Reverse Mathematics of Combinatorial Principles (19w5111)

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1 Introduction

Mathematicians prove theorems from basic assumptions called axioms. Today, the subject benefits from having "firm foundations", by which we usually mean axioms sufficient to prove virtually all of the theorems we care about. But given a particular theorem, can we specify precisely which axioms are needed to derive it? This is a natural question, and also an ancient one: over 2000 years ago, the Greek mathematicians were already asking it about the axioms of geometry. Reverse mathematics provides a modern approach to this kind of question. A striking empirical fact repeatedly demonstrated in this area is that the vast majority of mathematical propositions can be classified into just five main types, roughly corresponding to five general mathematical principles that crop up all across mathematics, regardless of whether we are looking at algebra, calculus, geometry, or many other areas.

But there are exceptions, and they include some very important mathematical theorems. One of these is a famous theorem due to F. P. Ramsey, which can be colorfully stated as follows: at any dinner party with infinitely many guests, it is possible to find either infinitely many of the guests that all know each other, or infinitely many of the guests none of whom knows any of the others. This is a profound result in the area of combinatorics, with numerous applications in mathematics and computer science. And as it happens, it falls outside the five main types mentioned above. Understanding why this theorem, and others like it, behave differently from the vast majority of others, sheds light on the capacities and limitations of different ways of reasoning in mathematics, particularly in combinatorics, and in so doing, gives us a better picture of the underpinnings of mathematics as a whole.

The workshop "Reverse Mathematics of Combinatorial Principles" at CMO/BIRS was dedicated to exploring the state of this area, which in its modern form encompasses computability theory, proof theory, computable analysis, and other areas of mathematical logic. Coinciding with the launch of a new collaborative grant from the U.S. National Science Foundation dedicated to the study of the intersection of logic and combinatorics, the Oaxaca workshop was also a key conference for organizing a broad, coordinated, longterm scientific effort aimed at the key questions in this subject and exploring future directions and applications.

The workshop was extremely successful, both in terms of scientific outcomes and in terms of community building. The location was superb, and the staff and conference support were excellent. In addition, all participants very much enjoyed the excursion and getting to learn more about Oaxaca and its history.

2 General overview of the program

2.1 Background

There is a deep and fascinating interaction between the complexity of describing the solutions of a given mathematical problem, and the strength of axioms needed to prove that solutions exist. Consider a typical such problem, having the form "for every A of some kind, there exists a B with some properties". Intuitively, the more complex the "solution" B must be relative to the "instance" A, the more difficult the statement ought to be to prove. Two prominent frameworks for formalizing this intuitive idea, for the relatively wide-ranging case when A and B can be coded by subsets of the natural numbers, are effective mathematics on the complexity side, and reverse mathematics on the proof-theoretic side. The analysis of mathematical principles provided by these complementary frameworks has provided deep insights into the fundamental techniques underlying different areas of mathematics, and identified new connections between them.

The setting for reverse mathematics is second-order arithmetic, which is a formal system strong enough to encompass (countable analogues of) most results of classical mathematics. It consists of the usual algebraic axioms for the natural numbers, together with the comprehension (i.e., set existence) axiom scheme, asserting that the set of all $x \in N$ satisfying a given formula exists. Fragments of this system obtained by weakening the comprehension scheme are called subsystems of second-order arithmetic. There are five that one encounters most frequently, each allowing a stronger form of comprehension, and thereby offering more that can be proved in it. In turn, each stronger level of set existence has a natural and well-understood computational analog. The first (RCA_0) is a system roughly corresponding to computable (constructive) mathematics, and serves as our base theory. The second system (WKL_0) includes the following weak version of König's Lemma: every infinite binary branching tree has an infinite path. It essentially adds the power of compactness arguments to the elementary effective procedures available in RCA_0. Computationally, it corresponds to the Jockusch-Soare low basis theorem that bounds the complexity of the path in terms of its halting problem. The third system (ACA_0) asserts that every set definable in (first order) arithmetic exists. It corresponds to being able to solve the halting problem (construct the Turing jump) relative to any set X. The last two systems (ATR_0 and Π^1_1-CA_0) are more powerful systems with second order existence axioms. The first of them corresponds to (effectively) iterating the Turing jump into the transfinite. The second adds the power to determine whether a given linear ordering is well founded.

