Neostability theory

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1 Overview of the Field, Recent Developments and Open Problems.

"Neostability theory" (a subfield of pure model theory characterized by the use of ideas and methods from stability theory in more general "tame" and sometimes "untame" settings) has become one of the main areas of pure model theory. The development of neostability has been propelled by both problems internal to the subject and applications to other branches of mathematics. The theory has achieved a high degree of coherence and technical results and constructions reappear in various contexts.

During the last decade the field has taken shape with many of the most promising young model theorists working around and making very important contributions. Such developments, which we could only hope for when we proposed the first BIRS meeting in 2009, have been influenced and driven in a very significant manner by the previous BIRS meetings in 2009, 2012, and 2015. The fourth BIRS meeting successfully continued to further this pattern.

Starting in the late Sixties and for all of the Seventies and Eighties, stability theory played a central role in model theory. Inaugurated with Morley's celebrated proof of his theorem on theories categorical in an uncountable cardinality, the theory reached a high degree of sophistication with Shelah's classification theory and was then developed by Shelah, Lascar, Poizat and others into what has arguably been the deepest and most applicable of the branches of model theory.

Stability theory reached an apex with the geometric stability theory of, especially, Hrushovski and Zilber. In the late 1970s Zilber introduced the group configuration in his work on totally categorical theories and then Hrushovski generalized the group configuration theorem well beyond these logically perfect theories. In so doing, it was revealed that structures of algebraic or algebraic geometric origin explain the complexity of some very general theories in which there is no apparent geometry or algebraic structure. This approach of analyzing a stable structure according to the geometry of the types was incredibly fruitful with results such as the trichotomy theorem for Zariski geometries by Hrushovski and Zilber and the latter applications to diophantine number theory (including the Mordell-Lang Conjecture for fields of positive characteristic) by Hrushovski and later by Scanlon.

Since the early 2000's, developments started looking to use the fundamental ideas and objects of study of stability theory, such as definable types, in other contexts. These developments prompted us to organize the first BIRS meeting on the subject in 2009. We will now mention what has happened with some of the main subfields in the last decade, emphasizing the more recent developments.

1.1 Recent Developments

The theory of non forking in dependent theories has manifested itself in disparate areas of mathematics. A good example of this is Hrushovski's breakthrough in additive combinatorics which is now known as the "non-abelian Freiman theorem", which used the idea that non forking could be developed relative to any good notion of smallness (in the sense of ideals or measure zero sets) together with a generalization of the generically presented group theorem to complete the proof. This idea of applying (infinite) model theoretic techniques to problems in combinatorics (which are by essence finite contexts), was then used by Malliaris and Shelah to obtain a stronger version of Szemerédi's Regularity Lemma for stable graphs, and was then generalized to a regularity lemma with respect to arbitrary Keisler measures by Malliaris and Pillay. On a different approach, Terry also has some very interesting extremal graph results using 0-1 laws of both Fraïssé limits and ultraproducts. Finally, work of Chernikov and Starchenko showed that in many regularity theorems one needed some form of non stability, and that the use of the orderability of the real numbers was needed in the proof of Erdős-Szekeres Theorem, bringing the concept of "distality" as an important tool for applications of model theory to combinatorics. This is a very active field with new research by, among others, Chernikov, Peterzil and Starchenko showing more interactions between combinatorics and model theory.

Pseudofinite model theory (the theory of ultraproducts of finite structures) has seen a very strong development in recent years. Besides the regularity theorems mentioned before, Pseudofiniteness of the triangular-free random graph (or the Urysohn sphere) is a long-standing open question, and possibly the work of Conant and Terry on the Urysohn space (where they characterized dividing and forking in continuous logic) may be an important tool to attack this problem. Kruckman gave a criterion for pseudofinitness of a countably categorical theory. Finally, foundational work of García, Steinhorn and Macpherson gives necessary condition for stability or simplicity, and describes the relation between forking and dimension in pseudofinite structures. Among the recent work, we mention results of Bukh, Hrushovski and Zimmerman on proper intersections and modularity, Wagner on the existence of big abelian subgroups in pseudofinite groups with almost chain condition on centralisers, and García and Wagner on unimodularity in the stable context.

A different line of study, which is also related to the existence of finitely additive measures on definable sets, is amenability which has been particularly useful in studying the topological dynamics of both definable groups and automorphism groups. Definable amenability of a definable group G, meaning the existence of a translation invariant finitely additive probability measure on the Boolean algebra of definable subsets of G, played a decisive role in Hrushovski, Peterzil, and Pillay's proof of Pillay's Conjecture for definably compact groups in o-minimal expansions of the real field. The use of definable amenability was recently extended by Montenegro, Onshuus and Simon to NTP_2 theories where a stabilizer theorem was proved and used to show that amenable groups in geometric fields were isogenus to algebraic groups.

