Tilting Theory, Singularity Categories, & Noncommutative Resolutions

Osamu Iyama (Nagoya University),
Jose Antonio de la Pena (UNAM),
Henning Krause (Bielefeld University),
Michael Wemyss (University of Glasgow)

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The conference 'Tilting Theory, Singularity Categories, & Noncommutative Resolutions' (19w5161) took place 1st September – 6th September 2019 at the Casa Matemática Oaxaca. The organisers were Osamu Iyama (Nagoya), José Antonio de la Peña (UNAM), Henning Krause (Bielefeld) and Michael Wemyss (Glasgow). In the original application Ragnar Buchweitz (Toronto) was also one of the main organisers, however he very sadly passed away in November 2017. Held in very high regard by all, his scientific influence on the field was huge. The depth of the mathematics on display at many times reflected his pioneering work on Cohen–Macaulay modules. The diversity of the scientific programme across commutative algebra, representation theory, homological algebra and algebraic geometry was in large part testament to the encouragement and mentoring he gave others, and the many synergies that have since been created.

1 Overview of the Field, Recent Developments and Open Problems

Tilting theory arose in two different areas of mathematics independently: First in the representation theory of finite dimensional algebras, where Gelfand and Ponomarev introduced "Coxeter functors" in 1970 to study the four-subspace problem, "BGP reflections" were introduced in the work of Bernstein, Gelfand and Ponomarev in 1973, then generalized by Auslander, Platzeck and Reiten (1979) to "APR-tilting functors", before finally Brenner and Butler axiomatized these constructions in 1980. Independently, tilting theory arose in Algebraic Geometry in 1978 through Beilinson's seminal work on the derived category of projective spaces. This led to the introduction and study of "helices" and "exceptional collections" of vector bundles on other Fano varieties in the Rudakov seminar. The crucial step was then made by Bondal, who showed that the algebraic and geometric sides were equivalent.

Nowadays, tilting theory is considered more broadly as providing exact equivalences between a triangulated category $\mathcal{T}$ and the derived category of modules over an algebra A, the endomorphism algebra of a tilting object. If one can find a tilting object in $\mathcal{T}$ one gains an extraordinary amount of information in that classical homological algebra can be brought to bear on the structure of $\mathcal{T}$. Of particular importance are those tilting objects whose algebra A is finite dimensional over a field and as a ring is of finite global dimension. The interpretation then is that $\mathcal{T}$ behaves like the category of coherent sheaves on a smooth and proper scheme, even though no natural scheme in the classical sense may be associated with $\mathcal{T}$.

Recent developments of tilting theory, especially application to categorification of cluster algebras of Fomin-Zelevinsky [FZ02], revealed the hidden combinatorial structure in tilting theory. It is controlled by mutation, an operation to construct a new silting object from a given one by replacing a direct summand. In particular, “König-Yang bijections” [KY14] and “Ingalls-Thomas bijection” [IT09] established by a number of authors gives bijections between important landmarks (“tilting structure”) in module categories, derived categories and cluster categories. It has a strong connection to Bridgeland’s space of stability conditions of triangulated categories [Br07].

For classical smooth projective varieties there is Dubrovin’s conjecture that predicts when exactly the triangulated category of coherent sheaves on such a variety carries a tilting object. However, with an arbitrary noetherian scheme X one can naturally associate two triangulated categories: the bounded derived category of coherent sheaves and the triangulated subcategory of perfect complexes on X. If the variety X is smooth, then these two categories coincide. For singular varieties this is no longer true, and the quotient of the bounded derived category of coherent sheaves by the full subcategory of perfect complexes captures many properties of the singularities of X. In 2003, Orlov introduced this quotient as the ”triangulated category of singularities” [Or04]. There are a number of variations and alternative descriptions of this category in the Gorenstein case, studied first by Buchweitz some 30 years ago under the name “stable derived category” [Bu]. For instance, one can identify the triangulated category of singularities with stable categories of Cohen-Macaulay modules or, in the case of hypersurfaces, with the homotopy category of matrix factorizations, a topic studied in Landau-Ginzburg models of String Theory. Theorems of Orlov [Or09] relate the bounded derived category of a generalized (noncommutative) projective variety to its triangulated category of singularities on the one hand and, if the associated homogeneous coordinate ring is Gorenstein, to the singularity category of that ring. Due to the existence of tilting objects, interesting applications arise from the categories of coherent sheaves on weighted projective lines [GL87]. Other important examples of singularity categories are given by cluster categories, which played a key role in categorification of cluster algebras. In fact they have a realization as singularity categories of certain dg algebras [Ke05].

