Ordinary and symbolic powers of ideals

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1 Overview of the Field

Problems involving the differences between ordinary and symbolic powers of ideals have arisen in several fields of mathematics, and they are currently a particularly active focus of research. The subject has a distinguished history. As Harbourne and Huneke note in [25], comparing the two types of powers is the content of a conjecture of Eisenbud and Mazur [17] relating to an important step in the proof of Fermat's Last Theorem. The comparison is also central to work of Chudnovsky, Waldschmidt, and Skoda (see [6, 43, 39]) from the 1970s on ideals of points which is related to recent work on fat points in algebraic geometry. As well, the comparison has proven vital in recent research using commutative algebra to study graphs and hypergraphs (e.g, [20, 23]).

Two special cases are especially illuminating, and provide enough details to get the idea if one is unfamiliar with symbolic powers. If I is a squarefree monomial ideal in $S = k[x_1, \ldots, x_n]$, then the m^{th} symbolic power of I is $I^{(m)} = \bigcap_{P_i} P_i^m$, where the P_i range over the associated primes of I. If instead I is the ideal of distinct fat points p_i, each with multiplicity r_i, and $I(p_i)$ is the ideal of forms vanishing at p_i, then $I^{(m)} = \bigcap_{p_i} I(p_i)^{mr_i}$. In general, $I^m \subset I^{(m)}$ with equality occurring rarely. One is often interested in determining for what values of r and s one has $I^{(r)} \subset I^s$. This problem is sometimes called the *containment problem*. A related problem is to determine for what r, j, and s one has $I^{(r)} \subset \mathfrak{m}^j I^s$, where $\mathfrak{m}$ is the maximal homogeneous ideal.

We identify three ways in which comparing ordinary and symbolic powers has arisen recently. First, it is a central topic in combinatorial commutative algebra. Understanding the difference between the two powers is key to studying combinatorial objects algebraically and vice versa, a burgeoning method of research in each area. For example, suppose G is a graph, and let J_G be the cover ideal of the graph. The cover ideal is an ideal generated by monomials that represent vertex covers of the graph. The ideal J_G is also the Alexander dual of the edge ideal of the graph, a monomial which is generated by degree-two monomials corresponding to the edges. For the four-cycle C_4, we have $J_{C_4}^2 = J_{C_4}^{(2)}$. However, the situation is different for the five-cycle C_5. If the vertices of C_5 are $\{x_1, \ldots, x_5\}$, then $x_1 \cdots x_5 \in J^{(2)} \setminus J^2$. There is a combinatorial reason for this: the symbolic square is generated by monomial double covers, that is, monomials that cover each edge at least twice. However, monomials in the ordinary square are only those double covers that can be split into two separate single covers. Note that $x_1 \cdots x_5$ is a double cover but cannot be partitioned into two single covers. Algebraically, this difference leads to an extra associated prime for $J_{C_5}^2$ compared to $J_{C_5}^{(2)}$.

The organizers have used algebraic and combinatorial methods to give an algebraic characterization of when a graph is perfect [20]. These criteria are simple to state and are independent of the Strong Perfect Graph Theorem; they are a characterization of the associated primes of ordinary powers of the cover ideal. Understanding the difference between the two types of powers in this combinatorial context is also vital for determining the associated primes of powers of arbitrary squarefree monomial ideals. Given any ideal I in S, Brodmann proved 40 years ago that the set of associated primes $\{\text{Ass}(S/I^r)\}$ stabilizes for $r \gg 0$ [4]. A much more delicate question is under what circumstances the associated primes of powers of the same ideal form a chain. That is, when is $\text{Ass}(S/I^r) \subseteq \text{Ass}(S/I^{r+1})$ for all $r \geq 1$? This property is known as the persistence of associated primes. Unfortunately, the containment is false in general, although it holds for large classes of ideals that have combinatorial significance. Many authors, including Herzog, Rauf, Vladoius [26], Hien, Lam, Trung [27], and Martinez-Bernal, Morey, Villarreal [33], have published papers on this topic and related important questions about the depth of ideals. While many algebraists believed for a long time that all squarefree monomial ideals have the persistence property, recent work in 2013 by a group of graph theorists led to a rather mysterious counterexample [30], making the area ripe for further investigation.

The study of the containment problem, popularized by Bocci and Habourne [2], is the second way in which the properties of symbolic and ordinary powers have arisen. A series of conjectures by Harbourne and Huneke (see, e.g., [25]) on containment questions involving ordinary and symbolic powers of ideals sparked a flurry of research several years ago that continues today. One of their motivations was a result of Waldschmidt and Skoda, which gives a lower bound on the least degree of a polynomial in the m^{th} symbolic power of a radical ideal I of points in $\mathbb{P}^N$ in terms of the same invariant for I itself. (While this work was in the context of complex analytic and algebraic geometry, work of Ein, Lazarsfeld, and Smith [16] as well as Hochster and Huneke [28], yields the same bound for all homogeneous ideals.) Their bound follows from the property that $I^{(Nm)}$ is always contained in I^m. Harbourne and Huneke wondered if there was a similar containment result that implied Chudnovsky's subsequent conjectural improvement of the work of Waldschmidt and Skoda in $\mathbb{P}^n$. Consequently, they made a number of conjectures about ordinary and symbolic powers of ideals of fat points in $\mathbb{P}^N$. For example, Harbourne and Huneke conjectured that $I^{(nr)} \subseteq \mathfrak{m}^{r(n-1)}I^r$, where $\mathfrak{m}$ is the maximal homogeneous ideal, and I is an ideal of points in $\mathbb{P}^n$. They proved this result for points in general position in order to prove Chudnovsky's improved bound in special cases. It is interesting to note that Waldschmidt and Skoda's original work used analysis, while algebraic techniques comparing ordinary and symbolic powers implies it straightaway. Harbourne and Huneke give a number of conjectures in their paper, proving them in special cases, that invite many opportunities to optimize when, for example, $I^{(r)} \subset \mathfrak{m}^j I^s$, and other similar types of containments.

