Low Dimensional Topology and Gauge Theory

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1 Overview of the Field

The study of manifolds of dimensions at least 5 had remarkable success in the 1960's, with the resolution of fundamental problems about existence and uniqueness of smooth or PL structures by a mixture of handlebody theory, surgery theory, and homotopy theory. These methods, while they tell us much about lower dimensional manifolds, do not apply in the same generality in dimensions 3 and 4. Starting in the 1970's, new tools were developed, revealing a whole new universe of phenomena that do not appear in higher dimensions.

The study of 3-manifolds, originally based on combinatorial methods, came to be dominated by the geometric methods introduced by Thurston. His remarkable geometrization program was finally completed by Perel'man [32, 33, 34] using the analytical machinery of Ricci flow. The study of 4-manifolds has also seen remarkable progress, although one still hopes for a broad vision comparable to Thurston's, and classification comparable to what is achieved via geometrization. A new and unexpected aspect in dimension 4 was the divergence of the topological and smooth categories, due to almost simultaneous breakthroughs by Freedman and Donaldson in 1981. Subsequent revolutions in gauge theory, described below, have greatly expanded the power of these initial insights, leading to diverse applications to topology and geometry in dimension 4.

Based on earlier work of Casson [2], Freedman showed [11] that the tools of high-dimensional topology could be applied to simply connected *topological* 4-manifolds, leading to a complete classification in terms of the intersection form in that case. He subsequently [12] extended these topological results to manifolds with *good* fundamental groups, where to say a group G is 'good' roughly speaking means that G contains no non-abelian free group. An important aspect of Freedman's work, that will reappear in different guise in the smooth case, is the interaction between 3-manifolds and the 4-manifolds they bound. For instance, Freedman's simply connected theory implies that any 3-manifold with the homology of the 3-sphere bounds a homology ball (in fact a contractible manifold); his non-simply connected results show that certain classes of knots in the 3-sphere are *topologically slice*, i.e. bound locally flat disks in the 4-ball. The smooth versions of these sorts of investigations are connected to classical topics; for instance questions about the structure of the smooth homology cobordism group (now resolved by Manolescu [28]) were seen to be equivalent to the problem of existence of triangulations of high-dimensional manifolds.

Months after Freedman announced his results, Donaldson [3] made the amazing discovery that the study of Yang-Mills moduli spaces leads to restrictions on the intersection form of smooth 4-manifolds. Coupled with Freedman's classification and construction results, Donaldson's theorem was seen to imply the existence of exotic smooth structures on $\mathbf{R}^4$. No such phenomenon occurs in any other dimension: smooth structures on $\mathbf{R}^n$ are unique for $n \neq 4$. (Later work of Taubes [35] using manifolds with periodic ends extended this to give uncountably many smooth structures on $\mathbf{R}^4$.) Other applications of Donaldson's theorem showed that not every homology 3-sphere bounds a smooth homology ball; a refined version shows that in fact the *homology cobordism group* Θ_3^H is infinitely generated [17, 9]. Similarly, Donaldson's theorem showed a large gap between the smooth and topological concordance groups (knots modulo smoothly or topologically slice knots).

Donaldson's theorem seemed like an isolated result until Donaldson [5, 6] used Yang-Mills theory in an equally surprising way to give invariants of smooth manifolds, and to establish the non-triviality of these invariants for complex Kähler surfaces. Donaldson's invariants were very powerful, leading for instance to the discovery of infinitely many smooth structures on closed manifolds (again, something that does not happen in other dimensions), but still rather difficult to calculate even in the case of complex surfaces. In 1994, Seiberg and Witten [37] introduced a new set of equations that were much easier to work with, leading to many new examples and to the resolution [22] of the famous Thom conjecture on the genus of an embedded surface in complex manifolds such as $\mathbf{C}P^2$. The advent of the Seiberg-Witten equations (and the invariants they define) brought forth an explosion of work in the area. An important aspect of this was that the Seiberg-Witten equations on symplectic manifolds (whose topological study was also creating great interest around the same time) could be solved fairly explicitly. Fundamental work of Taubes [36] connected these solutions to the study of pseudo-holomorphic curves, introduced by Gromov as a bridge between complex and holomorphic geometry.

