in any genus. There were discussions related to K-theoretic invariants in quasi-map theory for GIT quotients $X/\!\!/G$. There were also discussions on the LG/CY correspondence.

Equivariant methods not only appear as a computational tool, the theory has made contact into several affine areas. Gauged Floer theory has been used to detect displaceability of Lagrangians [46]; enumeration of certain equivariant holomorphic maps in toric varieties has given a connection [4, 16] between closed GW invariants and open invariants in the sense of disc counting [12]. Equivariant tools have been central in recent developments in the relation of GW theory and quantum groups [31] as well as in current work on Donaldson-Thomas theory, where the relation of DT invariants and GW invariants has been the subject of seminal work [32, 33]. Progress understanding the correspondence between vortices and gauged maps [45] gives evidence that gauged GW invariants can aid vortex counting problems, which in turn relates to knot theory [9]. The moduli spaces of vortices has been the focus of physical interest as a starting point to incorporate non-perturbative effects into correlators of supersymmetric QFTs. The correspondence of vortices with gauged maps is expected to facilitate computations which we hope will be of interest in physics. We also discussed some relations between gauged invariants and physics.

1.4 Geometric Langlands

The Geometric Langlands Conjecture for a curve C and a reductive group G is one of the most important conjectures in mathematics. This can be understood as a non-abelian generalization of the relation between a curve and its Jacobian. It claims the existence of Hecke eigensheaves on the moduli of G-bundles on C.

One of the newest advances in this theory is the so called *parabolic Geometric Langlands Conjecture* an extension of the usual GLC to curves with punctures. There were discussions on how this can be proved by using non-abelian Hodge theory.

2 Recent Developments and Open Problems

Some of the talks in Donaldson-Thomas theory gave us insights on the novel developments for a definition of Donaldson-Thomas invariants for 4-folds. There are several approaches leading to construction of a virtual fundamental class for moduli spaces of sheaves on such varieties, and it is still open to show some properties of invariants induced by such virtual cycles.

In the context of geometric Langlands conjecture, Ron Donagi gave a description on a generalization to the parabolic case, and how Wobbly bundles play an important role. These conjectures are open for general cases and there are many open problems, such as extending the parabolic GLC for higher genus.

We also saw several enhancements of cohomological invariants to K-theoretical invariants, these include K-theoretical GW invariants and also DT and PT invariants. There was a conjecture on a DT/PT type correspondence.

3 Presentation Highlights

The workshop consisted of twenty two lectures on advances in current research.

Deformation quantization of coherent sheaves and their morphisms. Vladimir Baranovsky presented some examples on deformation quantization of sheaves and morphisms between them. This is motivated by the following problem. In some geometric situations, the moduli spaces of coherent sheaves can be realized as a degenerate intersection of two shifted Lagrangian subspaces (shifted in the sense of derived symplectic structures) in a shifted symplectic space. Computations using obstruction theory is one way of dealing with degenerate intersections. A potential alternative is to deform the geometric picture in a non-commutative direction, which is the approach used for these examples.

Donaldson-Thomas theory of non-commutative projective schemes Kai Behrend presented joint work with Yu-Hsiang Liu and Atsushi Kanazawa on the study of non-commutative projective varieties in the sense of Artin-Zhang, which are given by non-commutative homogeneous coordinate rings, which are finite over their centre. He described the construction of a special type of moduli spaces of stable modules for these. He described the construction of a symmetric obstruction theory for Calabi-Yau 3-folds. This of course produces deformation invariants of similar to Donaldson-Thomas invariants. He presented as an example the Fermat quintic in the quantum projective space, where the coordinates commute only after carefully choosing 5-th roots of unity.

He provided possible further projects to study, for instance the moduli theory of finite length modules. In this case one can apply techniques that come from both, the Hilbert scheme of commutative 3-folds, and the representation theory of quivers with potential.