A third way in which comparing ordinary and symbolic powers has arisen is in work on the Conforti-Cornuéjols Conjecture in combinatorial optimization [7]. Roughly speaking, one starts with a hypergraph G and forms an incidence matrix A. One then looks at the dual linear programming problem

$$\max\{\langle \mathbf{1}, \mathbf{y}\rangle \mid \mathbf{y} \in \mathbb{R}^m_{\geq 0}, A\mathbf{y} \leq \mathbf{c}\} = \min\{\langle \mathbf{c}, \mathbf{z}\rangle \mid \mathbf{z} \in \mathbb{R}^n_{\geq 0}, A^{\mathrm{T}}\mathbf{z} \geq \mathbf{1}\}.$$

We say G has the packing property if the system has integer optimal solutions $\mathbf{y}$ and $\mathbf{z}$ for all vectors $\mathbf{c}$ with components equal to 0, 1 and $+\infty$, and G has the max-flow-min-cut (MFMC) property if the system has integral optimal solutions $\mathbf{y}$ and $\mathbf{z}$ for all nonnegative integer vectors $\mathbf{c} \in \mathbb{Z}^n_{\geq 0}$. While it is clear that if G satisfies the MFMC property, then it has the packing property, the converse direction is much harder. Conforti and Cornuéjols conjectured that a hypergraph has the packing property if and only if it has the MFMC property. This problem has proven extremely difficult (with a prize of \$5,000 offered for its solution).

The connection to commutative algebra is this: If I is the edge ideal of a hypergraph G, then G satisfies the MFMC property if and only if $I^{(q)} = I^q$ for all $q \geq 0$. That is, the ordinary and symbolic powers are all equal or, equivalently, the associated primes of S/I^q are the same as those of S/I for all $q \geq 0$. This reformulation of the Conforti-Cornuéjols conjecture allows one to study the problem by using the associated graded ring, the notion of m-grade (the maximum length of a regular sequence of monomials in I), and other algebraic notions. In the case of graphs, this allows one to prove that G satisfies the MFMC property if and only if G is bipartite. Several authors have made progress on the conjecture via the algebraic reformulation, including Hà and Morey, who studied properties of a hypothetical minimal counterexample from the algebraic perspective [23].

2 Recent Developments and Open Problems

In the past five to ten years, there have been a number of advances in our understanding of symbolic and ordinary powers of ideals. We give some sample highlights of these developments.

- In the 1970s, Chudnovsky [6] posed a conjecture on the smallest degree of a form that passed through a set of fat points of multiplicity m in $\mathbb{P}^n$. Chudnovsky was able to prove the conjecture for $n = 2$. Waldschmidt and Skoda used analysis to get a similar weaker bound, while Chudnovsky used algebraic geometry. Since an asymptotic version of Waldschmidt and Skoda's bound has arisen recently in new problems in algebraic geometry and commutative algebra, there has been renewed interest in this conjecture. In particular, Dumnick [13] verified the conjecture for general sets of points in $\mathbb{P}^3$, and more recently, Fouli, Mantero, and Xie [19] proved the conjecture for large classes of points in $\mathbb{P}^n$.

- The study of the sets of associated primes of I^s when I is a squarefree monomial ideal has seen a large number of advances. For example, Martinez-Bernal, Morey, and Villarreal [33] showed that all edge ideals of finite graphs satisfy the persistence property. It is interesting to note that a key step in their proof relied on a classical result about matchings due to C. Berge from the 1960s. This result is a good illustration of the interaction between graph theory and commutative algebra. The organizers [20] also gave a combinatorial interpretation of the elements in $\mathrm{Ass}(S/I^s)$, and posed a graph theory conjecture that would imply that all squarefree monomial ideals have the persistence property. This graph theory conjecture inspired the work of [30]; the example of this paper lead to an infinite family of squarefree monomial ideals without the persistence property (also see [24]).

- Given a homogeneous ideal $I \subseteq S = k[x_0, x_1, x_2]$, results of Ein, Lazarsfeld, and Smith, and Hochster and Huneke imply that $I^{(2r)} \subseteq I^r$ for all $r \geq 1$. As a consequence, for any ideal I of points in $\mathbb{P}^2$, we always have $I^{(4)} \subseteq I^2$. Huneke asked if this bound can be improved; e.g., is it true that if I is an ideal of points, do we also have $I^{(3)} \subseteq I^2$? This surprisingly simple question has resulted in some very interesting geometry. In particular, there are sets of points in $\mathbb{P}^2$ whose defining ideal I has the property that $I^{(3)} \not\subseteq I^2$, and some of the counterexamples are related to classical configurations of points first studied by Klein (in the 1870s) and Wiman (in the 1890s), among others. See the work of Dumnicki, Szemberg, and Tutaj-Gasínska [14] for the first counterexample to Huneke's question. This question continues to motivate current research.

- Nagel and Trok [36] recently settled a twenty year old conjecture on the Segre bound for the regularity of fat points in $\mathbb{P}^n$. For more details on this result, see the summary of Nagel's talk given below.

As is evident from the partial list of results given above, much progress has been made in the study of symbolic and ordinary powers of ideals. Even though there has been much progress, there are still many questions we would like to answer. In fact, one of the main goals of our session was not to just review past results, but to actively facilitate future developments in the field.