It is a profoundly fascinating empirical fact that (the countable analogues of) most mathematical theorems are either provable in the base theory, RCA_0, or else equivalent over RCA_0 to one of the other four subsystems, WKL_0, ACA_0, ATR_0, or Π^1_1-CA_0. Indeed, much of the early tendency in the subject, after its original development in the 1970s and 1980s by Friedman and Simpson, was a classificatory one, of finding for each theorem of ordinary mathematics which of these five categories it fits into. This can be regarded as a formalization and confirmation of the common intuition among mathematicians that theorems in very disparate areas of mathematics can be strongly evocative of each other, and that the same broad patterns of problem-solving crop up in solutions to apparently unrelated problems.

One of the most fruitful programs of research in computability theory over the past thirty years has been the investigation of the logical strength of combinatorial principles such as Ramsey's Theorem. One version of this principle, denoted RT^n_k ($n, k \in N$) asserts that every coloring of n-tuples of natural numbers by k many colors has a homogeneous set, i.e., an infinite set, on the n-tuples of which the coloring is constant. This is a far-reaching result, broadly asserting that in any configuration of objects, some amount of order is necessary. Understanding this order has been the objective of much research in combinatorics and logic, and in computability theory specifically, going back to the crucial work of Jockusch in the 1970s, where it has blossomed into a long and fruitful line of research. An important aspect of this work has been the uncovering of a wealth of combinatorial principles that fall outside of the five systems mentioned above. More recent work has revealed further examples of theorems from outside of combinatorics, whose strengths are nevertheless governed by combinatorial properties, and as such fit into the increasingly intricate structure (affectionately called the "reverse mathematics zoo") of relationships between different combinatorial principles. Examples include model-theoretic principles (Hirschfeldt, Shore, and Slaman, 2009), the finite intersection principle from set theory (Dzhafarov and Mummert, 2013), and a variant of Birkhoff's Recurrence Theorem from ergodic theory (Day, 2016).

2.2 Objectives of the workshop

The past five years have seen the introduction of a number of new ideas and methods that are proving successful in answering longstanding questions on the effective and reverse-mathematical content of various combinatorial questions. Part of this has come from a new research link between reverse mathematics and computable analysis. A central notion in the latter subject is that of Weihrauch reducibility, which provides a very natural formalism for reducing one kind of mathematical problem to another. Developed by Weihrauch in the early 1990s, it has been widely deployed in the effective study of various results in analysis, chiefly by Brattka and his co-authors, from around 2000 onwards. Some early applications of this tool to reverse mathematics were noted already by Gherardi and Marcone (2009), but it was only recently that this connection really took off, when Weihrauch reducibility was rediscovered and re-developed in the context of computable combiantorics in a series of papers Dzhafarov (2015), Dorais, Dzhafarov, Hirst, Mileti, and Shafer (2016), and Hirschfeldt and Jockusch (2016). These elaborated on the notion in various ways to obtain a refinement and, in many ways, extension of the traditional framework of reverse mathematics. The subsequent use of Weihrauch reducibility, and the numerous ancillary notions developed in computable analysis, have led to truly new insights into reverse-mathematical questions (Dzhafarov, 2016; Dzhafarov, Patey, Solomon, and Westrick, 2017). Conversely, the incorporation of techniques from computability theory and computable combinatorics are increasingly leading to solutions of and insights into longstanding problems in computable analysis. One goal of the workshop was thus to further explore this fruitful exchange, bringing together computability theorists and computable analysts, to find other problems of mutual interest, particularly from combinatorics, and in turn, to find additional applications of each of the areas to the other.

