Tropical methods in real algebraic geometry

Erwan Brugallé (Université de Nantes, France.),
Lucía López de Medrano (UNAM, México.),
Kristin Shaw (University of Oslo, Norway).

9th-13th September 2019

This workshop focused primarily on tropical methods in real algebraic geometry, particularly in regards to applications to enumerative geometry and topology of real algebraic varieties. Additional related topics included topology of tropical varieties, (real) analytifications and tropicalizations, amoebas of algebraic varieties, and extremal properties of real algebraic varieties.

1 Overview of the field, recent developments and research directions

Tropical geometry is a field of mathematics that emerged around the year 2000, and provides a polyhedral version of algebraic geometry. One of its main motivations is to study algebraic varieties via associated polyhedral complexes, known as tropicalizations. Tropicalizations are often simpler objects which retain a lot of relevant information about algebraic varieties. The foundational ideas of tropical geometry can already be found in the pioneering texts of R. Bieri and J. Groves and also in the O. Viro's patchworking technique. However, it was only at the end of the 90's that the theory began to consolidate itself.

Recent years have seen a tremendous development in tropical geometry that both established the field as an area of its own right and unveiled its deep connections to numerous branches of pure and applied mathematics. One can cite for example applications of tropical geometry to real and complex enumerative geometry ([Mik05], [BrMi07], [IKS13], [ABLdM11], [MMS18] [NPS18]), mirror symmetry ([KoSo06], [GrSi06], [GrSi14], [Yue16]), integrable systems ([Soi10], [InIw12]), Hodge theory ([Kon10], [AHK18], [IKMZ19]), analytic geometry in the sense of Berkovich ([KoSo06], [Gub07], [ABBR15], [NPS18]), construction of real algebraic varieties ([Vir01], [Shu12], [BrLdM12], [BLdMR13], [ABLdM14]), singularity theory ([AIL10], [Shu12], [MMS18]), algorithmic geometry ([DSS05], [JSY07]), biomathematics ([PaSt05]), and economics ([BaKl14], [Jos17]). For an introduction to tropical geometry, we suggest [BIMS15] or [MaSt15].

As mentioned above, one of the roots of tropical geometry lies in Viro's patchworking technique. This theory, invented in the late seventies, provides a powerful method to construct real algebraic varieties with a prescribed topology. It initiated a breakthrough in the study of the topology of real algebraic varieties, leading to the achievement of several classification results ([Vir84], [Ore02]) and to counterexamples of long-standing conjectures ([Ite93], [Bih01], [Bru06]).

In its turn, tropical geometry provides new insights to Viro's patchworking construction. Indeed, following the work of I. Gelfand, M. Kapranov and A. Zelevinsky [GKZ94], O. Viro [Vir01], and G. Mikhalkin [Mik05], Viro's theorem can be interpreted as a "quantization" of tropical hypersurfaces, which are limits of amoebas of complex algebraic varieties. One interesting feature of the tropical version of Viro's patchworking is that tropical varieties can be quantized beyond the case of hypersurfaces, suggesting many ways

to generalize the original patchworking techniques of Viro. Another promising direction has recently been proposed by A. Renaudineau and K. Shaw [RS18] who unveiled new relations between tropical homology and patchworking of non-singular tropical hypersufaces.

Patchworking is intimately related to amoebas and coamoebas. For instance, amoebas and coamoebas of algebraic varieties provide one of the bridges relating tropical geometry with non-archimedean, complex and real algebraic geometry. The (co)amoebas of hypersurfaces of toric varieties are quite well studied ([PaRu04],[PPT13]). However, (co)amoebas of varieties of higher codimension still remain mysterious and constitute an active field of current research. For example, effective algorithms for computations of (co)amoebas is a challenging problem for which recent progress has been made ([FMMW19]). Other recent directions of investigation in this area are non-archimedean valuations with values in an arbitrary ordered group ([Aro10], [Ban15]), and with the generalization of amoebas proposed by I. Krichever ([Kri14], [Eli16]).