On the first day of the conference, we held a brainstorming session and produced a list of open problems on ordinary and symbolic powers of ideals. We grouped the problems together by topic and asked participants to join group(s) that interested them (with the understanding that they could move among groups during the workshop). The workshop participants had several hours each day to work with their groups and made substantial progress on a number of problems. We reproduce an edited list of the problems below with the hope that this may help direct further research on the topics of the workshop.

1. (Francisco-Hà-Van Tuyl) Let I be a squarefree monomial ideal. An ideal I has the persistence property if $\mathrm{Ass}(I^r) \subseteq \mathrm{Ass}(I^{r+1})$ for all $r \geq 1$. An example from 2013 arising in graph theory shows that there exist squarefree monomial ideals that do not have the persistence property [30]. Question: Which squarefree monomial ideals have this property?

2. (Francisco-Hà-Van Tuyl) Let I be a squarefree monomial ideal. It seems that $\mathrm{Ass}(I^2) \subset \mathrm{Ass}(I^r)$ for all $r \geq 2$. (The recent counterexample to persistence is from $r = 3$ to $r = 4$, so this would be best possible.) Is this always true?

3. (Faridi) If I is a monomial ideal with the persistence property, does the polarization of I have the persistence property? [*Note: The group working on persistence property answered this in the negative during the conference.*]

4. (Herzog) The equality $I^{n+1} : I = I^n$ holds if I is normal. This equality implies that $\mathrm{Ass}(I^r) \subseteq \mathrm{Ass}(I^{r+1})$ for all $r \geq 1$. Are there similar conditions that imply $\mathrm{depth}(R/I^r) \geq \mathrm{depth}(R/I^{r+1})$ for all $r \geq 1$?

5. (Terai) Let I be a squarefree monomial ideal. R/I can be Cohen-Macaulay over some fields k but not others. Work of Terai and Trung shows: If R/I^t is Cohen-Macaulay for some $t \geq 3$ over some field k, then R/I^t is Cohen-Macaulay over all fields. Question: If R/I^2 is Cohen-Macaulay over some field k, then is it Cohen-Macaulay over all fields?

6. (Faridi) A König ideal is a squarefree monomial ideal $I = (m_1, \ldots, m_q)$ such that the maximum number of pairwise disjoint monomials $m_1, \ldots, m_q$ is equal to the height of I. Do König ideals have persistence? (Edge ideals of bipartite graphs form a large class of König ideals.)

7. (Hà) Let I be a squarefree monomial ideal in polynomial ring R. Question: If $I^{(n)} = I^n$ for all $n \leq$ bigheight of I, then is the equality true for all n? (This would imply the Conforti-Cornuéjols Conjecture.) Find N such that if equality holds for all $n \leq N$, then equality holds for all n. Note a related theorem of Roberts, Reid, and Vitulli: If I is a monomial ideal, and $\overline{I^n} = I^n$ for all $n \leq \dim R - 1$, then equality holds for all n.

8. (Hà) Let P be a prime ideal in a polynomial ring R. Does there exist N such that if $R_s[Pt]$ is finitely generated, and $P^{(n)} = P^n$ for all $n \leq N$, then $P^{(n)} = P^n$ for all n? Can we choose $N = \dim R$?

9. (Polini) Let $I \subset R$ be any ideal. Do depth$\overline{R[It]} \geq \dim R$ and $\overline{I^n} = I^n$ for all $n \leq \dim R - 1$ imply equality for all n?

10. (Dao) Let $R = k[x_1, \ldots, x_n]$. We have $\ell(H^1_{\mathfrak{m}}(R/I^n)) < \infty$ if and only if Ass(I^n) does not contain any dimension one prime. Also, $\ell(H^1_{\mathfrak{m}}(R/I^n)) < \infty$ for all $n \gg 0$ if and only if Ass(I^n) does not contain any dimension one prime for all $n \gg 0$.

    Suppose I is an edge ideal of a graph G. Then $\ell(H^1_{\mathfrak{m}}(R/I^n)) < \infty$ for all $n \gg 0$ if and only if $G -$ star(x) has a bipartite component for each $x \in G$. Does the following limit exist?

    $$\lim_{n \to \infty} \frac{\ell(H^1_{\mathfrak{m}}(R/I^n))}{n^d}$$

11. (Dao) Let I be a squarefree monomial ideal. What conditions give depth$R/I^n \geq k$ for all $n \gg 0$? It is known that for I the edge ideal of a graph, the eventual depth of R/I^n is the number of bipartite components of the graph. Is there a similar combinatorial characterization for other I? Faridi asks: Could such a characterization be related to the number of components of the König or packing properties?

12. (Harbourne) Explore (new) possibilities for unexpected curves. Szpond, Szemberg, Guardo, Harbourne, Migliore, Nagel all have interest in this. What line arrangements are good to look at? With multiple points, in what degree should one look?

13. (Szemberg) Consider $\prod_{0 \leq i < j < N} (x_i^n - x_j^n)$. This leads to higher-dimensional Fermat configurations of flats. Work of Malara and Szpond shows: If I is the ideal of codimension two flats with multiplicity at least three, then $I^{(3)} \not\subseteq I^2$. There is an explicit formula for I. Study codimension k flats with multiplicity $\geq 2k - 1$.