As mentioned above, of particular importance is the case when the endomorphism ring of the tilting object is of finite global dimension, and this property should be seen as indicating ”smoothness” of the associated categorical geometry. Indeed, in representation theory, it is basic to study modules whose endomorphism algebras have finite global dimension. Apart from their occurence in connection with tilting they appear naturally in many situations, e.g. in the Auslander correspondence and theory of representation dimension [Iy03], the quasi-hereditary algebras of Cline-Parshall-Scott [CPS88, Kr17], Rouquier’s dimensions of triangulated categories [Ro08], and cluster tilting in higher dimensional Auslander-Reiten theory [Iy07]. Such endomorphism rings are also called noncommutative resolutions of singularities, and are studied in commutative ring theory and algebraic geometry after Van den Bergh’s work in birational geometry [Va04a, Va04b]. Typical examples of noncommutative resolutions occur for Gorenstein rings of Krull dimension at most one, certain hypersurface singularities, determinantal varieties, and Stanley-Reisner rings.

1.1 Tilting Structure and Mutation

Recent developments of tilting theory make it clear that the notion of silting objects due to Keller and Vossieck [KV88], a generalization of tilting objects, is central. “König-Yang bijections” established by a number of authors (see [KY14, BY13] and references therein) show that silting objects correspond bijectively with several important landmarks in the triangulated category, including t-structures, co-t-structures and simple-minded collections. Mutation is an operation to construct a new silting object from a given one by replacing a direct summand, and gives the arrows in the Hasse quiver of the partially ordered set of silting objects [AI12].

A special class of silting objects called two-term corresponds bijectively with several important landmarks in the module category [AIR14, MS17], including support τ-tilting modules defined by the classical Auslander-Reiten translation functor τ, torsion classes, wide subcategories, and universal localizations. Among others, in a special case, they correspond bijectively with cluster tilting objects, which categorify the cluster algebra of Fomin-Zelevinsky. In the classical case of quivers, some of them are called “Ingalls-Thomas bijections” [IT13], which involve important combinatorial objects called noncrossing partitions in Coxeter combinatorics. Support τ-tilting modules have an infinitely generated analog called silting modules [AMV16], which makes the whole theory more fruitful.

Nowadays mutation is thought to be a fundamental tool to analyze the combinatorial structure of a given triangulated/module category. This has been successfully studied for several important examples including Brauer graph algebras [AAC18], preprojective algebras [AM17, IRRT18], Ginzburg DG algebras [QZ17] and derived-discrete algebras [BPP16, QW18].

1.2 Geometric Tilting and Singularity Categories

If R is a positively graded connected Gorenstein algebra that is finite over its centre and has an isolated singularity at its ideal of elements of strictly positive degree, then the corresponding singularity category is smooth and proper in the sense described in the short overview above. What is not yet understood is when these singularity categories carry a tilting object. Using Orlov's results one knows that this singularity category is related to the derived category of the Serre construction on the category of graded modules that one thinks of as the derived category of coherent sheaves on a "noncommutative projective variety". The two categories differ by a finite exceptional collection of objects, in the Calabi–Yau case they are actually equivalent. What is not known but deserves to be investigated is when the existence of a tilting object, say, on the geometric side, implies the existence of a tilting object on the singularity side. Through recent examples we know that this is certainly not always the case, but we also know that the answer is affirmative when R is a reduced commutative ring of Krull dimension 1 [BIY].