Deeply connected to patchworking and amoebas, simple Harnack curves are fascinating and extremal objects that play an important role in real algebraic geometry. They were introduced and classified in [Mik00], and have had a deep impact on subsequent developments in this field. Since then, they turned out to appear in connection with several areas of mathematics ([KOS06]). It has been an important and challenging problem to identify suitable higher dimensional generalizations. Some progress have recently been made toward such a generalization ([BMRS18]), however this direction of research is still largely to be explored.

One approach that the latter works utilize is the concept of real fibered morphisms between real algebraic varieties. Following the work of Passare and Risler [PaRi10], Harnack curves can be equivalently characterized as having a real fibered logarithmic Gauss map. Hyperbolic varieties, which have been proposed as anti-Harnack varieties, can also be characterized via real fibered morphisms to projective space [KuSh16]. In the hypersurface case, these real varieties have an intimate relationship with optimization [Gue97]. The separating semigroup of a real algebraic curve recently introduced by M. Kummer and K. Shaw ([KuSh17]) open a promising directions of investigation regarding extremal real algebraic varieties.

During the last fifteen years, tropical methods in complex and real enumerative geometry have turned out to be extremely fruitful. Not only do they provide a powerful tool to compute enumerative invariants of algebraic varieties, but they have also lead to the discovery of new and striking phenomena. Examples of such progress include the construction of Block-Göttsche invariants ([ItMi13],[BlGo16]), the invention of the floor decomposition technique ([BrMi07], [Bru15]), and piecewise polynomiality results concerning double Hurwitz and Gromov-Witten numbers ([CJM11], [ArBr17]).

A fascinating aspect of tropical enumerative geometry is the existence of tropical invariants that do not have any known classical analogue. This started with the discovery of tropical Welschinger invariants by I. Itenberg, V. Kharlamov, and E. Shustin in [IKS09]. Remarkably, these invariants satisfy the same type of recursive formulas, of so-called Caporaso-Harris type, as the corresponding (relative) Gromov-Witten invariants. This similarity led L. Göttsche and V. Shende to conjecture the existence of refined invariants, Laurent polynomials interpolating between Gromov-Witten and tropical Welschinger invariants, and satisfying a Caporaso-Harris type formula [GoSh14]. These polynomials are conjecturally related to Hodge polynomials of relative Hilbert schemes of points on the underlying complex surface. As a by-product of the invariance of the number of tropical curves equipped with the polynomial Block-Göttsche multiplicities, come many new integer-valued tropical invariants, which turn out to be mysterious from the complex enumerative perspective. New insight into Block-Göttsche invariants has recently been provided by quantum indices of real algebraic curves introduced by Mikhalkin ([Mik17]), by the work of Nicaise, Payne, and Schröeter based on motivic integration and Berkovich spaces ([NPS18]), and by their relations to the so-called Abramovich-Bertram type formulas discovered by E. Brugallé ([Bru18]).

In algebraic and symplectic geometries, several new real and open enumerative invariants have been constructed recently [KhRa15, GeZi18, SoTu16]. Developing tropical techniques to compute and relate these new invariants is an important research direction in this field.

2 Presentation Highlights

Following up on our statement of objectives we list the major research themes anticipated in the workshop and the contributions of participants. As likely to be expected there were some unanticipated topics presented

that had developed since the time of our proposal.

- (tropical) Generalizations of Viro's patchworking; applications to real linear series, to real parabolic locus of affine real hypersurfaces, ...

  The talks **Bounding the Betti numbers of real tropical varieties** and **Combinatorial patchworking and large Betti numbers** respectively by Arthur Renaudineau and Charles Arnal explored the recently discovered relation between tropical homology and patchworking of real algebraic varieties. Angelito Camacho explained in **On the number of transversal special parabolic points in the graph on a real polynomial** how to adapt the original patchworking construction to construct real parabolic locus of real affine hypersurfaces. These talks were nicely complemented by a talk by Oleg Viro, the inventor of the patchworking construction, entitled **Monomial hyperfields and non-combinatorial patchworking**.