14. (Galetto) There is interest in the Betti numbers of symbolic powers of star/matroid/hypersurface configurations. Following Geramita, Harbourne, Migliore, Nagel, suppose $f_1, \ldots, f_s$ are forms in the ring $k[x_0, \ldots, x_n]$. Assume that for $1 \leq c \leq n$, for all $1 \leq i_1 < \cdots < i_c \leq s$, $f_{i_1}, \ldots, f_{i_c}$ is a regular sequence. Let I be the intersection of the $(f_{i_1}, \ldots, f_{i_c})$ for all possible sets of indices. This is generated by all $f_{i_1} \cdots f_{i_{s-c+1}}$ such that $1 \leq i_1 < \cdots < i_{s-c+1} \leq s$. We have that $I^{(m)}$ is the intersection of each $(f_{i_1}, \ldots, f_{i_c})^m$. There is a ring map from $k[y_1, \ldots, y_s]$ to $k[x_0, \ldots, x_n]$ sending y_i to f_i which "preserves" Betti numbers. Thus the problem reduces to studying Betti numbers of the ideals $I^{(m)}_{s,c} = \bigcap (y_{i_1}, \ldots, y_{i_c})^m$.

15. (Bocci) Consider two ideals $I, J \subset R = K[x_0, \ldots, x_N]$. In the ring

    $$K[x_0, \ldots, x_N, y_0, \ldots, y_N, z_0, \ldots, z_N],$$

consider the ideals

$$\begin{aligned} I(\mathbf{y}) &= \text{ image of } I \text{ under the map } x_i \to y_i,\ i = 0, \ldots, N \\ J(\mathbf{z}) &= \text{ image of } J \text{ under the map } x_i \to z_i,\ i = 0, \ldots, N \\ L_{I,J} &= I(\mathbf{y}) + J(\mathbf{z}) + \langle x_i - y_i z_i,\ i = 0, \ldots, N \rangle. \end{aligned}$$

Then the Hadamard product, $I \star_R J$, of I and J is the elimination ideal

$$I \star_R J = L_{I,J} \cap K[x_0, \ldots, x_N].$$

Is it true that for P, Q points in $\mathbb{P}^N$, $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$?

16. (Ulrich) Let R be a regular local ring containing $\mathbb{Q}$. Suppose I is prime (or possibly just radical). The Eisenbud-Mazur Conjecture is that $I^{(2)} \subset \mathfrak{m}I$. This conjecture is true and easy in the graded case: $(f) \subset \mathfrak{m}(\frac{\partial f}{\partial x_i}) \subset \mathfrak{m}I$, where the first inclusion uses characteristic zero, and the second uses that $f \in I^{(2)}$. Weaker question: Is it true that $\overline{I^2} \subset \mathfrak{m}I$? More generally, is it true that $\overline{I^{n+1}} \subset \mathfrak{m}\overline{I^n}$ for all n?

17. (Núñez-Betancourt) Let $R = k[x_1, \ldots, x_d]$, and let I be a radical homogeneous ideal in R. The regularity of I^n has the form $cn + e$ for $n \gg 0$. Does there exist a constant c such that the regularity of $I^{(n)}$ is at most cn for all n? This is known if I is monomial or if $\dim(R/I) \leq 2$. Does there exist $c \in \mathbb{N}$ such that the regularity of $(\text{in}(I^n)) \leq cn$ for all n? It is known that if $t = \dim(R/I)$, then $a_t(R/I^{(n)}) \leq cn$ for some $c \in \mathbb{R}$.

18. (Morey) Let $R = k[x_1, \ldots, x_n]$. What conditions on I force, for all i and s,

$$x_i I^s \cap (I^{s+1} : x_i) \subseteq I^{s+1}?$$

(This arises in a paper of Villarreal *et al.* in showing equality of certain ordinary and symbolic powers.)

19. (Schenck) Characterize failure of WLP quadratic monomial ideals due to injectivity.

20. (Villarreal) Let G be a directed graph with weights at the vertices, and add a (directed) whisker at each vertex of G resulting in a new weighted directed graph G'. Form the weighted directed edge ideal I of G'. Show: I is Cohen-Macaulay if and only if for every whisker xy (where $x \in G$, and y is the new vertex), if the direction is $y \to x$, then the weight of x is 1.

21. (Mermin) Möbius powers of squarefree ideals, defined below, interpolate between symbolic and ordinary powers:

$$\overline{I^s} \subseteq I^{[s]} \subseteq I^{(s)}.$$

When are these containments proper, and what's going on?

A product of prime powers $I = \prod p^{e_p}$ decomposes as an intersection $I = \bigcap p^{a_p}$, where $a_p = \sum_{q \subseteq p} e_p$. Similarly, an intersection of prime powers $I = \bigcap p^{a_p}$ can be written as a product, $\bigcap p^{a_p} = \prod p^{e_p}$, where the exponents are obtained from the as by Möbius inversion (and negative e_p correspond to a colon ideal).

For example, $I = (ab, ac, bc) = (a, b) \cap (a, c) \cap (b, c) = (a, b) \cdot (a, c) \cdot (b, c) : (a, b, c)$, but also $I = (ab, ac, bc) = (a, b) \cap (a, c) \cap (b, c) \cap (a, b, c)^2 = (a, b) \cdot (a, c) \cdot (b, c) : (a, b, c)^3$.

The Möbius powers are obtained by taking powers of everything on the right. So the Möbius square of I is either $I^{[2]} = (a, b)^2 \cdot (a, c)^2 \cdot (b, c)^2 : (a, b, c)^2 = I^2$ or $I^{[2]} = (a, b)^2 \cdot (a, c)^2 \cdot (b, c)^2 : (a, b, c)^6 = I^{(2)}$.