New connections with classical (pure) combinatorics are of course also essential. One such connection has recently been provided by the study of Hindman's Theorem (HT), which states that for every coloring of the natural numbers with finitely many colors, there is an infinite set S, all nonempty sums of distinct elements of which have the same color. One reason this problem is particularly interesting from the point of view of reverse mathematics is that there are several proofs of it in the literature, using a range of combinatorial, set-theoretic, and ergodic-theoretic methods. The effective content of a number of these has been investigated by Blass, Hirst, and Simpson (1987), Hirst (2004), and Towsner (2011, 2012). The established bounds on logical strength still leave a significant gap. One recent approach for narrowing it has been to look at restricted forms of HT, motivated in part by the following question in pure combinatorics: Hindman, Leader, and Strauss (2003) asked whether every proof of Hindman's Theorem restricted to sums of length at most two is already a proof of the full Hindman's Theorem (for arbitrary sums). In reverse mathematics, this question can be made more precise by asking if the restricted form of HT implies the full version over RCA_0. This remains open, but already partial results are leading to new insights that point to the need for powerful new methods. As a case in point, Csima, Dzhafarov, Hirschfeldt, Jockusch, Solomon, and Westrick (2020) obtained a partial answer to the question of Hindman et al. using an effective version of the Lovász Local Lemma from probability theory. There is thus clearly a case for wider use of probabilistic methods in combinatorial and computability-theoretic arguments, the exploration of which was another goal of the workshop.

An important task in understanding principles is to characterize their first-order consequences, i.e., to determine what they can say purely about the natural numbers. Combinatorial principles are important in this regard, as they serve as a proving ground for heuristic insights and technical innovations. Perhaps the landmark problem in the area is to determine the first-order consequences of RT^2_2. The seminal work on this question by Cholak, Jockusch, and Slaman (2001), who gave an upper bound in terms of weak forms of induction, which was subsequently improved by Chong, Slaman, and Yang (2014) in their work on the proof-theoretic version of the RT^2_2 vs. SRT^2_2 problem.

In what is one of the most exciting recent results in the area, Patey and Yokoyama (2017) showed that RT^2_2 cannot prove any more Π^0_3 statements (i.e., those that have the form $\forall\exists\forall\cdots$) about the natural numbers than are already provable in RCA_0. This is a surprising fact with important foundational consequences, as described to a general audience in a recent article by Wolchover in Quanta Magazine entitled “Mathematicians Bridge Finite-Infinite Divide”. Beyond its foundational significance, their result develops entirely new techniques for studying first-order consequences of combinatorial statements, which we would obviously like to further investigate at the workshop. There is a palpable sense that a precise characterization of the first-order consequences of RT^2_2 is now within striking distance, which the workshop also explored.

3 Activities and progress made during workshop

3.1 General structure

The workshop consisted of fourteen 50-minute talks over the five days, and one 90-minute open problem session. Several talks were partly or entirely expository in nature, e.g., Dobrinen’s and Hirst’s, presenting surveys of known results in conjunction with open problems. The open problem session itself featured a large number of ad hoc presentations of different problems, including ones that are not necessarily part of ongoing larger projects but may themselves open avenues for future research.

The schedule was organized to facilitate a mix between presentations and collaboration time for participants. Wednesday was left entirely open for research time and the conference excursion. In this way ours was very much a working workshop, with a number of projects begun, continued, or completed during the week (e.g., Dzhafarov, Hirscheldt, and Reitzes, to appear; Dzhafarov, Goh, Hirschfeldt, Patey, and Pauly, to appear; Fiori-Carones, Marcone, Shafer, to appear).

3.2 Presentation Highlights

3.2.1 Overview of the solution to the SRT^2_2 vs. COH problem

For many years, a central problem surrounding the logical analysis of versions of Ramsey’s Theorem has been whether Ramsey’s Theorem for pairs (RT^2_2) is implied by an important weaker version known as Stable Ramsey’s theorem for pairs (SRT^2_2) over RCA_0. The question was answered in its original form by Chong, Slaman, and Yue (2016). However, their proof makes essential use of nonstandard models of (fragments of) arithmetic. That is, they construct a model in which SRT^2_2 is true and RT^2_2 is false, but in which the set of “natural numbers” has some unusual properties that the actual natural. The question of what happens over *standard* models remained open, and proved in many ways to be much more difficult. The question was finally solved just shortly before the workshop, by Monin and Patey (to appear).

At the workshop, Patey presented a comprehensive overview of his and Monin’s recent breakthrough solution to this problem. Although the techniques in their argument are extremely novel and sophisticated, Patey’s presentation of them was via a gradual, largely chronological progression from well-established ones, explaining each new elaboration in turn and thereby giving an overall clear impression of the proof. Several participants noted after the talk how illuminating it was.