Classical preprojective algebras, introduced by Gelfand-Ponomarev [GP79], are a well-studied topic of the representation theory of quivers. They appears in the classical picture of the (homological) McKay correspondence in dimension two. Recently "higher preprojective algebras" were introduced in the study of cluster categories in higher dimensional Auslander-Reiten theory [IO11, Ke11]. They also appear in geometric tilting theory: If a smooth projective Fano variety X of dimension d carries a tilting object of minimal possible global dimension, equal to d, then the pullback of that tilting object onto the canonical affine line bundle over X is still tilting and the endomorphism algebra of the pullback is the higher preprojective algebra. It provides a noncommutative resolution of singularities of the singular commutative ring that is the cone over the anti-canonical morphism on X. Moreover, by work of Amiot-Iyama-Reiten [AIR15], the corresponding singularity category can be realized as the cluster category of a finite-dimensional algebra A, while the singularity category of graded modules even admits a tilting object.

Another challenge in the analysis of singularities categories arises from the study of thick and localising subcategories. The goal is a classification of such subcategories, following the work of Hopkins and Neeman for perfect complexes over commutative noetherian rings. Such classification results exist for singularity categories (or stable module categories) arising from representations of finite groups [BCR97, BIK11] or from complete intersections [St14]. Also, categories of finite type have been considered, using the concept of a non-crossing partition.

1.3 Quasi-Hereditary Algebras and Noncommutative Resolutions

The endomorphism algebra of a tilting object often comes with some additional structure because the algebra is quasi-hereditary. The notion of a quasi-hereditary algebra was introduced by Scott and then further studied by Cline-Parshall-Scott in the context of highest weight categories [CPS88]. In fact, the module category of a finite dimensional algebra A is a highest weight category if and only if A is quasi-hereditary. In that case the so-called standard objects form an exceptional sequence with the extra property that the objects having a standard filtration form an exact category that is derived equivalent to the full module category [Kr17].

Highest weight structures provide another instance where parallel developments happened in Algebraic Geometry and Representation Theory. Exceptional sequences are studied intensively since the pioneering work on vector bundles of the Rudakov school. On the other hand, the notion of a highest weight category is one of the central concepts in the representation theory of Lie algebras.

Thus it seems to be important to find out when there are tilting objects such that the endomorphism algebra is quasi-hereditary. Interesting examples arise in work of Hille and Perling on rational surfaces [HP14]. For further examples of this connection, relating derived categories of Grassmannians and modular representation theory, see [BLV10, BLV15, BLV16, Ef17].

It is known that any module category is "locally quasi-hereditary" in the following sense: For any module M over an artin algebra there is a "complement" N such that the endomorphism ring of $M \oplus N$ is quasi-hereditary [Iy03]. It would be interesting to see how far this approach can be pushed to construct noncommutative resolutions that are quasi-hereditary.

Another striking connection arises from the representation theory of bocses (bimodules over categories). There are interesting cases when the highest weight category has the additional property that the objects with standard filtration form a hereditary category. This situation is modeled by a special kind of bocses [KKO14] and makes it possible to use reduction techniques.

2 Presentation Highlights

The scientific programme consisted of 3 lecture series, which surveyed and presented recent developments, 16 individual lectures, plus 4 short talks (of 30 minutes each) given by more junior participants. The topics of the lectures can be grouped into the following themes:

2.1 Singularity Categories and Related Topics

Keller opened the conference with a lecture on Tate-Hochschild cohomology, the singularity category and applications, based on the papers [Ke18, HK]. After outlining various versions of Hochschild cohomology and Tate-Hochschild cohomology, and remarking on the open problem as to whether Tate-Hochschild cohomology admits a Gerstenhaber bracket, Keller explained how the dg-$\mathbb{Z}$-graded singularity category, with its natural enhancement, recovers the Tjurina algebra and hence (after fixing the dimension) the singularity itself. A consequence is that the derived contraction algebra recovers the singularity, hence establishing a weakened version of the Donovan–Wemyss conjecture regarding the classification of flops [DW16].