22. (Van Tuyl) When I is a squarefree monomial ideal, the Waldschmidt constant $\widehat{\alpha}(I)$ can be computed via a linear program constructed from the primary decomposition of I. Can one do something similar for $\rho(I)$, the resurgence of a squarefree monomial? (See the synopsis of Seceleanu's talk below for discussion of the Waldschmidt constant and resurgence.)

3 Presentation Highlights

We took a somewhat different approach from many workshops and focused on getting participants working on open problems during the week at CMO rather than emphasizing talks. In order to ensure a significant amount of time for discussion, we limited the number of presentations to eight. We asked our speakers to talk about an area related to ordinary or symbolic powers of ideals, and where possible, to highlight open problems and possible directions in order to advance the field. We selected the eight presenters to ensure that we had a balance among algebraic, combinatorial, and geometric perspectives on ordinary and symbolic powers of ideals.

We provide brief summaries of the talks of our eight speakers:

1. Brian Harbourne (University of Nebraska)

   Harbourne gave a talk on unexpected curves based on joint work with David Cook II, Juan Migliore, and Uwe Nagel [8]. He began by describing the open problem to classify all line arrangements in $\mathbb{P}^2_{\mathbb{C}}$ with no simple crossings, noting the four kinds of examples that are known. He pointed out the strong relevance of this problem to issues surrounding ordinary and symbolic powers, particularly the fact that the nontrivial cases of the known examples yield unexpected curves. More generally, unexpected curves arise in the following way. Suppose Z is a finite set of points in $\mathbb{P}^2$, and let P be a general point. Suppose m is an integer such that mP does not impose the expected number of conditions on the linear system of curves of degree $m+1$ containing Z. Then one says Z admits an unexpected curve of degree $m+1$. (One can also give this condition numerically in terms of the expected dimensions of vector spaces.) Harbourne explained how unexpected curves can sometimes be built from configurations of points that fail the containment problem, and he explained connections to the SHGH Conjecture and to Terao's Conjecture, the latter of which is related to work of Di Gennaro, Ilardi, and Vallès [12]. He also raised several open problems. For example, if the point set Z has an unexpected curve of degree d, does Z impose independent conditions on forms of degree d as in the examples that have been studied? Additionally, in these examples, if Z has an unexpected curve, the lines dual to Z are not supersolvable. Does this always hold?

2. Rafael Villarreal (Cinvestav-IPN)

   Villarreal discussed a number of problems related to the properties of powers of (squarefree) monomial ideals. His talk was broken into two parts. The first part was focused on the problem of when all the ordinary powers of a squarefree monomial ideal equal its symbolic powers. He reviewed an important conjecture of Conforti-Cornuéjols about hypergraph that states the equivalence of max-flow-min-cut property (MFMC) and the packing property. It is known that the symbolic powers and ordinary powers of a squarefree monomial ideal are all equal if the associated hypergraph constructed from the generators of the ideal satisfies the MFMC. This suggests a possible approach to solving the Conforti-Cornuéjols conjecture by showing that the packing property implies the equality of ordinary and symbolic powers of the corresponding monomial ideal. Villarreal outlined a number of reductions that one can make to solve this important problem.

   The second part of Villarreal's talk raised an interesting question of when the irreducible decomposition of a monomial ideal is unique (in general, the irreducible decomposition fails to have this property). He described some of his recent work using directed graphs to give some infinite families of monomial ideals with this property [42]. This approach leads to a number of interesting questions about what families of graphs are Cohen-Macaulay. In fact, one of the working groups at the conference proved some partial results in this area.

3. Craig Huneke (University of Virginia)

   Huneke described his talk as a conversation about the problem of comparing ordinary and symbolic powers of ideals. One theme of his talk was to encourage explorations of generalizations of some of the problems being discussed in the workshop. As an example, he suggested that it might be fruitful to explore an F-pure variation of the packing problem. If k is a field of characteristic p, we say that an ideal $I \subseteq R$ is F-pure if the Forbenius map $F : R/I \to R/I$ splits. Using Fedder's criteria for checking F-purity, it is not difficult to verify that all squarefree monomial ideals are F-pure. Huneke suggested that it might be interesting to understand how to interpret the packing problem with the language of F-purity. He outlined a couple of ideas that one should check to see if this program is even viable. Additionally, Huneke discussed a possible notion of an F-König ideal, an extension of the definition of a König ideal, asking, for example, whether the two notions are equivalent in the case of a squarefree monomial ideal in characteristic p.

In the second half of his talk, Huneke discussed the containment problem in both the regular and non-regular cases. One interesting question, suggested in work of Harbourne, was whether or not for a fixed ideal, the containment $I^{(cn-c+1)} \subseteq I^n$ holds for all n large enough, where c is the codimension of I. Note that it is known that this fails for small n, but it may hold for large n. Huneke stated and proved a theorem in this direction in joint work with Grifo: If $(R, \mathfrak{m})$ is local, and R/I is F-pure, then $I^{(cn-c+1)} \subseteq I^n$ for all $n \geq 1$, where c is the bigheight of I. Huneke concluded with brief remarks about the non-regular case. He asked whether, for $(R, \mathfrak{m})$ a complete local domain, there exists a uniform b such that for each $P \in \operatorname{Spec}(R)$ and each $n \geq 1$, $P^{(bn)} \subseteq P^n$. This is true in the regular case with $b = \dim R$ [16, 28, 32].