  In his talk **Motivic specialization and rationality problems**, Johannes Nicaise gave a very nice application of patchworking complex birational geometry. Two talks were also devoted to application of patchworking to the study of real linear series on real algebraic curves: **On the real inflection points of linear (in)complete series on real (hyper)elliptic curves** by Cristhian Garay, and **The combinatorics and real lifting of tropical bitangents to plane quartics** by Hannah Markwig.

  Beyond combinatorial patchworking strictly speaking, Matilde Manzaroli illustrated in **Real algebraic curves on real minimal del Pezzo surfaces** new developments in application of degeneration methods in real algebraic geometry, and Josephine Yu presented a Berkovich analytification facet of semi-algebraic geometry in **Real tropicalizations and analytification of semi-algebraic sets**.

- Extremal aspects of real algebraic varieties.

  The notion of total reality of morphisms between real algebraic varieties has recently been developed in new interesting directions, in particular through the work of Mario Kummer, who spoke about **Some aspects of total reality** during the conference. In the case of curves, Grigory Mikhalkin, in **On canonical divisors of real type I curves**, completely classified separating semi-groups of hyper-elliptic real algebraic curves.

- Geometry of amoebas of complex algebraic varieties, topology of tropical varieties; tropical homology theory.

  The topology of tropical varieties, as well as their approximation by amoebas of complex algebraic hypersurfaces constituted a central topic in the conference. In his talk **Spines for amoebas of rational curves**, Johannes Rau gave an approximation theorem by amoebas of rational tropical curves in any dimension. This constitutes a subsequent improvement of former approximation results. Omid Amini presented in **Hodge isomorphism for matroids** his recent work establishing an isomorphism between the tropical homology of a matroidal fan and the Chow ring of a matroid. The aforementioned talk by Arthur Renaudineau also provided a striking link between tropical hodge numbers of a non-singular tropical hypersurface and Betti numbers of the real part of its patchworkings.

- Tropical enumerative geometry; relation with real and complex enumerative geometries; tropical refined invariants of algebraic surfaces.

  Two talks were specifically devoted to refined tropical invariants. Andres Jaramillo Puentes gave a talk on **Göttsche conjecture for tropical refined invariants**, that provided a new insight to the celebrated Göttsche conjecture related to enumeration of curves in complex algebraic surfaces. In the talk **The moment problem and its quantization**, Thomas Blomme generalized Mikhalkin's approach of quantum indices of tropical plane curves to curves in higher dimensions.

  Eugenii Shustin exposed a refreshing local interpretation of Welschinger invariants in **Local and global cuspidal invariants of Welschinger type**. This new interpretation is likely to provide a better understanding of global real invariants, even if those have been quite studied for almost two decades. The enumeration of surfaces is much less developed that the case of curves, and Madeline Brandt gave a partial correspondence theorem in **A tropical count of binodal cubic surfaces**.

We also had two talks with more a "classical" perspective, nevertheless providing new potential connections with tropical geometry. Xujia Chen, in **A geometric interpretation of Solomon-Tukachinsky's open Gromov-Witten invariants**, exposed very nicely the geometric pictures behind J. Solomon and S. Tukachinsky's paper. Ilia Itenberg explains how to count **Planes in four-dimensional cubics** via Hodge theory and Niemeyer lattices.

- Topics not anticipated

  Several talks also addressed some related topics that had developed since the time of our proposal. Jens Forsgård explained how to use **The fewnomial approach to nonnegativity** to produce new nonnegativity certificates for polynomial functions. In **Monodromy of rational curves in toric surfaces**, Lionel Lang determined the action of the monodromy on the set of nodes of rational algebraic curves in a given toric surface. In her talk **Universal cluster algebras and the totally positive tropical Gr(2,n)**, Lara Bossinger related cluster algebras to tropicalizations of the positive part of Grassmaniann of planes. Finally, the set of preorders on a group can be endowed with several natural topologies, and Julie Decaup reported on **The compacity of the set of preorders on a group** for all these topologies.