The definition of Donaldson's invariants rested on a detailed analysis of the Uhlenbeck compactification of the moduli space. That analysis led to further restrictions [4] on the intersection forms of spin 4-manifolds, yielding the most basic case of the fundamental $11/8$ inequality (the ratio between $b_2(X)$ and the signature of X is at least $11/8$) for spin manifolds, conjectured in the 1970s by Kas and Kirby. In a fundamental work, Furuta [16] showed how the stronger compactness properties of the Seiberg-Witten equations could be used to attack this problem, and established a lower bound of $10/8$. His ideas subsequently gave new approaches to questions about group actions on 4-manifolds, as well as a important reinterpretation of the Seiberg-Witten invariant as an element of an equivariant stable homotopy group. In recent years this approach has led, in the hands of Manolescu and others, to new directions in Floer theory and the resolution of many hard problems; see section 2 for a discussion.

The role of symplectic geometry in 4-manifold theory extends even beyond the work of Taubes. Donaldson [7] showed that symplectic manifolds in all dimension admit the structure of a Lefschetz pencil, with Gompf [18] (building on work of Thurston-Winkelnkemper) providing a converse result. This means that symplectic 4-manifolds can be studied using the mapping class group of surfaces, a well understood object. Conjectures from the time suggested that perhaps symplectic manifolds could be the building blocks for all simply connected manifolds, in the same sense that hyperbolic 3-manifolds are the building blocks in dimension 3. Work of Szabó and many others shows this to be overly optimistic, but the search for an analog of Thurston's geometrization program remains intriguing.

A parallel development to Donaldson's theory was the development of Floer homology [10], originally for homology 3-spheres. Formally, Floer homology is the Morse homology associated to the Chern-Simons functional on the space of connections (modulo automorphisms) on a bundle over 3-manifold. Among other nice properties, the Donaldson invariants (originally defined for closed manifolds) of a manifold with boundary live in the Floer homology of the boundary [8]. This property, along with gluing theorems generalizing Donaldson's connected sum theorem, became a crucial tool in calculating Donaldson invariants. Because the critical points of the CS functional are representations of the fundamental group of the 3-manifold, one starts to see connections between gauge theory and the geometric approach to 3-manifolds.

The Seiberg-Witten equations also have good gluing properties, which (among other properties) led to the rapid rapid progress in the theory. It took some time before the corresponding Floer theory was developed by Kronheimer and Mrowka [24]. Their *monopole homology* groups are technically sophisticated and have the great advantage of applying to arbitrary 3-manifolds. The structure of monopole homology was greatly influenced by the introduction of Heegaard Floer homology groups by Ozsváth-Szabó [29, 30] around 2000. This remarkable development grew out of an effort to understand Seiberg-Witten theory in a more combinatorial fashion, and has numerous applications in 3 and 4-dimensional topology, indeed more than we can sensibly discuss here.

Some of the strongest applications of Heegaard Floer theory in fact occur on that interesting borderline between dimensions 3 and 4 alluded to above. For instance, the Ozsváth-Szabó *correction terms* (or d-invariants) [31], analogous to similar invariants defined by Frøyshov [13, 14, 15] via Floer theory and monopole theory, lead to very strong results about rational homology cobordisms. A host of new Heegaard Floer knot invariants gives a great deal of new information about the smooth concordance groups. For instance, Hom [19, 20] defined such an invariant to produce a $\mathbf{Z}^{\infty}$ summand in the kernel of the map between the smooth and topological concordance groups.

2 Recent Developments and Open Problems

Recent developments in the field include work by many authors. Bauer-Furuta extended the Seiberg-Witten invairants to a cohomotopy invariant of smooth 4-manifolds that can detect inequivalent smooth structures on some reducible 4-manifolds. Manolescu developed a verison of Pin(2) Floer theory based on a spectra. This new version of Floer theory resolved the long-standing triangulation conjecture by showing that there are topological manifolds in every dimension greater than 4 that are not homeomorphic to any simplicial complex. Lin developed an alternative approach to the Pin(2) theory based on the monopole version of Floer theory. This year Feehan and Leness completed the proof at Witten's conjecture relating the Seiberg-Witten invariants to the Donaldson polyomial.