4. Alexandra Seceleanu (University of Nebraska)

   Seceleanu's talk focused on asymptotic invariants related to the powers of ideals. Particularly, she discussed the Waldschmidt constant, the resurgence, and the asymptotic resurgence of an ideal. The Waldschmidt constant of an ideal I is $\hat{\alpha}(I) = \inf\left\{\frac{\alpha(I^{(m)})}{m} : m \geq 1\right\}$, where $\alpha(I)$ is the initial degree of I. The resurgence of I is $\rho(I) = \sup\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\}$. The asymptotic resurgence of I is $\hat{\rho}(I) = \sup\{\frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} : \text{for all } t \gg 0\}$. These three invariants give tools to compare regular powers with symbolic powers. Unfortunately, they are quite hard to compute in general, and there are many wide open questions. For example, Nagata has conjectured that for $n \geq 10$ general points in $\mathbb{P}^2$, the Waldschmidt constant is $\sqrt{n}$. This is known when n is a perfect square. If the conjecture is correct then there are infinitely many examples of ideals with irrational Waldschmidt constants. However, no such example is currently known.

   For most of her talk, Seceleanu focused on monomial ideals, giving an overview of what is known about the Waldschmidt constant in this setting. She described some of the work of Cooper, Embree, Hà, and Hoefel [9], including their symbolic polytope approach, and her own work with a number of other co-authors [3]. In the case of squarefree monomial ideals, computing the Waldschmidt constant is equivalent to solving a linear program. An advantage of this interpretation is that it allows one to prove a Chudnovsky-like conjecture that allows us to bound the Waldschmidt constant in terms of the smallest degree of a form in the ideal and the bigheight. Seceleanu noted that for monomial ideals, the Waldschmidt constant is always rational, and it is always algorithmically computable. She asked in what other contexts these properties should hold. Finally, Seceleanu concluded with a summary of known results about the Waldschmidt constant and the (asymptotic) resurgence of an ideal in the setting of configurations of lines and points.

5. S. Dale Cutkosky (University of Missouri)

   Cutkosky spoke about symbolic algebras of monomial curves. He began by stating the finite generation problem: If P is a prime ideal contained in a polynomial ring over k, when is $\bigoplus_{n \geq 0} P^{(n)}$ a finitely generated k-algebra? While some thought this may always be true, Roberts found a counterexample in dimension three (in characteristic zero), using an example of Nagata and giving a new counterexample to Hilbert's fourteenth problem [37]. Recently, Sannai and Tanaka found a counterexample to the finite generation over any algebraically closed field of characteristic $p > 0$, using a prime ideal in a 12-dimensional polynomial ring [38].

   Having given this background, Cutkosky then moved to discuss settings in which finite generation might hold or might fail. Let k be an algebraically closed field, and let P be the kernel of the map from $k[x, y, z]$ to $k[t]$ defined by $f \mapsto f(t^a, t^b, t^c)$, where a, b, and c are (without loss of generality, pairwise relatively prime) positive integers. Previous work of Huneke gave a necessary and sufficient condition for finite generation in this context, and it was showed explicitly that finite generation holds for $c \leq 4$ and all a and b [29]. Cutkosky mentioned two additional explicit positive results, when

$(a + b + c)^2 > abc$ [10] and when $c = 6$ [40]. He described the method used to obtain the result in [10], working in the language of weighted projective spaces.

After giving two criteria for the failure of finite generation, including a discussion of the important role of negative curves, Cutkosky explained an application of one of these criteria: a theorem of Goto, Nishida, and Watanbe, giving an (analytically irreducible) example in characteristic zero (with $a = 7N - 3$, $b = 8N - 3$, $c = (5N - 2)N$, where $N \geq 4$ and $3 \nmid N$) that is not finitely generated [22]. He then outlined the argument behind a characteristic zero example of failure of finite generation, due to González and Karu [21], when $a = 15$, $b = 26$, and $c = 7$, in which the authors used toric geometry to attack the problem (and prove much broader results than just this example). Cutkosky concluded with an open question, asking whether there exists an example that is not finitely generated and also does not have a negative curve. This would give a characteristic p example of non-finite generation of a space curve that is analytically irreducible.

6. Claudia Polini (University of Notre Dame)

   Polini talked about a joint work with Andy Kustin and Bernd Ulrich on degree bounds for local cohomology [31]. Let $R = k[x_1, \ldots, x_d]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, and let M be a finitely generated graded R-module. The general problem is to find "upper bounds" for the local cohomology modules $H^i_{\mathfrak{m}}(M)$, which are Artinian. In particular, what are reasonable upper bounds on $a_i(M) = \text{topdeg } H^i_{\mathfrak{m}}(M)$ and $b(H^i_{\mathfrak{m}}(M)) = \text{topgendeg } H^i_{\mathfrak{m}}(M)$ (the top degree of a generator)? In her talk, Polini focused the top generating degree invariant and in particular on the case $i = 0$ with a view toward applications. She explained the following main theorem (specialized to a setting that is convenient to explain): Let R be a positively graded Cohen-Macaulay k-algebra, $d = \dim R$, and $(C_\bullet, \partial_\bullet)$ an "approximate resolution" with $M = H_0(C_\bullet)$. Set $t = \text{depth}(M/H^0_{\mathfrak{m}}(M))$ and $a(R)$ the a-invariant of R. Assume that $\dim H_j(C_\bullet) \leq j$ for all $1 \leq j \leq d-1$ and that the C_j are maximal Cohen-Macaulay for all $0 \leq j \leq d$. Then
   $$b(H^0_{\mathfrak{m}}(M)) \leq b(C_{d-t}) + \Theta_t + a(R),$$
   where $\Theta_t = t$ if R is standard graded; otherwise, take a minimal homogeneous system of generators of $\mathfrak{m}_R$, and then Θ_t is the sum of the t highest degree generators.