Counting sheaves on singular curves and surfaces Amin Gholampour presented work on a construction of a relative perfect obstruction theory. More precisely, given a virtually smooth quasi-projective scheme M, and a morphism from M to a nonsingular quasi-projective variety B, he showed that it is possible to find an affine bundle M'/M that admits a perfect obstruction theory relative to B. Then, he presented how to relate the resulting virtual cycles on the fibers of M'/B to the image of the virtual cycle $[M]^{vir}$ under refined Gysin homomorphisms.

As an application of this technique, he focused in the case when M is a moduli space of stable codimension 1 sheaves on a nonsingular projective surface or a Fano threefold.

Lagrangian correspondence between Hitchin and de Rham moduli space. Olivia Dumitrescu discussed a conjecture established by Gaiotto in 2014, where one predicts the existence of a Lagrangian correspondence between holomorphic Lagrangian of opers in the Dolbeault moduli space of Higgs bundles and the de Rham moduli space of holomorphic connections.

She presented ideas on how this conjecture was solved in 2016 for holomorphic opers in work with Fredrickson, Kydonakis, Mazzeo, Mulase and Neitzke. Then she explained how, using a similar analysis Collier and Wentworth extended the correspondence for more general Lagrangians consisting of stable points.

She presented an entirely algebraic geometric description of the Lagrangian correspondence of Gaiotto, based on the work of Simpson.

Moduli stacks of sheaves on Calabi-Yau four-folds as critical loci. Dennis Borisov described joint work with A. Sheshmani and S-T. Yau. He started by introducing the (-2)-shifted symplectic structures introduced by Pantev, Toen, Vezzosi, Vaquie on the moduli stacks of sheaves on Calabi-Yau 4-folds. Using techniques in differential and derived geometry one can construct Lagrangian foliations relative to these symplectic structures on these stacks. The quotients by the foliations are perfectly obstructed derived stacks. Moreover they are equipped with globally defined (-1)-shifted potentials, whose critical loci are the original moduli stacks.

Stringy Kähler moduli, mutation and monodromy. Will Donovan discussed the derived symmetries associated to a 3-fold admitting an Atiyah flop which may be organized into an action of the fundamental group of a sphere with three punctures, thought of as a stringy Kaehler moduli space. He extend this construction to the case of general flops of irreducible curves on 3-folds in joint work with M. Wemyss. This used certain deformation algebras associated to the curve and its multiples, with applications to Gopakumar-Vafa invariants.

On the Geometric Langlands Conjecture and Non-Abelian Hodge Theory. Ron Donagi gave a great overview of the Geometric Langlands Conjecture (GLC) for a curve C and a group G. He explained that this can be thought as a non-abelian generalization of the relation between a curve and its Jacobian. The conjecture claims the existence of certain Hecke eigen-sheaves on the moduli of G-bundles on C.

A generalization, of this is the parabolic GLC, which is a further extension to curves with punctures. He gave several illuminating examples of these conjectures and he gave an outline for an approach to proving them using non-abelian Hodge theory.

A key geometric ingredient in doing so is the locus of wobbly bundles: bundles that are stable but not very stable. He finally provided instances where this program has been implemented, in particular GLC for $G = GL(2)$ and genus 2 curves which is joint work with T. Pantev and C. Simson, and parabolic GLC for $\mathbf{P}^1$ with marked points jointly with T. Pantev.

Integral transforms and quantum correspondences Mark Shoemaker gave us an overview and novel interpretations of a collection of well-known comparison results in genus-zero Gromov-Witten theory. The main objective was to relate these to integral transforms between derived categories. In turn, he explained that this implies that various comparisons between Gromov-Witten and FJRW theory are compatible with Iritani's integral structure. He sketched a proof of a version of the LG/CY correspondence relating quantum D-modules with Orlov's equivalence using a modified version of the so called crepant transformation conjecture.

K-theoretic generalized Donaldson-Thomas invariants Young-Hoon Kiem described a novel construction regarding obstruction theories. He introduced the notion of an *almost perfect obstruction theory*, which lies in between a semi-perfect obstruction theory and an honest perfect obstruction theory. He showed that an almost perfect obstruction theory allows to construct the virtual structure sheaf and hence K-theoretic virtual invariants.