Hua talked about joint work in progress with Keller [HK2], on analytic superpotential algebras, and the relationship to motivic Donaldson–Thomas theory. He first outlined some impressive generalisations of commutative results, such as the Mather–Yau theorem, to the formal noncommutative setting in the case when the complete Jacobi algebra is finite dimensional. Motivated by Toda's Ext-quiver local description of moduli spaces, Hua then established similar results in the analytic setting. The talk ended with the main result that, given a quiver without loops or 2-cycles, an analytic superpotential algebra is preserved under quiver mutation.

Hirano spoke on stability conditions for 3-fold flops, based on his joint work with Wemyss [HW]. These turn out to be completely controlled by noncommutative resolutions, and their variants. The combinatorics of these objects is described in [IW], and are controlled by certain new hyperplane arrangements. Hirano explained how stability conditions in this geometric setting give a regular covering of the associated hyperplane arrangement, and how further controlling the autoequivalence group gives the first computation of the Stringy Kähler Moduli Space (SKMS) for general 3-fold flops.

Kalck talked on his joint work with Pavic and Shinder [KPS]. Motivated by a recent paper of Kawamata [Ka] (itself motivated in part by noncommutative deformations), he introduced a general notion of a Kawamata semi-orthogonal decomposition. The very nice property this has is that is descends to both perfect complexes, and the singularity category. Kalck outlined a series of obstructions to such a decomposition existing, with some being the existence of loops in the deformation theory associated to the curves. The upshot is that Kawamata's result may be the optimal, with understanding the obstructions leading to new results in their own right.

Ueda gave a talk on moduli of A_∞-structures on a generator of the (graded) singularity category for a deformation of an exceptional unimodal singularity [LU]. Deforming the unimodal singularity loses the property of having a tilting module, and so A_∞ methods come to the fore. Ueda also outlined the relationship to the moduli space of K3 surfaces, and homological mirror symmetry.

Thibault spoke on graded singularity categories of Gorenstein algebras with levelled Beilinson algebras. A main problem is to find conditions on a noetherian AS-regular algebra A and an idempotent $e \in A$ for which the graded singularity category $\mathrm{Sing}^{\mathrm{gr}}(eAe)$ admits a tilting object. Of particular interest is the situation in which A is a graded skew-group algebra $S\#G$, where S is the polynomial ring in n variables and $G < SL(n,k)$ is finite, and $e = \frac{1}{|G|}\sum_{g\in G} g$, so that $eAe \cong S^G$. A tilting object was found by Amiot, Iyama and Reiten [AIR15] in the case where A has Gorenstein parameter 1. Generalizing the work of Iyama and Takahashi [IT13], Mori and Ueyama [MU16] obtained a tilting object in $\mathrm{Sing}^{\mathrm{gr}}(S^G)$, provided that S is a noetherian AS-regular Koszul algebra generated in degree 1 and G has homological determinant 1. Thibault discussed certain silting objects and then specialised to the setting in which the Beilinson algebra is a levelled algebra, giving a generalisation of the result of Mori and Ueyama.

Kvamme introduced a far reaching generalization of Gorenstein rings. He introduced the notion of a Nakayama functor relative to an adjunction, generalising the classical Nakayama functor for a finite-dimensional algebra. This can be characterized in terms of an ambidextrous adjunction of monads and comonads. He also discussed this concept from the viewpoint of Gorenstein homological algebra. In particular he obtained a generalization of the equality of the left and right injective dimension for a finite-dimensional Iwanaga-Gorenstein algebra, and for a module category he showed that this property can also be characterized by the existence of a tilting module.

2.2 Noncommutative Resolutions

Faber gave a two-lecture overview of her joint work with Buchweitz and Ingalls [BFI] on noncommutative resolutions for reflection groups. The first lecture gave background on the classical McKay correspondence, and Auslander's version of it. The second lecture introduced the main topic, namely that of discriminants of reflection groups, and Faber outlined how to construct a noncommutative resolution by factoring the skew group ring by an appropriate idempotent. The subtle point is that to understand this factor involves relating a noncommutatively defined map to a more classical one involving the normalisation and the conductor ideal.