   Polini discussed applications of this work to Rees rings, affine monomial curves, points in projective space, and hyperplane sections. She ended by posing some questions of Dao-Montaño and Dao-Nuñez-Betancourt about the initial degree of the local cohomology of R/I^n and $R/I^{(n)}$. Assuming the length of $H^i_{\mathfrak{m}}(R/I^n)$ (resp., $H^i_{\mathfrak{m}}(R/I^{(n)})$) is finite for $n \gg 0$, does there exist an $\alpha \in \mathbb{N}$ such that the initial degree of the local cohomology of $H^i_{\mathfrak{m}}(R/I^n)$ (resp., the initial degree of $H^i_{\mathfrak{m}}(R/I^{(n)})$) is greater than or equal to αn for all $n \gg 0$? The case of symbolic powers is known when I is monomial and when Proj R/I is a locally complete intersection and normal; it is otherwise wide open.

7. Juan Migliore (University of Notre Dame)

   Migliore discussed old and new problems related to Lefschetz conditions and powers of linear forms. We say that R/I has the Weak Lefschetz property (WLP) if for a general linear form L and every positive integer i, the multiplication map $\times L : [R/I]_{i-1} \to [R/I]_i$ has maximal rank. Additionally, R/I has the Strong Lefschetz Property if $\times L^k : [R/I]_{j-k} \to [R/I]_j$ has maximal rank for all k. There are a number of general types of questions one can ask: For which k does $\times L^k : [R/I]_{j-k} \to [R/I]_j$ has maximal rank for all j? Alternatively, given k, for which j does $\times L^k : [R/I]_{j-k} \to [R/I]_j$ have maximal rank? When do the properties hold for particular families of algebras, such as Gorenstein algebras? (For example, whether every codimension three Gorenstein algebra has the WLP is a wide open question.) What role does the characteristic of the underlying field play?

   As Migliore noted, the majority of the work has focused on the WLP (i.e., $k = 1$). The origin of these questions can be traced back to work of Stanley in 1980 and Watanbe in 1987. One current stream of research concerns understanding the ideals of the form $(L_1^{a_1}, \ldots, L_s^{a_s})$ where L_i are linear forms, and $s \geq n + 1$, where n is the number of variables in the ring. A nice result in this direction is the work of Schenck and Seleceanu which states that when $n = 3$, then for all L_i and a_i, the quotient ring has the WLP. This leads to questions about what happens for higher powers, that is, in the direction of the SLP. Migliore discussed some very recent work he has done with Uwe Nagel to generalize some of the recent results. One of the main results is that if $I = (L_1^2, L_2^{a_2}, \dots, L_s^{a_s})$ is an ideal of powers of general linear forms in three variables, then R/I has the SLP. The proof relies on a clever use of the Snake Lemma.

Migliore ended his talk with a number of open problems. He asked about the complete classification of maximal rank for $\times L^4$ and $\times L^5$ for $k[x, y, z]$, which would extend recent work of his with Miró-Roig. In addition, one would like to be able to understand $\times L^k$ for $k \geq 2$ in four or more variables. Another problem is to determine when almost complete intersections have the WLP. A number of partial results are known, many from [34] and [35], and Migliore presented a conjecture from [34] that would cover the remaining cases. Migliore also explained a conjecture of Harbourne, Schenck, and Seceleanu on the non-ACI case.

8. Uwe Nagel (University of Kentucky)

   Nagel talked about his work with William Trok on Hilbert functions of fat point schemes [36]. Let $Z = \{P_1, ..., P_s\}$ be finite set of reduced points in $\mathbb{P}^n$. A fat point scheme X supported on Z is the scheme defined by the ideal $I_X = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s} \subset R = K[x_0, \dots, x_n]$. We have the following bound on the Hilbert function:

   $$h_X(j) \leq \min \left\{ \binom{n+j}{n}, \sum_{i=1}^{s} \binom{n+m_i-1}{n} \right\}.$$

   When the points in Z are general, we expect equality to hold, though it does not always do so. Thus one open problem is, for general Z, to classify n, $(m_1, \dots, m_s)$, and j such that equality fails. Alexander-Hirschowitz solved the case of all $m_i = 2$ in 1995. When $n = 2$, and m_i arbitrary, this is one way of stating the SHGH conjecture.

   Allowing now Z to be arbitrary, one way to look at Hilbert function questions is to study the regularity index $r(X)$, which is the minimum j such that $h_X(j) = \deg X$ (equivalently, $\operatorname{reg}(X) - 1$). Around 2000, Fatabbi and Lorenzini [18] and, independently, Trung proposed a conjecture on the regularity index:

   $$r(X) \leq \max \left\{ \left\lceil \frac{-1 + \sum_{P_i \in L} m_i}{\dim L} \right\rceil : L \subset \mathbb{P}^n \text{ linear subspace of } \dim > 0 \right\}.$$

   This bound was first proven by Segre in 1962 with $n = 2$ and no three colinear points. As a result, we call the quantity on the right-hand side the Segre bound, denoted by $\operatorname{Seg}(X)$. Nagel mentioned a number of other known cases, including work from [1, 5, 11, 18, 41]. After giving this overview, Nagel spent much of the rest of his talk by discussing his new result with William Trok. They prove the following: If Z is arbitrary, then $r(X) \leq \operatorname{Seg}(X)$. Additionally, if $L \subset \mathbb{P}^n$ is linear with $\left\lceil \frac{-1 + \sum_{P_i \in L} m_i}{\dim L} \right\rceil = \operatorname{Seg}(X)$, and $Z \cap L$ lies on a rational normal curve of L, then $r(X) = \operatorname{Seg}(X)$. Thus the bound is optimal. Nagel gave a sketch of the proof. A key new ingredient in their proof was the use of matroids. In particular, they need to generalize a classical result of Edmonds [15] on partitions arising from matroids. Nagel concluded with some remarks about the Waldschmidt constant for ideals of sets of $n + 2$ and $n + 3$ general points in $\mathbb{P}^n$ in characteristic zero.