3.2.2 Problems from computable analysis

As mentioned in the previous section, there has been a strong interest in recent years in interactions of reverse mathematics with computable analysis, which is an area straddling mathematical logic and computer science. Several talks at the workshop discussed research in this direction, including the ones by Marcone and Pauly, who between the two of them gave a very thorough introduction to Weihrauch reducibility. Pauly discussed the Weihrauch analysis of closed choice principles on various represented spaces (Brattka, de Brecht, and Pauly, 2012; Le Roux and Pauly, 2015), and applications thereof, such as in the proof of a computable version of the Hausdorff-Kuratworski theorem (Pauly, 2015).

Marcone presented recent results with Valenti investigating the uniform computational content of the open and clopen Ramsey theorems in the Weihrauch lattice. While these are known to be equivalent to ATR_0 from the point of view of classical reverse mathematics, there is not a canonical way to phrase them as instance-solution problems in the style discussed above. Thus, here Marcone and Valenti present different multivalued variants and study their degree from the point of view of Weihrauch, strong Weihrauch and arithmetic Weihrauch reducibility. This is a fine example of the sense in which Weihrauch reducibility can be regarded as a refinement of the traditional framework of reverse mathematics, highlighting distinctions that are undetectable over RCA_0 alone.

3.2.3 Connections with set-theoretic investigations of infinitary combinatorics

Natasha Dobrinen presented developments in the set-theoretic investigation of infinitary combinatorics, including her recent solution to a longstanding question concerning colorings of triangle-free graphs. She included a number of very fascinating questions that are perfectly suited for study in reverse mathematics, including finding the computational and proof-theoretic content of Milliken's theorem for trees and for the Rado graph. Milliken's tree theorem considers finite colorings of certain finite subgraphs of the full binary tree, and asserts the existence of a so-called *strong* perfect subtree which is monochromatic with respect to this coloring. The adjective "strong" refers to the fact that the the solution subtree has two key properties: 1) if two nodes appear on the same level of the subtree, then they are at the same level in the full binary tree; 2) the subtree is closed under meets, meaning that if two nodes appear in the subtree then so does their longest common prefix. Combinatorially, these properties make the construction of suitable solutions here rather more complicated than in the case of Ramsey's theorem or the so-called Tree Ramsey's theorem, which was previously studied in reverse mathematics. Dobrinen's questions at the Oaxaca workshop inspired a recent "Research in Paris" at the Institut Henri Poincaré between Anglès d'Auriac, Cholak, Dzhafarov, Monin, and Patey, in which a detailed computability-theoretic analysis of Milliken's theorem was accomplished. Dobrinen's talk thus highlights the fact that many of the combinatorial questions being worked on in set theory and in computability theory are deeply related, and are worthy of mutual inspection.

3.2.4 First-order analyses

Two talks, by Yokoyama and Kołodziejczyk, addressed recent developments in first-order reverse mathematics, i.e., in the proof-theoretic consequences of various problems. Yokoyama discussed his recent work with Patey, mentioned in the previous section, as well as separate work with Slaman on the strength of Ramsey's theorem for pairs. In particular, an exciting new result by Slaman and Yokoyama (to appear) is a precise characterization of the first-order strength of the principle $RT^2_{<\infty}$. This is Ramsey's theorem for arbitrary finite colorings of pairs of integers: i.e., $(\forall k)RT^2_k$. While for each fixed (standard) k, the principle RT^2_k is easily provable from RT^2_2, quantifying over the number of colors allows k to be possibly nonstandard, and this increases the complexity of the principle, as already noted by Hirst (1987) and Cholak, Jockusch, and Slaman (2001). The result of Yokoyama and Slaman is that the first-order strength of $RT^2_{<\infty}$ aligns with the Bounding Scheme for Σ^0_3 (three-quantifier) formulas ($B\Sigma^0_3$), which is an axiom strictly in-between induction for Σ^0_3 and Σ^0_4 formulas of arithmetic.