This class of NTP_2 theories has established itself as a very adequate generalization of both simple and dependent theories. Recent work has shown that in many cases one can generalize common results on simple and dependent theories to this context. Chernikov and Kaplan's proof of Kim's Lemma about the behavior of forking, Hempel's work on groups with the almost chain condition on centralizers, and existence of definable abelian "envelopes" of infinite abelian subgroups are good examples.

In a different direction, work of Chernikov and Ramsey give a criterion for NSOP_1 based on the existence of an equivalence relation satisfying the independence theorem. Very recently, Ramsey showed that any NSOP_1 theory admits a symmetric notion of independence which satisfies the independence theorem. This gives good hope of being able to generalize certain techniques from simple theories to NSOP_1, which would open up a new set of structures approachable by model theory. This tools are now being used by many young research, and developments of NSOP_1 theories were some of the main highlights of the meeting.

Finally, the foundational study of unstable structures has continued to expand and our understanding of various classes of theories has grown substantially. Shelah now has a "(re-)counting of types" characterization of dependent theories, while efforts towards understanding the different properties of non forking in dependent theories have continued with work by Chernikov, Kaplan, Simon and Usvyatsov, with a highlight in Simon's proof of a decomposition theorem for types in dependent theories into a stable and a distal-like part.

1.2 Open Problems

We list a few outstanding conjectures and open problems.

Stable forking conjecture and related problems: For the last decade or so, no progress has been made towards settling some of the open problems in simplicity theory. The stable forking conjecture (even in the supersimple case), elimination of hyperimaginaries in simple theories, the equivalence between forking and thorn forking are all questions that seem as far away from being solved as we were ten years ago.

Minimal fields: Another famous open problem in the field is whether or not a minimal field (one where every definable set is finite or cofinite) is algebraically closed. It is known that the results holds if the characteristic is positive or if no partial order is defined (this was partially solved at a previous Neostability meeting). Solving this conjecture (by Podewski) is an important goal for our field.

Equivalence of $NSOP_1$ and NTP_1: As reported previously, both the concepts of NTP_1 and $NSOP_1$ complement NTP_2 as the "other side" of generalizing simple theories (a theory is simple if and only if it has NTP_2 and either NTP_1 or $NSOP_1$). However, this branch of neostability has not been as active as NTP_2, to the extent where we do not know yet whether any NTP_1 has $NSOP_1$. This equivalence (or non equivalence) together with an understanding of definable sets in such theories, is an area which should be explored.

2 Testimonials

It was generally felt that this has been an excellent meeting, with talks of very high quality, at the same time giving an opportunity to speak to everyone who wanted to. The impression was given of "pure" model theory being a very dynamic area with major contributions coming from the younger generation. A particular highlight were the series of talks on $NSOP_1$, underlining the robustness of the notion and the scope for further development.

Some of the comments we received from the people who attended the workshop:

- **Enrique Casanovas:** I found the talks of Pierre Simon, Itay Kaplan, Gabe Conant and Nick Ramsey extremely good. I am deeply interested in NSOP1 theories, so I was very happy with the many talks dedicated to this stuff. I had the opportunity to talk to Byunghan Kim and we planned some joint work, which in fact we are finishing now. On the other hand I have some work done with Silvia Barbina on Steiner Triple Systems (the Fraïssé limit has a NSOP1 theory) and we plan to continue our work. For this it was interesting to see what John Baldwin had to say on strongly minimal Steiner systems.

- **Lynn Scow:** I was able to extend my results at the meeting as well as get a lot of excellent feedback from other researchers and possibly start a new collaboration. I heard about a lot of new directions and new work on old directions and it was fantastic. I thought the meeting was excellent and I really appreciated the ability to attend.

- **Dugald MacPherson:** It was a really good meeting, excellent talks, good opportunity for discussion, excellent organisation.

  For me, the big thing was a chance to push joint work with Sylvy Anscombe and Charlie Steinhorn really useful discussions pushing things along for what I think is a substantial paper on our notion 'multi-dimensional asymptotic classes. I had useful shorter discussions with a number of other people, though I am not sure if any will lead to a specific collaboration.

- **Assaf Hasson:** The meeting was excellent. I found the choice of speakers and subjects to be very good with an excellent balance between longer talks on new exciting results (such as the talks by P. Simon, Y. Peterzil, K. Krupinski) and shorter talks focused on more local results (some of them on results entirely new to me, such as A. Kruckman's talk). The (natural) focus on pure model theory gave an excellent overview of the new directions and trends in the subject (e.g., the series of talks on NSOP1).

- **Zoe Chatzidakis:** I very much enjoyed the talks of Kruckman and Jimenez, and appreciated very much their contents (which were new to me).

- **Kobi Peterzil:** Some replies

  1. Progress made :Made significant progress in my work with Starchenko and Chernikov on Elekes-Szabo type of theorems in stable and o-minimal settings.

  2. Some Highlights: Topological dynamics, P-adic groups, NIP and linear orders

- **Byunghan Kim:** I was particularly impressed by achievements made by young scholars, and happy to confirm that the future of model theory is bright and promising. In addition I found model theory is growing in South American region.