4 Scientific Progress Made and Outcomes of the Meeting

The brainstorming session on the first day of the conference was one of the most useful activities. The participants produced a list of problems spanning algebraic, combinatorial, and geometric aspects of ordinary and symbolic powers, many of which can be studied from multiple perspectives. Creating the list sparked interesting discussions that made the questions more precise and helped participants understand better what is already known. As is clear from several of the testimonials, many participants began new collaborations as a result of the work done during the meeting. Having the participants break into groups by topic of interest allowed young researchers to work with more senior faculty members and encouraged new collaborations that otherwise would likely not have arisen. We are particularly happy that most of the groups included members at different career stages and from different continents.

We anticipate that a number of papers will result from this workshop, and several groups have told us that they are already working on drafts. To provide some specifics, we shall describe some preliminary results that participants found during the meeting itself.

The group working on the persistence property found examples of monomial ideals that are not squarefree that have the persistence property but whose polarization does not have the persistence property. This answers one of the open problems from the brainstorming session. It is interesting in part because the usual monomial ideals in the literature lacking the persistence property have polarizations that do have persistence. The counterexample is a modification of the ideal associated to the graph found in 2013 work of graph theorists Kaiser, Stehlík, and Škrekovski to give a counterexample to a graph-theoretic conjecture of the organizers.

The group working on the resurgence found an upper bound which, together with previous work [3], gives the exact value for the resurgence of edge ideals of graphs that possess a lot of symmetries; for example, odd cycles and odd anti-cycles. It is expected that the same method would work for any graph and, thus, give the exact value for the resurgence of the edge ideal of any graph.

The group working on integral closure focused on the following question: Let R be a regular local ring, and let I be a reduced (prime) ideal. For which i need one check that $\overline{I^i} = I^i$ to imply that equality holds for all i, where the bar indicates integral closure? The first open case is for $\dim R = 3$ and the integral closure of the Rees algebra of I is almost Cohen-Macaulay (i.e., when $\operatorname{depth}\overline{R[It]} = 3$).

The group working on the weighted directed edge ideal of graphs solved the problem that Villarreal proposed. Let G be any weighted directed graph on $x_1, \ldots, x_n$, let $x_1y_1, \ldots, x_ny_n$ be whiskers (with arbitrary direction and weights at the y_is) at the vertices of G, and let the resulting graph be D. The group was able to show that the following conditions are equivalent: (a) The weighted directed edge ideal $I(D)$ is Cohen-Macaulay; (b) The weighted directed edge ideal $I(D)$ is unmixed; and (c) For each whisker xy, where $x \in G$ and y is the new vertex, if the direction of the edge is $y \to x$, then the weight of x must be 1.

The group working on Hadamard products was interested in the problem of whether for any two points $P, Q \in \mathbb{P}^N$, one has $I(P)^m \star I(Q)^n = I(P \star Q)^{m+n-1}$, where the coordinates of $P \star Q$ is given by componentwise product of the coordinates of P and Q. The group found partial evidence and some useful reductions. The group members also discussed ideas for using these configurations to build new configurations that fail the containment problems that were central to the workshop.

The group investigating regularity and initial ideals had some good progress. Using characteristic p methods, they demonstrated that if I is a radical ideal with bigheight h, then $\mathrm{in}(I^{(hn)}) \subseteq \mathrm{in}(I)^n$. A result of Sullivant shows that if k is an algebraically closed field of characteristic zero, and $\mathrm{in}(I)$ is radical, then $\mathrm{in}(I^n) \subseteq \mathrm{in}(I)^{(n)}$. The group generalized this to the case in which k is perfect and led to the containment that $\mathrm{in}(I^{(hn)}) \subseteq \mathrm{in}(I)^{(n)}$.

A large group looked at the graded Betti numbers of powers of monomial ideals related to star configurations. This group also made some important discoveries. The group found that each degree slice of the symbolic powers of these ideals is polymatroidal, and the symbolic powers have linear quotients. They have expected values for the Betti numbers of the last strand. In addition, the group raised some questions for further research. For example, if an ideal has componentwise linear quotients, is that equivalent to the ideal itself having linear quotients? If I is matroidal, does that imply that $I^{(m)}$ is componentwise polymatroidal? Every strand of the Betti table seems to stabilize eventually; when does this happen?

There was also one group studying unexpected curves, looking at topics raised in Harbourne's talk. Suppose that $X = m_1P_1 + \cdots + m_rP_r$, where the P_i are general points, and $Z = n_1Q_1 + \cdots + n_sQ_s$ in $\mathbb{P}^2$. We want to find X and Z with t such that $\dim I(X+Z)_t = 1 > \dim(I(Z)_t - \sum \binom{m_i+1}{2})$. To find $\langle F \rangle = I(X+Z)_t$ with F irreducible, the group wanted to look at blowups and tried to use some of the available numerical data. All currently known examples have $r = 1$, and ultimately, they would like to find an example with $r \geq 2$.

Apart from the groups working on the specific problems outlined in the problem session, Huneke, Migliore, and Nagel told us in June that they will likely have a paper coming out on the weak Lefschetz property based on work done in Oaxaca.

In summary, each group at the meeting has made progress on the problems on which they were working, and they have been communicating electronically since the end of the workshop. We look forward to seeing the papers that result from these collaborations.

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