Transversal, Helly and Tverberg type Theorems in Geometry, Combinatorics and Topology III

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October 23 –October 28, 2016

1 Introduction

Helly's Theorem and its relatives, the theorems of Radon and Caratheodory have played an important role in Discrete and Convex Geometry, there are numerous generalizations and variations of them that have been studied from different perspectives and points of view.

The classical lemma of Radon for instance states that any $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two classes with intersecting convex hulls. This theorem can be rephrased in the following equivalent way:

Given any affine map: $f : \Delta^{d+1} \to \mathbb{R}^d$, been Δ^{d+1} the $d + 1$ simplex in $\mathbb{R}^d$ there are two disjoint faces σ_1 and σ_2 of Δ^{d+1} such that $f(\sigma_1) \cap f(\sigma_2) \neq \emptyset$.

During this workshop one of the main philosophical aspects was related with the question of up to what extent this classical theorems really depend on convexity or linearity, i.e., whether or not there are some more fundamental topological principles that underly them For instance topological generalizations of Helly's theorem, topological generalizations of the classical theorems in discrete and convex geometry of Boros-Füredi (d=2) and of Bárány.

One of the most beautiful theorems in combinatorial convexity is due to Tverberg, that is the r-partite version of Radon's Lemma. To be more precise,Tverberg's theorem states that every $(d+1)(r-1)+1$ points in Euclidean d-space $\mathbb{R}^d$ can be partitioned into r parts such that the convex hulls of these parts have nonempty intersection. This theorem still remains central and is one of the most intriguing results of combinatorial geometry. It has been shown that there are many close relations between Tverberg's theorem and several important results in mathematics, such as; Rado's Central Theorem on general measures, the Ham Sandwich Theorem and the Four Color Theorem, just to mention some examples. The original Tverberg Theorem now has several different proofs, including those by Tverberg, Roudneff, Sarkaria and more recently due to Zvagelśkii. A specially elegant proof is due to Zarkaria with further simplifications of Onn. And could be rephrased in the following way.

Theorem 1.1. Tverberg Theorem. *Let $r \geq 2$ and $d \geq 0$ be integers and let $N = (r-1)(d+1)$. Then given any affine map $f : \Delta^N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ^{d+1} such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.*

Tverbergs theorem is closely connected with the multiplied or colorful versions of the theorems of Helly, Hadwiger and Caratheodory, first studied by Bárány and Lovasz. In fact, there is a topological version of the Tverberg's theorem that has received lots of attention in the last decades. During the 90's techniques and ideas of algebraic topology were used in a relevant a deep manner to study this version, and nowadays, due to the influence of Gromov's topological ideas the late developments of this problem become an important area of research.

In part, this workshop was very much connected with several of the recent developments and variations of this theorem and also to the modern techniques in algebraic topology used to study Tverberg Theorem as well as some other important theorems in the field.

2 The topological Tverberg Theorem

The first strong indication of the connection of Tverberg's Theorem with algebraic topology was the observation of Bajmóczy and Bárány [2] that Radon's Theorem is not only true for affine maps but actually for every continuos map $f : \Delta^{d+1} \to \mathbb{R}^d$. The proof follows from the Borsuk-Ulam Theorem. In general, extending Tverberg's Theorem 1.1 to continuous maps turned out to be a major problem. However, this was accomplish first for r a prime number by Bárány, Shlosman and Szűcs in [3] and more generally for r a power of prime by Özaydin in [13].

Theorem 2.1. Topological Tverberg Theorem, Bárány, Shlosman, SzűcS, Özaydin. *Let $r \geq 2$ be a prime power, $d \geq 0$ an integer and let $N = (r-1)(d+1)$. Then, given continuous map $f : \Delta^N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ^{d+1} such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Özaydin already showed that the obstruction used to prove the topological Tverberg Theorem for r a prime power vanishes for all other r. The statement of Theorem 2.1 for every r is known as the topological Tverberg conjecture and counterexamples to this conjecture came about when Mabilliard and Wagner [12] developed a major theory which extends fundamental theorems from classical obstruction theory for embeddability to an obstruction theory for r-fold intersection of disjoint faces in maps from simplicial complexes to Euclidean spaces. In particular, they proved that for simplicial complexes of dimension at most $d-3$ (and other technical conditions) the vanishing of the r-fold Van Kampen obstruction implies the existence of a continuous map that does not identify points from r-pairwise disjoint faces.