In addition to these high quality talks, the meeting provided a convivial atmosphere suitable for fruitful scientific exchanges.

3 Outcome of the Meeting

From our perspective, the event had better results than we expected. Conferences were of high academic level and well presented, and the participants seemed satisfied and happy. The working conditions were excellent. There was a considerable percentage of women, not only as participants, but also as speakers. The average age of the participants was lower than expected and the participation of the students was very relevant. For several of them, this was the first occasion to present their work to the expert of the mathematical community in these matters. Being able to carry out these events in a city like Oaxaca adds an important tourist and cultural component that all participants could take advantage of.

As we stated in our objectives, it was important that there be interaction between the international community expert in the topics related to the workshop and the Mexican mathematical community. Along these lines, two satellite events were organized, one in Cuernavaca (Morelos) and another in Puerto Escondido (Oaxaca):

- "Tercer Encuentro Tropical". (http://www.matcuer.unam.mx/actividades/3erEncuentroTropical/)

  Organizer: Luca Lpez de Medrano (UNAM).

This event was organized with the support of LASOL (Solomon Lefschetz Laboratory-Joint Unit CNRS-UNAM-CONACyT) and was held at the Cuernavaca Unit of the Institute of Mathematics (UCIM) of the UNAM from September 2 to 6, the week before workshop at the Oaxaca Mathematical House. There were participants from 6 universities in France, Israel, Mexico and Norway. During the meeting each participant presented the progress of their current work, time was left to work in collaborations and talks were given for the entire UCIM community.

In this meeting participated:

- Ilia Itenberg (IMJ-PRG, Sorbonne University);
- Julie Dacaup (Instituto de Matemticas, UNAM);
- Andrs Jaramillo (Nantes University);
- Luca Lpez de Medrano (Instituto de Matemticas, UNAM);
- Matilde Manzaroli (University of Oslo);
- Daniel Meza (Universidad Autnoma del Estado de Morelos);

- Anh Nguyen (Nantes University);
- Arthur Renaudineau (Lille University);
- Kristin Shaw (University of Oslo);
- Evgenii Shustin (Tel-Aviv University).

- "Algebraic Geometry in Mexico 2019". Puerto Escondido, Oaxaca.

  Organizer: Ernesto Lupercio (UNAM).

A mexican researcher and two mexican students met in Puerto Escondido, Oaxaca, to work with Grigory Mikhalkin (Geneva University), who in addition to discussing possible collaborations, gave two keynote lectures.

In this meeting participated:

-Pablo Cesar Cruz Martinez, (Cinvestav);
-Cristhian Garay (CIMAT);
-Grigory Mikhalkin (Geneva University);
-Jenifer Viafara Chanchi, (Cinvestav).

References

[AHK18] K. Adiprasito, J. Huh, E. Katz : Hodge Theory for Combinatorial Geometries. Ann. of Math. (2) 188 (2018), no. 2, 381452.

[ABBR15] O. Amini, M. Baker, E. Brugallé, and J. Rabinoff : Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta. Res. Math. Sci. 2 (2015), 2:7

[Aro10] F. Aroca : Krull-tropical hypersurface., Annales de la faculté des sciences de Toulouse Mathématiques, 19 (2010), pp. 525-538.

[AIL10] F. Aroca, G. Ilardi, and L. Lopez de Medrano : Puiseux power series solutions for systems of equations. Internat. J. Math. 21 (2010), no. 11, 1439-1459.

[ABLdM11] A. Arroyo, E. Brugallé, and L. Lopez de Medrano : Recursive formulas for Welschinger invariants of the projective plane. Int. Math. Res. Not. 2011, No. 5, 1107-1134 (2011).

[ABLdM14] A. Arroyo, E. Brugallé, and L. López de Medrano : On maximally inflected hyperbolic curves. Discrete and Computational Geometry. Volume 52, Issue 1 (2014), Page 140-152.