Špenko gave a two-lecture overview on her spectacular work, joint with Van den Bergh [SV17], that produces noncommutative resolutions of quotient singularities for reductive groups. This far-reaching generalisation of the finite group case was illustrated first in the easiest example of the conifold, then in the next easiest example of a two-dimensional torus, where the complexity of the problem is already startling. Špenko outlined the general algorithm, linked the construction to Smith's talk (see below), and then outlined some of the more recent developments to the NC Bondal–Orlov conjecture, and to schobers.

Smith gave a lecture on her joint work with Faber and Muller [FMS] which constructs noncommutative resolutions for all (normal) toric varieties. The construction is independent of Špenko–Van den Bergh, but ultimately recovers the same algebra. The point however is that the toric setting allows us much finer control of the information, and for example the precise value of the global dimension can be recovered. Smith also outlined some speculations in characteristic p, the first which tries to generalise the characteristic zero theorem of Stafford–Van den Bergh which says that the centre of a module-finite homologically homogeneous ring has rational singularities.

2.3 Ubiquity of Algebras of Finite Global Dimension

2.3.1 Quasi-hereditary Algebras

Külshammer gave two introductory lectures on quasi-hereditary algebras. Exceptional collections frequently arise in algebraic and symplectic geometry. Since the work of Beilinson on coherent sheaves on projective space, exceptional collections have been used to construct derived equivalences. In representation theory, exceptional collections appear for the class of quasi-hereditary algebras. In the first talk, Külshammer gave an overview over the theory of exceptional collections and quasi-hereditary algebras. In the second talk, he focus on an approach to quasi-hereditary algebras using exact Borel subalgebras, corings, and A_∞-Koszul duality [KKO14].

Flores Galicia gave a talk on quasi-hereditary algebras based on a joint work with Kimura and Rognerud. The notion of a quasi-hereditary algebra depends on a partial order given to the set of simple modules. In particular an algebra may be quasi-hereditary for one partial order but not for another one, even in the hereditary case. He introduced an equivalence relation on the set of all partial orders giving a quasi-hereditary algebra, calling the equivalence classes quasi-hereditary structures. In the case of the path algebra of an equioriented quiver of type A, he classified all its quasi-hereditary structures in terms of tilting modules, highlighting its nice combinatorial properties. Then he generalised this classification to any orientation. As a complementary example he discussed a class of quiver algebras with a unique quasi-hereditary structure.

2.3.2 Auslander Algebras and Mirror Symmetry

Jasso and Dyckerhoff gave two talks on their joint work with Lekili, which reveals remarkable triangle equivalences between the perfect derived categories of higher Auslander algebras of type A, partially wrapped Fukaya categories of symmetric products of disks, and the d-dimensional Waldhausen S-construction [DJW19].

Jasso explained triangle equivalences between the perfect derived categories of higher Auslander algebras of type A, partially wrapped Fukaya categories of symmetric products of disks, and the d-dimensional

Waldhausen S-construction of k. Fukaya categories of marked Riemann surfaces are now understood to be described combinatorially in terms of (graded) gentle algebras. Jasso explained how Iyama's higher Auslander algebras of type A [Iy11] relate to Fukaya categories of symmetric products of disks. He also explained how to leverage this relationship to provide an alternative proof of a result of Beckert concerning certain derived equivalences between higher Auslander algebras of type A in different dimensions.

Dyckerhoff explained that factorization homology provides a means of integrating algebraic structures over a space equipped with suitable geometric decorations thus producing interesting invariants. He discussed how to use this framework to integrate higher Auslander algebras of Dynkin type A over a framed surface X and explained how the resulting invariants relate to Fukaya categories of symmetric products of X.