Kronheimer-Mrowka resolved Property P using a combination of symplectic topology and gauge theory. Older techniques in gauge theory are finding new results and applications. Kronheimer-Mrowka used a version of Instaton theory to show that Khovaov homology detects the unknot. There is a philosophical link between Floer theories and categorified knot polynomials via a transition from exact triangles in the Floer theory to skein relations in knot theory. Several researchers are exploring this connection. One of the most amazing developments is a gauge theory approach to 4-color conjecture by Kronheimer-Mrowka.

Constructions and botany of small 4-manifolds remains an active area of research. Such constructions of use a combination of techniques (FS knot surgery, Luttinger surgery, fiber sums). More recently, mapping class group techniques are finding application construction of exotic structures of small manifolds. The same mapping class group techniques also apply to Lefschetz pencils and symplectic 4-manifolds.

Many results about 4-manifolds generalize to results about surface in 4-manifolds (knotted surfaces). Wall's classical result about the stable equivalence of smooth structures on homeomorphic 4-manifolds (first proved in the simply-connected case and later generalized by Gompf to general fundamental groups) makes periodic appearances in current research. The analogous questions for surfaces have received a fair amount of attention in recent years. One large break-through in this area is Gabai's light bulb theorem. The theory of corks is also developing. Examples of finite and infinite order corks have been constructed, and the original cork theorem has been generalized to see that any finite collection of smooth structures on a given topological 4-manifold are related by a single finite-order cork. In contrast, it has also been shown that for many manifolds no single cork can generate all smooth structures.

The following are a few of the outstanding open problems in the theory of 4-manifolds.

1. Construct exotic structures on the smallest 4-manifolds (or prove that such structures do not exist). Examples of interest include: S^4, $\mathbf{C}P^2$, $S^2 \times S^2$, $\mathbf{C}P^2 \# \bar{\mathbf{C}P}^2$ as well as $S^1 \times S^3$, and $T^2 \times S^2$. The case of S^4 is perhaps the most famous question in the field. This is the smooth Poincaré conjecture in 4 dimensions.

2. Do any two homeomorphic, simply-connected 4-manifolds become diffeomorphic after one stabilization, i.e., taking connected sum with $S^2 \times S^2$.

3. Does every smooth 4-manifold with $b_2^+ > 2$ have simple type?

4. Does every $\mathbf{Z}[\mathbf{Z}]$-homology $S^1 \times S^3$ admit a homology sphere section?

5. Which contractible manifolds admit a Stein structure?

3 Presentation Highlights

The first talk of the workshop was given by David Gabai on the 4 dimensional light bulb theorem. This theorem represents a promising breakthrough in the field. It is interesting because the result and proof only require much older techniques. It could have been proved using techniques from the early 1970s. In particular it does not rely on Casson handles, Freedman's work, or any gauge theory. An antecedent of this theorem is the 1-stable equivalence result form 2013 of Auckly, Kim, Melvin and Ruberman [1]. The AKMR paper used the light bulb idea in a specific family of examples.

Roughly (ignoring some hypothesis), the light bulb theorem states that two smooth surfaces with a common geometric dual sphere are isotopic. The exact statement follows:

Theorem 3.1 *Let M be an orientable 4-manifold such that $\pi_1(M)$ has no 2-torsion. Two embedded 2-spheres with common transverse sphere are homotopic if and only if they are ambiently isotopic via an isotopy that fixes the transverse sphere pointwise.*

The reason this is such an exciting result is that it sheds light on the basic problem of stable topology. Namely, how many stabilizations are required. The case of pairwise stabilization is an immediate consequence. In other words, if (X, F_1) (X, F_2) are a pairs of a smooth 4-manifold with embedded surface such that F_1 and F_2 are homologous, then $(X, F_1)\#(S^2\times S^2, p\times S^2)$ is isotopic to $(X, F_1)\#(S^2\times S^2, p\times S^2)$. Indeed, the two stabilized surfaces have a common geometric dual. Two weeks before the conference, Auckly, Kim, Melvin, Ruberman, and Schwartz used the light bulb theorem to show that such surfaces become isotopic after one external stabilization.