[BaKl14] E. Baldwin and P. Klemperer : Tropical geometry to analyse demand. http://www.nuff.ox.ac.uk/users/klemperer/Tropical2014.pdf, 2014.

[Ban15] S. D. Banerjee : Tropical geometry over higher dimensional local fields. Journal fur die reine und angewandte Mathematik (Crelles Journal), 2015 (2015), pp. 71-87.

[Ber08] B. Bertrand : Real Zeuthen numbers for two lines. Int. Math. Res. Not. IMRN 2008, no. 8.

[BLdMR13] B. Bertrand, L. Lopez de Medrano, and J.-J. Risler : On the total curvature of Tropical Hypersurfaces. Algebraic and Combinatorial aspects of tropical geometry, proceedings de la conférence Tropical Geometry, CIEM, Castro Urdiales, (2011).

[Bih01] F. Bihan : Une sextique de l'espace projectif réel avec un grand nombre d'anses. Revista Matematica Complutense 14 no 2, (2001), 439-461.

[BlGo16] F. Block, and L. Göttsche : Refined curve counting with tropical geometry. Compos. Math. 152 (2016), no. 1, 115-151.

[BoKa12] T. Bogart, and E. Katz : Obstructions to lifting tropical curves in surfaces in 3-space. SIAM J. Discrete Math. 26 (2012), no. 3, 1050-1067.

[Bru06] E. Brugallé : Real plane algebraic curves with asymptotically maximal number of even ovals. Duke Math. J., 131 (3):575-587, 2006

[Bru15] E. Brugallé: Floor diagrams relative to a conic, and GW-W invariants of Del Pezzo surfaces. Adv. Math. 279 (2015), 438-500

[Bru18] E. Brugallé: On the invariance of Welschinger invariants. arXiv preprint arXiv:1811.06891, 2018.

[ArBr17] E. Brugallé, and F. Ardila : The double Gromov-Witten invariants of Hirzebruch surfaces are piecewise polynomial. Int. Math. Res. Not. IMRN 2017, no. 2, 614641.

[BIMS15] E. Brugallé, I. Itenberg, G. Mikhalkin and K. Shaw : Brief introduction to tropical geometry. Proceedings of the Gökova Geometry-Topology Conference 2014, 1-75, Gökova Geometry/Topology Conference (GGT), Gökova, 2015.

[BrLdM12] E. Brugallé, and L. Lopez de Medrano : Infection points of real and tropical plane curves. Journal of Singularities volume 4 (2012), 74-103.

[BrMi07] E. Brugallé, and G. Mikhalkin : Enumeration of curves via foor diagrams. Comptes Rendus de l'Académie des Sciences de Paris, série I, 345, 2007.

[BMRS18] E. Brugallé, G. Mikhalkin, J.-J. Risler, K. Shaw : Non-existence of torically maximal hypersurfaces. Algebra i Analiz 30 (2018), no. 1, 2031.

[BrPu13] E. Brugallé, and N. Puignau : Enumeration of real conics and maximal configurations. J. Eur. Math. Soc. (JEMS) 15 (2013), no. 6, 2139-2164.

[BrSh15] E. Brugallé, and K. Shaw : Obstructions to approximating tropical curves in surfaces via intersection theory. Canadian Journal of Mathematics 67(2015), no. 3, 527-572

[CJM11] R. Cavalieri, P. Johnson, and H. Markwig : Wall crossings for double Hurwitz numbers. Adv. Math. 228 (2011), no. 4, 1894-1937.

[DLSV12] J. De Loera, B. Sturmfels, and C. Vinzant : The central curve in linear programming, Found. Comput. Math. Volume 12 (4), 509–540, 2012.

[DSS05] M. Develin, F. Santos, and B. Sturmfels : On the rank of a tropical matrix. In Combinatorial and computational geometry, volume 52 de Math. Sci. Res. Inst. Publ., pages 213-242. Cambridge Univ. Press, Cambridge, 2005.