For instance, for partial desingularizations of the moduli stack of semistable sheaves on Calabi-Yau 3-folds, there are no perfect obstruction theories but only semi-perfect obstruction theories. While a semi-perfect obstruction theory is sufficient for the construction of virtual cycles in Chow groups, it seems insufficient for virtual structure sheaves in K-theory. He gave examples of DM stacks with almost perfect obstruction theories include the Inaba-Lieblich moduli spaces of simple gluable perfect complexes and the partial desingularizations of moduli stacks of semistable sheaves on Calabi-Yau 3-folds. This gives K-theoretic Donaldson-Thomas invariants of derived category objects and K-theoretic generalized Donaldson-Thomas invariants. All of this was joint work with Michail Savvas.

The punctured logarithmic maps Qile Chen presented joint work with Dan Abramovich, Mark Gross and Bernd Siebert on logarithmic Gromov-Witten theory. This is the virtual count of the number of holomorphic curves with prescribed tangency condition along boundary divisors. He introduced a variant of logarithmic maps called the punctured logarithmic maps. They naturally appear in a generalization of the gluing formulas of Li-Ruan and Jun Li in usual GW theory. The punctured invariants play the role of relative invariants in these classical gluing formulas. They extend logarithmic Gromov-Witten theory by allowing negative tangency conditions with boundary divisors.

K3 surfaces with symplectic group actions, enumerative geometry, and modular forms Jim Bryan gave a talk on $K3$ surfaces with a symplectic action of a group G. The idea generalizes the classical story of the Hilbert scheme parameterizing n points on a $K3$ surface X, where in the 90s it was discovered that the generating function for the Euler characteristics of the Hilbert schemes is related to both modular forms and the enumerative geometry of rational curves on X. He showed that the Euler characteristics of the G-fixed Hilbert schemes parametrizing G-invariant collections of points on X are related to modular forms of level $|G|$ and the enumerative geometry of rational curves on the stack quotient $[X/G]$. These ideas lead to some beautiful new product formulas for theta functions associated to root lattices. He also explained how this picture also generalizes to refinements of the Euler characteristic such as the χ_y genus and the elliptic genus. This in turn gives connections with Jacobi forms and Siegel modular forms.

Lie Algebra Representations and BPS numbers. Sheldon Katz's talk started with the discussion of how certain virtual correspondences are constructed between smooth moduli spaces of certain stable 1-dimensional sheaves on surfaces, leading to a representation of a Lie algebra on the direct sum of the cohomologies of these moduli spaces. This representation commutes with the $SL_2 \times SL_2$ representation described by Gopakumar and Vafa using M-theory, constructed mathematically via perverse sheaves and hard Lefschetz. In the case of a rational elliptic surface with a type II^* fiber (the E-string of physics), a representation of the affine E_8 Lie algebra is obtained. Since the cohomologies determine BPS numbers of the associated local surface, the generating function of these BPS numbers or their $SL_2 \times SL_2$ refinements is the character of a representation of the affine E_8. This was predicted by Huang, Klemm, and Poretschkin via physics. This is based on joint work with Davesh Maulik.

Stable higher rank flag sheaves on surfaces and Vafa-Witten invariants Artan Sheshmani gave a talk on his joint work with Shing-Tung Yau.

He studies the moduli space of holomorphic triples $f : E_1 \to E_2$, consisting of (possibly rank > 1) torsion-free sheaves (E_1, E_2) and a holomorphic map between them, over a smooth complex projective surface S. The triples are equipped with Schmitt's stability condition. He proved that when this Schmitt stability parameter becomes sufficiently large, the moduli space of triples benefits from having a perfect relative and absolute obstruction theory in some cases (depending on Chern character of E_1). He explained that there is a way to generalize this construction to higher-length flags of higher rank sheaves by gluing moduli spaces of triples, extending his joint work with Gholampur and Yau, where the obstruction theory of nested Hilbert schemes over the surface was studied. He first described how to extend earlier results to the moduli space of flags $E_1 \to E_2 \to \cdots \to E_n$, where the maps are injective (by stability). There is wall-crossing in a the master space, developed by Mochizuki, between the theory of such higher rank flags, and the theory of Higgs pairs on the surface, which provides the means to relate the flag invariants to the local DT invariants of any threefold given by a line bundle over the surface, $X := \mathrm{Tot}(L \to S)$. The latter DT invariants, when L is the canonical bundle of S, contribute to Vafa-Witten invariants.