2.3.3 Calabi-Yau Algebras and Higher Preprojective Algebras

Hille explained higher preprojective algebras in the context of geometric tilting theory. He reported on work jointly started with Buchweitz about the pull back of a tilting bundle to the total space of the canonical line bundle. Let X be an algebraic variety with a tilting bundle T, then he gave a criterion when its pull back to the total space Y of the canonical line bundle is also a tilting bundle. This is closely related to his previous work [Hi95] on distinguished tilting sequences and generalizes the results therein. Moreover, he computed the endomorphism ring of the pull back tilting bundle as the higher preprojective algebra. This leads to a geometric construction of those algebras. The construction needs tilting bundles T with endomorphism algebra of global dimension $\dim X$. He considered several examples for surfaces and compute the possible global dimensions of A.

For a field K of char $K = 0$ and a finite acyclic quiver Q, the preprojective algebra $\Pi(Q) = K\overline{Q}/(\rho)$ of an acyclic quiver Q is the path algebra $K\overline{Q}$ of the double quiver $\overline{Q}$ of Q with the mesh relation $\rho = \sum_{\alpha\in Q_1}(\alpha\alpha^* - \alpha^*\alpha)$. It is an important mathematical object having rich representation theory and plenty of applications. Minamoto gave a talk on joint work with Herschend about a new class of 3-Calabi-Yau algebras obtained by extending classical preprojective algebras. He introduced a cubical generalization $\Lambda(Q) := K\overline{Q}/([a,\rho] \mid a \in \overline{Q}_1)$ where $[-,+]$ is the commutator. This is a special case of algebras introduced by Etingof-Rains, which is a special case of algebras introduced by Cachazo-Katz-Vafa. The algebra $\Lambda(Q)$ of Herschend-Minamoto has remarkable properties, among other things it provides the universal Auslander-Reiten triangle for KQ. Using the natural grading of $\Lambda(Q)$ given by $\deg\alpha = 0, \deg\alpha^* := 1$ for $\alpha \in Q_1$, he introduced an algebra $A(Q) := \begin{pmatrix} KQ & \Lambda(Q)_1 \\ 0 & KQ \end{pmatrix}$ where $\Lambda(Q)_1$ is the degree 1-part of $\Lambda(Q)$. He explained that the algebras $A(Q)$ and $\Lambda(Q)$ are one-dimensional higher versions of the classical ones KQ and $\Pi(Q)$.

2.3.4 Global Dimension and Smoothness

A. Takahashi gave a talk on joint work with Kikuta and Ouchi [KOT]. He studied the scaling dimension (or the similarity dimension) of the perfect derived category of a smooth compact dg algebra called the Serre dimension. It is expected that the infimum of the Ikeda-Qiu's global dimension function [IQ] on the space of stability conditions [Br07] also gives another "good" notion of dimension. One of his results is that its infimum is always greater than or equal to the Serre dimension. Motivated by the ADE classification of the 2-dimensional $N = 2$ SCFT with $\widehat{c} < 1$, he also gave a characterization of the derived category of Dynkin quivers in terms of the Serre dimension and the global dimension function.

Schnuerer talked on smoothness of the derived categories of algebras, based on his joint work with Elagin and Lunts [ELS]. For any scheme X over a reasonable field, it turns out that the derived category $\mathrm{D^b}(\operatorname{coh} X)$ is always smooth in the dg sense, regardless of whether X has singularities or not. However, the perfect complexes of X are smooth (in the dg sense) if and only if X has no singularities. The main result of Schnuerer's talk was that this first fact translates into finite dimensional algebras: for a finite dimensional algebra Λ over a perfect field, the category $\mathrm{D^b}(\operatorname{mod}\Lambda)$ is always smooth, answering a question of Iyama. This statement also generalises to module-finite algebras, with mild assumptions on the centre.

2.4 Categorical Approach

Burban talked on joint work in progress, with Drozd, on Morita theory for non-commutative noetherian schemes. Generalising ideas of Gabriel, Burban first explained how to recover the underlying scheme X

from the category $\mathrm{Qcoh}(X, \mathcal{A})$, where $\mathcal{A}$ is an appropriate noncommutative structure sheaf on X. With this in hand, he then explained how to recover a global version of Morita theory. There were two main applications: a new proof of a conjecture of Caldararu, and a criterion for when two non-commutative projective curves are Morita equivalent.