Theorem 3.2 *If X is a smooth simply-connected 4-manifold and $\alpha \in H_2(X)$ is an ordinary class, then any two closed oriented surfaces F_0 and F_1 in X of the same genus representing α, both with simply-connected complement, are smoothly isotopic in $X\#(S^2\times S^2)$ (summing away from $F_0\cup F_1$). When α is characteristic, the same result holds if one stabilizes by summing with $S^2\tilde{\times}S^2$.*

This make one wonder if progress may now be made in the absolute case.

The second talk followed a line of ideas related to exotic smooth structures on small 4-manifolds. After Donaldson's original work, Kotchick constructed exotic structures on the projective plane blown up at eight points. For many years, this was the smallest known closed manifold to admit multiple smooth structures. Later, it was discovered that by replacing linear configurations of (complex projective) lines by smaller manifolds with the same boundary, exotic smooth structures could be constructed on smaller manifolds. This is known as the rational blow-down process. More recently, researchers have been considering other configurations of lines that could be surgered out to obtain exotic structures on smaller manifolds. Sumeyra Sakalli is a Ph.D. student of one of the organizers, Anar Akhmedov. She spoke about exotic structures on small manifolds obtained by surgering the configuration of all lines passing through four points in the plane.

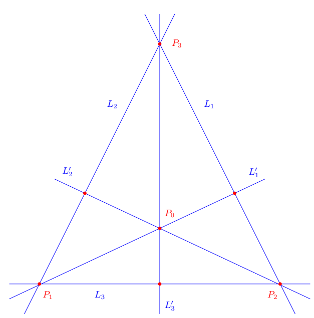

Figure: All lines through four points

Their result is

Theorem 3.3 *Let M be $(2n-1)\mathbb{C}P\#(2n-1)\overline{\mathbb{C}P}$ for any integer $n \geq 12$. Then there exist an infinite family of non-spin irreducible symplectic 4-manifolds and an infinite family of irreducible non-symplectic 4-manifolds that all are homeomorphic but not diffeomorphic to M.*

We will not review all of the talks from the workshop, but just pick some select highlights. The next talk we highlight is the one by Kouichi Yasui. There are many 4 manifolds that admit many smooth structures. A natural question is if there is some set of moves that can generate all smooth structures on a given manifold. The cork theorem gives one answer to this question. It states that any two smooth structures on simply-connected 4-manifold are related by a cork twist – remove the interior of a compact contractible piece and glue the piece back via some diffeomorphism of the boundary. The following figure displays the first cork that was discovered.

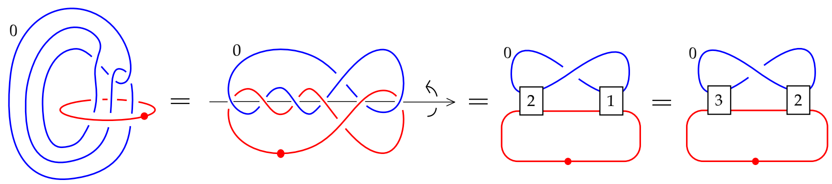

Figure: The Akbulut Cork

The cork theorem is analogous to Alexander's theorem on PL manifolds. Alexander's theorem gives an infinite list of moves that will transition from any combinatorial triangulation to any other combinatorial triangulation of the same problem. There are infinitely many contractible compact 4-manifolds and an infinite number of possible cork twists. Pachner showed that there is a finite set of moves that relate any two PL triangulations of the same manifold. Thus oe is led to ask if there is a finite set of corks that may be used to relate any two smooth structures on the same simply-connected 4-manifold. Yasui showed that in the simplest sense the answer to this question is no. One must use infinitely many corks. Specifically,

Theorem 3.4 *For each positive integer n, there exists an infinite family of homeomorphic, simply-connected, closed 4-manifolds such that, for any pair (X, W) with $b_1(W) < n$, the family can not be generated by twisting W.*