LG/CY correspondence for one-folds via modularity Yefeng Shen discussed joint work with Jie Zhou, and Jun Li, Jie Zhou. He started with the description of Gromov-Witten invariants of Calabi-Yau one-folds (elliptic curves and elliptic orbifold curves) as quasimodular forms, using some tautological relations and ordinary differential equations in the theory of quasimodular forms, with minimal calculations.

He applied the same method to the Fan-Jarvis-Ruan-Witten (FJRW) theory of simple elliptic singularities which he used to prove the LG/CY correspondence for all CY one-folds. He explained that this uses the Cayley transformation of quasimodular forms, where the GW/FJRW invariants appear as coefficients of Fourier/Taylor expansions of the same quasimodular forms.

DT/PT correspondence for Calabi-Yau 4-folds Martijn Kool talked about joint work with Y. Cao and S. Monavari. He started with a presentation of Hilbert schemes and stable pair moduli spaces on compact Calabi-Yau 4-folds. He recalled the construction of the virtual class, as constructed in special cases by Cao-Leung and in general by Borisov-Joyce. He presented a conjectural DT/PT correspondence for the resulting invariants. He also provided a toric Calabi-Yau fourfolds, we conjecture a K-theoretic enhancement of this correspondence using invariants which were recently discovered in the case of Hilbert schemes of points by Nekrasov. Using dimensional reduction, he recovers Nekrasov-Okounkov's K-theoretic DT/PT correspondence for toric threefolds.

Refined invariants of flopping curves Ben Davison discussed certain Jaconbi algebras, a class of algebras arising from a quiver with potential. To such algebras one can associate a collection of invariants, called *BPS invariants*, by lifting constructions from 3-dimensional complex geometry.

A conjecture of Wemyss and Brown states, approximately, that in fact the finite-dimensional Jacobi algebras are in bijective correspondence with local isomorphism classes of flopping curves in 3-dimensional complex varieties. A consequence of the conjecture would be that for such algebras, all of the refined invariants are positive (in marked contrast with general Jacobi algebras). He discussed a recent proof of this positivity result, along with a monodromy/purity conjecture for the cohomologically refined Gopakumar-Vafa invariants of flopping curves.

K-theoretic quasimap wall-crossing for GIT quotients Ming Zhang presented porgress on work with Yang Zhou on quasi-map theory for GIT quotients $X = W//G$ as introduced by Ciocan-Fontanine-Kim-Maulik. Zhang discussed the theory of epsilon-stable quasimaps and wall-crossing formulas of cohomological epsilon-stable quasimap invariants for all targets in all genera, a result that has been recently proved by Yang Zhou. Zhang introduced permutation-equivariant K-theoretic epsilon-stable quasimap invariants with level structure and prove their wall-crossing formulae for all targets in all genera, this has applications in physics, where these invariants are related to the $3d\ \mathcal{N} = 2$ supersymmetric gauge theories studied by Jockers-Mayr, and the wall-crossing formulae can be interpreted as relations between invariants in the UV and the IR phases of the $3d$ gauge theory.

Virtual Cycle on the Moduli Space of Maps to a Complete Intersection. Rachel Webb discussed the classical question in Gromov-Witten theory relating the GW invariants of a complete intersection to the GW invariants of the ambient variety. In genus-zero this can often be done with a *twisted theory,* that is by considering twisted invariants by certain Euler classes. However this fails in higher genus. Several years ago, Chang and Li presented the moduli space of p-fields, as part of the solution to the higher-genus problem. They provided a mechanism of constructing the virtual cycle on the space of maps to the quintic 3-fold as a cosection localized virtual cycle on a larger moduli space (the space of p-fields). Their result is analogous to the classical statement that the Euler class of a vector bundle is the class of the zero locus of a generic section. In her talk, Webb discussed joint work with Qile Chen and Felix Janda where they extend Chang-Li's result to a more general setting, a setting that includes standard Gromov-Witten theory of smooth orbifold targets and that alos works for quasimap theory of GIT targets.