[Eli16] Y. Eliyashev : Geometry of generalized amoebas. arXiv:1608.06077

[FMMW19] J. Forsgård, L. F. Matusevich, N. Mehlhop, T. de Wolff : Lopsided Approximation of Amoebas. Math. Comp. 88 (2019), no. 315, 485500.

[GeZi18] P. Georgieva, A. Zinger : Real Gromov-Witten Theory in All Genera and Real Enumerative Geometry: Construction. Ann. of Math. (2) 188 (2018), no. 3, 685-752.

[GoSh14] L. Göttsche, and V. Shende : Refined curve counting on complex surfaces. Geom. Topol. 18 (2014), no. 4, 2245-2307.

[GrSi06] M. Gross, and B. Siebert : Mirror symmetry via logarithmic degeneration data I. J. Diff. Geom. 72, (2006).

[GrSi14] M. Gross, and B. Siebert : Local mirror symmetry in the tropics. Proceedings of the International Congress of Mathematicians Seoul 2014. Vol. II, 723744, Kyung Moon Sa, Seoul, 2014.

[Gub07] W. Gubler : Tropical varieties for non-Archimedean analytic spaces. Invent. Math., 169(2):321-376, 2007.

[Gue97] O. Güller : Hyperbolic polynomials and interior point methods for convex programming. Mathematics of Operations Research, 22(2):350-377, 1997.

[GKZ94] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky: Discriminants, resultants, and multidimensional determinants. Mathematics: Theory and Applications. Birkhauser Boston Inc., Boston, MA. 1994.

[InIw12] R. Inoue et S. Iwao : Tropical curves and integrable piecewise linear maps. Contemp. Math., 580, pp. 21-39 (2012).

[Ite93] I. Itenberg : Contre-exemples á la conjecture de Ragsdale. C. R. Acad. Sci. Paris, 317, Sér. I (1993), 277 - 282.

[IKS09] I. Itenberg, V.Khalramov, and E. Shustin : A Caporaso-Harris type formula for Welschinger invariants of real toric del Pezzo surfaces. Comment. Math. Helv. 84 (2009), no. 1, 87-126.

[IKS13] I. Itenberg, V.Khalramov, and E. Shustin : Welschinger invariants of real Del Pezzo surfaces of degree at least 3. Math. Annalen 355 (2013), no. 3, p. 849-878.

[IKMZ19] I. Itenberg, L. Katzarkov, G. Mikhalkin, and I. Zharkov : Tropical homology. Math. Ann. 374 (2019), no. 1-2, 9631006.

[ItMi13] I. Itenberg, and G. Mikhalkin : On Block-Göttsche multiplicities for planar tropical curves. Int. Math. Res. Not. IMRN 2013, no. 23, 5289-5320.

[Jos17] M. Joswig : The Cayley trick for tropical hypersurfaces with a view toward Ricardian economics. Homological and computational methods in commutative algebra, 107128, Springer INdAM Ser., 20, Springer, Cham, 2017.

[JSY07] M. Joswig, B. Sturmfels, and J. Yu: Affine buldings and tropical convexity. Albanian J. Math., 1(4):187-211, 2007.

[KOS06] R. Kenyon, A. Okounkov, and S. Sheffield. Dimers and amoebae. Ann. of Math. (2), 163(3):1019-1056, 2006.

[KhRa15] V. Kharlamov, and R. Rasdeaconu : Counting real rational curves on K3 surfaces. Int. Math. Res. Not. IMRN 2015, no. 14, 5436-5455.

[Kon10] M. Kontsevich : Tropical Hodge conjecture. Talk at Genève-Paris-Strasbourg seminar, 14/04/2010.

[KoSo06] M. Kontsevich, and Y. Soibelman : Affine structures and non-Archimedean analytic spaces. In The unity of mathematics, volume 244 de Progr. Math., pages 321-385. Birkhåuser Boston, Boston, 2006.