Hanna Schwartz showed (joint work with Paul Melvin) that given any finite collection of smooth structures there is a fixed cork that may be twisted relating all. This generalized a family of examples constructed by Auckly, Kim, Melvin and Ruberman. In fact, removing the compactness assumption, one may find a contractible bit that may be removed and twisted to generate all smooth structures. In outline, one considers a collection of 2-handlebody corks relating each smooth structure in the collection to a given one. By general position, the cores of the two handles may be taken to only intersect in finite collections of points. Repeating the proof of the cork theorem, one can find a single compact contractible manifold 2-handlebody that contains all of these corks. Furthermore the complement may be taken to be simply-connected. Since any contractible 2-handlebody embeds into $\mathbb{R}^4$. Thus one may take copies of the new big contractible bit in coordinate patches that are disjoint and disjoint from the one large contractible manifold. Twisting one copy of each of the original corks in one copy of the big contractible, taking an internal boundary sum and a diffeomorphism of the boundary cycling the factors completes the construction.

Bob Gompf talked about the analogous construction on exotic $\mathbb{R}^4$s showing that the diffeotopy groups of these spaces can be very complicated. At first thought, it may be surprising to hear that one can distinguish mapping classes of an exotic $\mathbb{R}^4$. The extra data in this situation is the action of the diffeomorphism on the end of the 4-manifold. This is analogous to considering the action of a diffeomorphism on the boundary of the manifold. Biji Wong demonstrated how Floer theory could be used to establish the non-triviality of the equivariant corks in the AKMR family.

There is some debate about terminology in this area. It is analogous to replacing one-to-one by the term injective. Selman Akbulut started his talk with exposition arguing for the importance of requiring corks to be Stein. This condition was part of the original definition of a cork, and some would call a non-Stein contractible, compact manifold with boundary mapping classes that do not extend a loose cork, while others would call such objects corks and the objects with more structure Stein corks. Selman continued with some very interesting examples that will certainly motivate further study. One such is in the picture from his lecture listed below. This picture is worth more than one thousand words to explain why small conferences in 4-manifold like the August 2017 one at Casa Matemática Oaxaca are so valuable. It is very difficult to write a complete description leading one to consider a Kirby diagram such as the one in the picture below, but much easier to describe the motivation in person. Right now, the only place researchers can go to learn this material is the lecture video on the conference website.

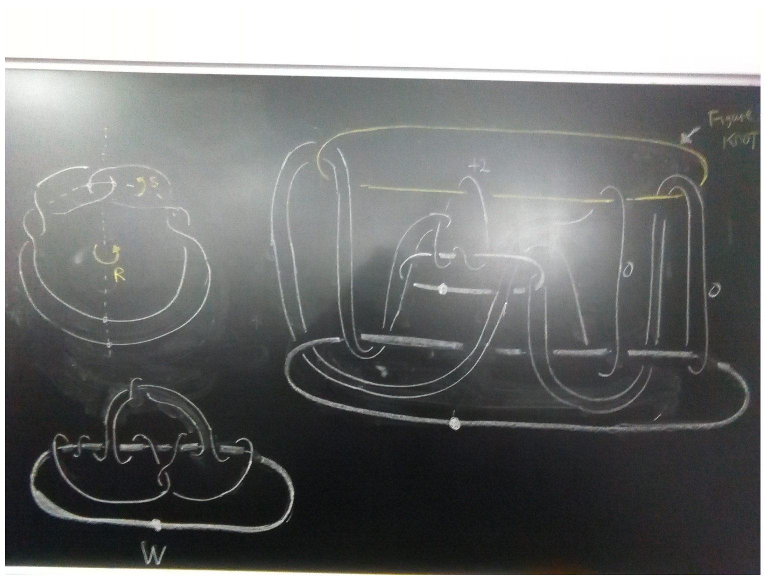

Figure: An interesting small 4-manifold

No matter the terminology, the question of the existence of compact contractible 4-manifolds with no Stein structure (in either orientation) is interesting. Tom Mark spoke about exactly this question.

Inanc Baykur spoke about applications of relations in the mapping class group to constructing exotic smooth structures on small 4-manifolds. This technique also yields results about Lefschetz fibrations and is complementary to the techniques based on configurations of lines. His talk also tied in to the talk of Naoyuki Monden on signatures of surface bundles.