The logarithmic gauged linear sigma model Felix Janda presented joint work with Q. Chen and Y. Ruan. These is important progress on the moduli space of *logarithmic R-maps*. Janda described how to construct virtual cycles and why these moduli and their invariants are the key ingredient toward an approach to proving many *mirror* conjectures involving higher genus Gromov-Witten invariants of quintic threefolds.

Stable pairs with a twist. Dori Bejleri reported work on joint work with G. Inchiostro. about moduli spaces of stable log varieties or stable pairs (X, D). These are the higher dimensional analogues of the compactified moduli of stable pointed curves. The existence of a proper moduli space has been established thanks to the last several decades of advancements in the minimal model program, he gave a small introduction on how this works. He explainded how the notion of a family of stable pairs remains quite subtle, and in particular that the deformation-obstruction theory for these moduli is not known. He discussed the case when the boundary divisor D is empty, which has been studied by Abramovich and Hassett with stable varieties, replacing X with an associated orbifold. In this setting, they show that the quite subtle notion of family of stable varieties becomes simply a flat family of the associated orbifolds. Dori extended this approach to the case where there is a nonempty but reduced boundary divisor D. He hopes that this will produce a deformation-obstruction theory for these moduli spaces. He finally explained an application, where he shows that this approach leads to functorial gluing morphisms on the moduli spaces. This approach generalizes the clutching and gluing morphisms that describe the boundary strata of the moduli of curves.

Counting sheaves on Calabi-Yau 4-folds Jeongseok Oh talked on the definition of a localised Euler class for isotropic sections, and isotropic cones, in $SO(N)$ bundles. He used his construction to give an algebraic definition of Borisov-Joyce's sheaf counting invariants on Calabi-Yau 4-folds. This is a new result explaining the algebraic nature of the Borisov-Joyce construction. He also shows that when a torus action is present, localisation formulas also hold, thus providing a computational tool for the invariants. This was joint work with R. P. Thomas.

Quantum cohomology for isotropic Grassmannians and Lefschetz exceptional collections John Alexander Cruz Morales presented joint work with A. Mellit, A. Kuznetsov, N. Perrin and M. Smirnov. He showed a new result proving that the the big quantum cohomology ring of isotropic Grassmannians $IG(2, 2n)$ is generically semisimple, contrasting to the small quantum cohomology ring, which is not. This is proven using the so called *generic smoothness*. He explained that non-semisimplicity leads to a decomposition of the small quantum cohomology ring that relates to a certain decomposition of the derived category of $IG(2, 2n)$ in a so-called Lefschetz exceptional collection.

4 Scientific Progress Made

There were several advances in the construction of K-theoretic Donaldson-Thomas invariants as well the construction DT-theory for CY 4-folds. Further there were several advances in defining a new set of invariants induced by moduli spaces of generalized quiver representations of sheaves with support on algebraic surfaces and curves, the latter provide connections to the well known Vafa-Witten theory in physics. Finally some new directions were discussed, regarding construction of moduli theories with aim to count rational algebraic surfaces in ambient algebraic 3 folds, using the methods of virtual cycle intersection theory.

5 Outcome of the Meeting

The meeting was quite successful in bringing different groups of people working on moduli theories and enumerative invariants. One of the key ingredients for enumerative theories is the construction of the *virtual fundamental class or virtual structure sheaf* on moduli spaces. There were several advances in the computation and construction virtual structure sheaves for DT theory for CY 4-folds, including localization formulas.

There were advances on the computations of higher genus GW invariants for complete intersections. There were multiple advances on applications of gauged sigma models for invariants associated to GIT quotients.

There were discussions on the progress for K-theoretic invariants, in particular work on invariants incorporating level structures in K-theory. Some wall-crossing formulas were presented. Also, there was a discussion on K-theoretic DT/PT correspondence.

On BPS invariants, there was progress on a positivity conjecture, stating that for some type of Jacobi algebras, the BPS invariants are positive.

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