- **Charlotte Kestner:** I really enjoyed the meeting, generally it got me up to speed with what is going on in the subject. I spent some time working with Sylvy. I also had some useful chats with Gabriel Conant and John Baldwin about strongly minimal graphs and the Urysohn sphere.

- **Léo Jimenez:** I found that the meeting was really nice and useful for me. It gave me a great opportunity to give my first real talk, and the other talks were fantastic in giving a broad overview of neostability. I also enjoyed the location, food, and friendly atmosphere.

- **Alex Kruckman:** I enjoyed all the talks very much. The biggest benefit of the meeting, for me, was talking to Silvain and Samaria, and separately to Christian d'Elbee, about work that they're doing which is similar in spirit to the subject of my talk. This should hopefully lead to collaborations and common generalizations down the road. While we're all currently busy with job applications and other work, I also had time to plan future projects with Gabe and Nick, who I have collaborated with before. So it was very productive overall.

3 Presentation Highlights

One of the most recurrent topics throughout the talks of the workshop was NSOP_1 theories. Kim gave a talk introducing the subject, followed by d'Elbee's example of an NSOP_1 theory consisting of an algebraically closed field with a generic subgroup of the additive group. Ramsey then spoke of Pseudo Algebraically closed fields and gave conditions for the field having not having SOP_1. Finally, a related result is Kruckman's report on work with Walsberg and Tran where they created a new tool for constructing models ("interpolative fusions") and proved that the interpolative fusion of NSOP_1 theories has NSOP_1.

Another of the remarkable result presented during the workshop, was Simon's theorem on linear orders in dependent theories. This construction of a V-definable linear order in any unstable set, together with the characterization of ω-categorical structures of thorn rank 1 is a very important step towards understanding more on ω-categorical structures.

Finally, many of the people attending the meeting found Krupinski's report on his work with Hrushovski and Pillay very impressive. The connections between topological dynamics and compactifications in definable groups and their types is a very nice continuation of Newelski's ideas of understanding the interactions between model theory and topological dynamics.

4 Scientific Progress Made

The main scientific progress of the neostability meetings, has been the diffusion and consolidation of the different efforts and results in the area, which has generated a community of researchers working on the field. People explicitly stated this in their testimonials, and the community building that has been achieved during the BIRS meetings is clear and significant when one sees the evolution of stability-theoretic methods in non stable contexts such as dependent, NTP_2, and NSOP_1 theories. The interaction we have been able to have at each of the BIRS meetings has allowed for the consolidation of a robust field of research, with many developments coming from young mathematicians. This was particularly clear at the 2018 meeting in Oaxaca and it was very fulfilling to see.

Some more specific results participants reported during the meeting are the following: MacPherson reported substantial progress in his work with Anscombe and Steinhorn on multi-dimensional asymptotic

classes. Peterzil reported significant progress in his work with Starchenko and Chernikov on Elekes-Szabo type problems. Onshuus and Simon reported progress in their work of classifying dependent ω-categorical structures of finite thorn-rank. And various people reported progress in different approximations to understanding NSOP_1 structures.

5 Gender (im)balance and plans for improvement

Although efforts were been made to reduce gender imbalance when inviting participants and inviting speakers, more can still be done in subsequent meetings particularly towards reducing the imbalance of people who actually agree to give talks. For instance, in an effort to broaden participation, we offered *all* of our invited participants an opportunity to speak. This had the unintended consequence of reducing the proportion of women speakers in that they disproportionately opted not to speak.

Not only will we take affirmative steps to ensure a better gender balance with future meeting, but we will also address the differential treatment of speakers by the audience. What we have in mind here is how some people, whether they be women, younger mathematicians, or people who are uncomfortable speaking in English, tend to be subjected to more aggressive questioning. Of course, in any particular case, because there may be very good neutral reasons for the audience's intervention, we do not mean to impute any discriminatory motives. However, even when some such correction is merited, the overall effect is to further discourage the participation of already marginalized people. For future events, we intend to remind our participants that while some level of skepticism and immediate requests for clarification are appropriate and healthy for our scientific exchange, people should remain respectful and avoid unnecessarily aggressive public questioning. We also intend to direct our session chairs to moderate the questioning.

6 Outcome of the Meeting

This meeting offered a very rich array of results. The contributions made by younger mathematicians, and the coherence displayed, shows that neostability theory is consolidating into a very exciting and fruitful area. Particularly striking is the robustness of the class NSOP_1, an area in which very little research was done in 2009 (when the first neostability workshop was organized). It is notable that during working sessions at the 2009 Banff meeting, Kim proposed studying the independence notion which now bears his name as a way to explain the tameness of some theories we now recognize to fall into the NSOP_1 classification.

The meeting had the effect of solidifying the body of work in neostability theory as a coherent project to discern robust divisions within the class of all theories and to develop stability theoretic methods in an appropriate level of generality.