Psaroudakis gave a talk on joint work with Oppermann and Stai. Let $\mathcal{T}$ be a triangulated category with coproducts and let X be a set of compact objects. Then X generates a certain t-structure, and in particular describes explicitly a left adjoint to the inclusion of the coaisle. Unfortunately, it does not make much sense to consider the naive dual of this setup; cocompact objects rarely appear in categories which occur naturally. Motivated by this, Psaroudakis introduced a weaker version of cocompactness called 0-cocompactness, and showed that in a triangulated category with products these objects cogenerate a t-structure. As an application, he provided explicit right adjoints between certain homotopy categories (i.e. "big" singularity categories in the sense of Krause). Moreover, under the presence of a relative Serre functor he showed how one can get 0-cocompact objects from compact ones.

2.5 Fuchsian Singularities

Lenzing explained bijections between several significant mathematical objects including weighted smooth projective curves, hereditary noetherian abelian categories with Serre duality, and 2-dimensional positively graded commutative affine Gorensetin algebras of Gorenstein parameter 1 which are graded isolated singularities [Le]. A fuchsian singularity is classically attached to a finitely generated, cocompact subgroup G of the automorphism group of the hyperbolic plane $\mathbb{H}$. It is the $\mathbb{Z}$-graded algebra of G-invariant differentials (automorphic forms) on $\mathbb{H}$. Lenzing extended the concept to algebraically closed fields of arbitrary characteristic. Then he discussed their relationship to mathematical objets of a different nature, and he provided a ring-theoretic characterization of fuchsian singularities. Finally he explored their singularity categories.

2.6 Quiver Grassmanianns and Moduli Stacks of Representations

Pressland talked on joint work with Sauter. A quiver Grassmannian is a projective algebraic variety parametrising the submodules of a given A-module M of a fixed dimension vector d. A famous result in geometric representation theory, obtained by several different authors, states that every projective variety X (over an algebraically closed field of characteristic zero) is isomorphic to such a quiver Grassmannian (e.g. [Re13]). He explained how one can use this algebraic description to construct a desingularisation of X. The construction is representation-theoretic, involving a tilt of an endomorphism algebra in $\operatorname{mod} A$, and the desingularising variety is again described in terms of quiver Grassmannians. In his work, he extended methodology of Crawley-Boevey and Sauter, and of Cerulli Irelli, Feigin and Reineke.

Chan talked on two recovery results, this time related to the fact that the category of coherent sheaves on a stack $\mathbb{X}$ does *not* in general allow to recover $\mathbb{X}$. Neither does the derived category, even for schemes. Enriching $\operatorname{coh} \mathbb{X}$ with extra information, such as a tensor structure, allows us to do this, and Chan outlined the main results of [AC], namely recovery theorems via tensor stable moduli stacks, and via moduli of refined representations.

3 Scientific Progress Made, and Outcome of the Meeting

The concept of the meeting was to bring together experts from different mathematical fields and to encourage their interaction. The topics were ranging from representation theory of algebras to noncommutative geometry, but many contributions of the participants were crossing boundaries and demonstrated beautiful connections, often using the common language of derived categories.

Extremely helpful were three series of introductory lectures on

- noncommutative resolutions of quotient singularities for reductive groups (Špenko),
- noncommutative resolutions for reflection groups (Faber),
- quasi-hereditary algebras and exceptional collections (Külshammer).

The high proportion of relatively young participants was another aspect of this meeting. Their contributions demonstrated the healthy state of the subject area of this meeting. This generation keeps alive the pioneering ideas of Ragnar Buchweitz, who was one of the initial organizers of this meeting, but sadly passed away in November 2017. In fact, the spirit of his interdisciplinary approach towards algebra and geometry was felt throughout this meeting and has been developed further during this week.

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