Kołodziejczyk's talk focused on the axioms needed to prove various results related to automata on infinite words and Büchi's theorem on the decidability of the monadic second-order theory of $(N, \leq)$. The main result was that Büchi's complementation theorem for nondeterministic automata on infinite words is equivalent to the induction scheme for Σ^0_2 formulas, as well as to the decidability of the depth-n fragment of the monadic second-order theory of $(N, \leq)$, for each $n \geq 5$. Connections were also drawn with the "bounded-width" version of König's Lemma, often used in proofs of McNaughton's theorem. The work highlights an interesting new connection between proof theory, computability theory, combinatorics, and complexity theory.

3.2.5 Open problem session

Questions were presented by Vasco Brattka, Natasha Dobrinen, Rupert Hölzl, Arno Pauly, Sam Sanders, Keita Yokoyama, and Bartosz Wcisło, covering the full spectrum of areas presented at the workshop.

4 Connections of workshop to broader research efforts

Reverse mathematics and computable combinatorics is entering an exciting new period of research activity, bolstered by a recently-awarded collaborative grant from the U.S. National Science Foundation entitled "Focused Research Group: Computability-theoretic aspects of combinatorial problems". This is a multi-institution grant between the University of California, Berkeley; the University of Chicago; the University of Connecticut; the University of Notre Dame; and Pennsylvania State University. The Oaxaca meeting, whose organizers are all principal investigators on the NSF grant, thus served as a kind of inaugural event to this concerted new research effort. The grant will support numerous collaborations and meetings over the next three years, which will all largely be organized around the themes of the Oaxaca workshop. A key focus will be on the open questions first presented there.

In addition, all of the participants at the Oaxaca meeting are collaborators or key collaborators on the NSF grant, and the Oaxaca workshop provided an opportunity for many of them to meet in person for the first time and to begin to discuss potential topics for joint research. As noted above, in this sense the workshop has already proved fruitful to a number of projects in progress and a number of forthcoming papers.

Our list of participants at the Oaxaca workshop, as well as on the NSF grant, includes a number of women and early career researchers, e.g., Marta Fiori Carones, Natasha Dobrinen, Jun Le Goh, Justin Lin, Corrie Ingall, Li Ling Ko, Manlio Valenti, and Linda Westrick. Going into the workshop, we were committed to actively promoting participation by members of both groups, and we believe the Oaxaca workshop has had a positive effect in this regard for the broader NSF project as well by setting a strong example. More generally, the Oaxaca workshop helped create a renewed sense of community among the participants that promises to have a long-lasting influence in terms of future collaborations and scientific outcomes.

5 Schedule of the workshop

Monday, September 16

10:00–10:10	Introduction and Welcome
10:10–11:00	Stephen G. Simpson: Reverse mathematics and the ascending chain condition
11:40–12:30	Alberto Marcone: The open and clopen Ramsey theory in the Weihrauch lattice
15:00–15:50	Jeff Hirst: Questions about Hindman's theorem
16:30–17:20	Leszek Kołodziejczyk: The reverse mathematics of Bchi's decidability theorem
17:20–19:00	Collaboration time

Tuesday, September 17

09:30–10:20	Chitat Chong: Cohesive trees

11:00–11:50	Natasha Dobrinen: Open Problems in the Reverse Mathematics of Ramsey Theory on Trees and Graphs
15:00–15:50	Ludovic Patey: SRT^2_2 does not imply COH in omega models
16:30–17:20	Benoit Monin: Lowness of the pigeonhole principle
17:30–19:00	Open problems session

Wednesday, September 18

Morning	Collaboration time
Afternoon	Excursion

Thursday, September 19

09:30–10:20	Linda Brown Westrick: Reverse math of the dual Ramsey theorem
11:00–11:50	Wei Wang: Some propositions between WWKL0 and WKL0
15:00–15:50	Keita Yokoyama: The first-order part of Ramsey's theorem for pairs
16:30–17:20	Arno Pauly: Weihrauch degrees of closed choice on finite spaces
17:20–19:00	Collaboration time

Friday, September 20

09:30–10:20	Paul Shafer: The reverse mathematics of an inside-outside Ramsey theorem
11:00–11:50	Noam Greenberg: Relationships between preservation properties of problems

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