There is a revival of classical gauge theory techniques. By classical we mean techniques developed prior to the introduction of the Seiberg-Witten equations. Examples include an approach to the 4-color problem, and framed instanton homology. One example of this revival is Steven Sivek's characterization of $\mathrm{SU}(2)$-cyclic surgeries via analysis of the pillowcase. The pillowcase is the representation space of $\mathbb{Z}^2$ into $\mathrm{SU}(2)$. Any representation will take a pair of generators to a pair of elements that commute. It is well known that any pair of commuting matrices in $\mathrm{SU}(2)$ may be simultaneously diagonalized by conjugating. Since the determinant is equal to one, such a matrix is determined by the $(1,1)$ entry. By possibly one further conjugation by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

one may assume that the $(1,1)$ term of the image of the first generator has argument in $[0,\pi]$. The argument of the image of the second generator will be in $[0,2\pi]$ and after identifying points on the boundary of this rectangle that correspond to the same representations, one sees that it is the $(2,2,2,2)$-orbifold. This is a shape that looks like a pillow-case. The same orbifold also appears as the representation space of the free group on two generators with certain trace restrictions. This is important in further versions of Instanton homology.

Of course the study of classical gauge theory is now informed by discoveries coming from Seiberg-Witten theory and Heegaard Floer theory. To first order these are all equivalent theories meaning something found in one theory is likely to have an analogue in the other theories. However there are differences. For example, the fundamental group is related to the classical model in a transparent way because the space of flat connections may be identifies with representations of the fundamental group. When the Seiberg-Witten equations were introduced, Witten conjectured an exact relation allowing one to compute the Seiberg-Witten invariants from the Donaldson invariants and vice-versa. While the physical argument connecting these two theories is still well out of reach, there was a mathematical approach that could (and did) lead to a proof of this conjecture. In a tour-de-force Paul Feehan and Tom Leness worked through the technical details of a mathematical program establishing Witten's conjecture. This work was started in 1994 and was just completed this year. Paul gave an overview of this body or work. The simple-type version of the result is as follows.

Theorem 3.5 *Let X be a standard four-manifold. If X has Seiberg-Witten simple type, then X has Kronheimer-Mrowka simple type, the Seiberg-Witten and Kronheimer-Mrowka basic classes coincide, and for any $w \in H^2(X;\mathbb{Z})$ and $h \in H_2(X;\mathbb{R})$ the Donaldson invariants satisfy*

$$\mathbf{D}_X^w(h) = 2^{2-(\chi_h - c_1^2)} e^{Q_X(h)/2} \sum_{\mathfrak{s}\in Spin^c(X)} (-1)^{\frac{1}{2}(w^2+c_1(\mathfrak{s})\cdot w)} SW_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}),h\rangle}. \tag{1}$$

The basic gauge theory invariant is computed as the number of solutions to a system of differential equations. If the expected dimension of the space of solutions is positive, one adds extra conditions to get a zero dimensional set of solutions that are then counted. When the expected dimension of the space of solutions is negative one could consider a family of spaces to generate a zero dimensional space of solutions for the family. The result is an invariant of families. This idea has been noted and used in the past. This past Spring Hokuto Konno introduced a complex of surfaces in a 4-manifold that is clearly a smooth invariant of the manifold and then defined cohomology classes of this complex using moduli spaces with expected negative dimension. This is a wonderful new idea in gauge theory, that is likely to have further applications. His talk was a very clear exposition of this idea.

It is worth explaining this nice new idea. Let $(X,(s))$ be a Spin^c 4-manifold. Now set

$$V(X) := \{\Sigma^2 \subset X \,|\, [\Sigma]\cdot[\Sigma] = 0, \text{ and } \max(0, 2g-2) < |c_1(\mathfrak{s})[\Sigma]|\}.$$

Let $K(X)$ be the simplicial complex with vertices $V(X)$ such that any set of embedded surfaces in $V(X)$ that are pairwise disjoint form a simplex. Clearly, this is a smooth invariant of the Spin^c 4-manifold X. It is a massive complex because there are an uncountable number of such embedded surfaces in X. A cohomology class will associate an integer to each simplex in $K(X)$. Given the surfaces representing a simplex, one may construct a family of metrics stretching the radius of the tubular neighborhood of each surface independently out to a long fixed length. When applied to this family, the a moduli space of negative expected dimension will give a zero-dimensional family of solutions. The signed count of these solutions is the integer one associates to the simplex. One shows that the result is a well-defined cohomology class independent of any extra information resulting in what Konno calls the cohomological Seiberg-Witten invariant.