[Kri14] I. Krichever : Amoebas, Ronkin function, and Monge-Ampère measures of algebraic curves with marked points. Topology, geometry, integrable systems, and mathematical physics, 265-278, Amer. Math. Soc. Transl. Ser. 2, 234, Amer. Math. Soc., Providence, RI, 2014.

[KuSh16] M. Kummer, and E. Shamovich : Real fibered morphisms and Ulrich sheaves. arXiv:1507.06760v3.

[KuSh17] M. Kummer, and K. Shaw : The Separating Semigroup of a Real Curve. arXiv:1707.08227

[MaSt15] D. Maclagan, and B. Sturmfels : Introduction to tropical geometry. Graduate Studies in Mathematics, 161. American Mathematical Society, Providence, RI, 2015. xii+363 pp.

[MMS18] H. Markwig, T. Markwig, and E. Shustin : Enumeration of Complex and Real Surfaces via Tropical Geometry. Adv. Geom. 18 (2018), no. 1, 69100.

[Mik00] G. Mikhalkin : Real algebraic curves, the moment map and amoebas. Ann. of Math. (2), 151(1):309-W326, 2000.

[Mik05] G. Mikhalkin : Enumerative tropical algebraic geometry in R2. J. Amer. Math. Soc., 18(2):313-377, 2005.

[Mik17] G. Mikhalkin : Quantum indices of real plane curves and refined enumerative geometry. Acta Math. 219 (2017), no. 1, 135180.

[NPS18] J. Nicaise, S. Payne, F. Schroeter : Tropical refined curve counting via motivic integration. Geom. Topol. 22 (2018), no. 6, 31753234.

[Ore02] S. Yu. Orevkov. Classification of flexible M-curves of degree 8 up to isotopy. *Geom. Funct. Anal.*, 12(4):723–755, 2002.

[PaSt05] L. Pachter and B. Sturmfels (eds.) : Algebraic statistics for computational biology, Cambridge University Press, 2005.

[PaRi10] M. Passare, J-J. Risler : On the curvature of the real amoeba, In Proceedings of the Gökova Geometry-Topology Conference 2010, (2011),129–134.

[PaRu04] M. Passare, and H. Rullgård : Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope. Duke Math. J. 121 (2004), no. 3, 481-507.

[PPT13] M. Passare, D. Pochekutov, and A. Tsikh : Amoebas of complex hypersurfaces in statistical thermodynamics. Math. Phys. Anal. Geom. 16 (2013), no. 1, 89-108.

[RS18] A. Renaudineau and K. Shaw : Bounding the Betti numbers of real hypersurfaces near the tropical limit. arXiv preprint arXiv:1805.02030. 2018.

[Shu12] E. Shustin : Tropical and algebraic curves with multiple points. Perspectives in Analysis, Geometry, and Topology (On the Occasion of the 60th Birthday of Oleg Viro). Series: Progress in Mathematics, Vol. 296. Birkhauser, 2012.

[Soi10] Y. Soibelman : Complex integrable systems, wall-crossing formulas and tropical geometry. Talk at Genève-Paris-Strasbourg seminar, 14/04/2010.

[SoTu16] J. Solomon, and S. Tukachinsky : Point-like bounding chains in open Gromov-Witten theory. arXiv:1608.02495.

[Vir01] O. Ya. Viro : Dequantization of real algebraic geometry on logarithmic paper. In European Congress of Mathematics, Vol. I (Barcelona, 2000), volume 201 de Progr. Math., pages 135-146. Birkhäuser, Basel, 2001.

[Vir84] O. Ya. Viro : Gluing of plane real algebraic curves and constructions of curves of degrees 6 and 7. In Topology (Leningrad, 1982), volume 1060 de Lecture Notes in Math., pages 187-200. Springer, Berlin,1984.

[Yue16] T. Yue Yu : Enumeration of holomorphic cylinders in log Calabi-Yau surfaces. I. Math. Ann. 366 (2016), no. 3-4, 1649-1675.