4 Scientific Progress Made

Perhaps the most important benefits of this workshop come from the interactions between the participants. Stories illustrate this point. Paul Feehan told the organizers that he had very productive discussions about the slice-ribbon conjecture with Hans Boden. In his talk he described the fact that the perturbations that he and Tom Leness used in their proof of Witten's conjecture are distinct from the holonomy perturbations that are used in other parts of Floer theory. He and Hans Boden discussed Instanton Floer theory with and without holonomy perturbations. On one hand such discussions could have taken place in the early 1990s because the underlying mathematics does not require the more recent developments of Seiberg-Witten theory, Heegaard Floer theory, or symplectic field theory. On the other had, this area of mathematics is undergoing a bit of a renaissance as seen in recent work of Boden, Daemi, Sivek, Hearld, Hedden, and Kirk. Such discussions and analysis may lead to new results, especially in framed instanton Floer homology, variants of Khovanov homology, and interactions between the fundamental group and smooth structures in four dimensions. One exciting proposal of Kronheimer and Mrowka is a approach to a computer-free proof of the 4-color theorem based on this old-school gauge theory.

A number of collaborators were present at the workshop. Auckly, Kim, Melvin and Schwartz were all in attendance and had opportunities to work. Lisa Piccirillo continued a collaboration with Tom Mark. Hans Boden and Cynthia Curtis were able to continue their long collaboration. Cagri Karakurt and Tom Mark had a very fruitful discussion discovering an important issue raised in Tom's talk. John Baldwin, Tom Mark and Cagri Karakurt were able to have many discussions. Auckly and Gompf discussed multiple nucli in elliptic surfaces.

Several graduate students and recent Ph.D.s were able to attend and network. These include Sergio Holguin-Cardona, Vincent Longo, Maggie Miller, Lisa Piccirillo, Katherine Raoux, Sumereyra Sakalli, Hanna Schwartz, Jonathan Simone and Biji Wong.

It generally takes about a year for an initial idea in this field to be developed into a result that can appear in a preprint. Thus, one can not point to breakthroughs that were initiated at this CMO meeting now. However, it is certain that such results will follow because the participants were all productive researchers in gauge theory and low dimensional topology. Furthermore, there were constant interactions between participants exploring various ideas in the field.

5 Outcome of the Meeting

The most immediate outcome of the meeting was the communication of new mathematical ideas to a collection of mathematical researchers from all over the world. The interaction and collaboration between participants is probably the most important outcome. Direct interaction is particularly important in 4-dimensional topology. This is because it is much easier to describe complicated topological constructions and diagrams in person than it is do do via writing or e-mail. Putting topologists in the same space for an extend period period of time is the only way to achieve these interactions. Blackboards around the institute were covered with diagrams similar to the one in the picture in the section on presentation highlights. These blackboard diagrams were evidence of the deep mathematical discussions taking place at the workshop.

As an international conference, the workshop Low Dimensional Topology and Gauge Theory brought together researchers from nine different countries to share ideas. This provided participants opportunities to disseminate scholarly work widely. The workshop engaged students and early career mathematicians in a small research setting with ample time for in-depth discussions. Conference organizers are aware of the benefits of broadening participation in the mathematical sciences and kept this in mind when recruiting and inviting participants to the workshop. Getting the broadest possible participation is a long-term project that will take many workshops over many years to achieve. Having this as an explicit goal of the workshop organizers for most workshops is a first step in making this an outcome of the full program of workshops run over an extended period of time.

Plans for more conferences and small group meetings, as well as possible new collaborations were discussed. Of course the exchange of ideas, work spent continuing collaborations and developing new collaborations will all lead to new research – research that will hopefully become the topic of a future meeting at the Banff International Research Station and/or Casa Matemática Oaxaca.

BIRS and CMO have a polished system for arranging the logistics of a math workshop. The smooth organization is apparent to all participants. Making it so easy to run a quality workshop will encourage program participants to propose and run future workshops, further disseminating and creating mathematical research at the very highest levels.

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