It was commonly believed that the topological Tverberg conjecture was correct. However, one of the motivations of Mibillard and Wagner for studying eliminations of higher order intersection was that this may lead to counterexamples via Özaydin. Mibillard and Wagner came close but there was a crucial assumption of large codimension in their theory, which seemed to prevent the application of the new theory to this case. It turned out that a simple combinatorial argument due to Florian Flick [7] allows to overcome the codimension problem. This together with a reduction also sketched earlier by Gromov, yields counterexamples to the topological Tverberg conjecture for every r that is not a prime power.

Florian Frick gives combinatorial arguments which allows to use Özaydin result in Mabilliars-Wagner's theory as a beautiful example of a powerful combinatorial method with other applications by Blagojevic, Frick and Ziegler. During our workshop Florian Frick on his talk "Tverberg Theorem for almost affine maps" discussed the new developments about this theorem along this line.

Now that the topological extension of Tverberg's theorem has been disproven, it should be interesting to investigate to which extent the topological Tverberg conjecture fails. On the one hand one can ask by how much one would have to increase the dimension of the simplex to ensure the existence of an r-fold Tverberg point, on the other hand one can ask for the simplest map that does not exhibit an r-fold Tverberg point while one would expect such a point for codimension reasons. Here we conjecture that the topological Tverberg conjecture should not turn out to be wrong in perhaps the simplest case imaginable:

Conjecture 2.2. *Let $N = (r-1)(d+1)$ and let $F \colon \Delta_N \to \mathbb{R}^{d-1} \oplus \mathbb{R}, x \mapsto (f(x), g(x))$ be a map such that f is linear and g is linear on the first barycentric subdivision of Δ_N. Then Tverberg's theorem holds, that is, there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that $F(\sigma_1) \cap \dots \cap F(\sigma_r) \neq \emptyset$.*

An affirmative answer to this conjecture would imply an affine version of the generalized Van Kampen–Flores theorem: given a set $X \subseteq \mathbb{R}^d$ of $(r-1)(d+2)+1$ points, there are r pairwise disjoint subsets $X_1, \dots, X_r$ of X with $|X_i| \leq \lceil \frac{r-1}{r} d \rceil + 1$ such that $\text{conv}\, X_1 \cap \dots \cap \text{conv}\, X_r \neq \emptyset$. This is only known for r a prime power due to the topological proofs of Sarkaria and Volovikov.

The conjecture is known to be true for r a prime power, since in this case it is even true for F continuous. The first open case is $r = 6$ and $d = 2$, when the conjecture is a graph-theoretic problem: can K_{16} be drawn in the plane with edges that are composed of two straight lines broken at the half point of its x-coordinates such that no vertex is surrounded by five vertex-disjoint triangles and no two vertex-disjoint edges intersect in a point surrounded by four vertex-disjoint triangles?

3 Gromov's influence in Discrete Geometry

As we saw in the previous section, the special case of linear maps from the simplex to $\mathbb{R}^d$ are generalized by the work of Gromov, Mabillard-Wagner, to a much more general topological setting of continuous maps, an arbitrary target manifold instead of $\mathbb{R}^d$, and an arbitrary high-dimensional expander instead of the simplex. This ideas also yields improved bounds even for the classical setting (Karasev).

The more general topological framework is now the study of self-intersections of maps from simplicial complexes to manifolds, which includes classical problems and results such as embeddability, the Whitney trick, etc.

Equally importantly, the notion of coboundary expansion (independently discovered by Linial and Meshulam in their work on random complexes and by Gromov) and Gromov's theorem form a beautiful bridge between discrete and convex geometry and other areas of mathematics, such as geometric group theory (Ramanujan complexes) and random structures/stochastic topology (random complexes). People had been investigating possible higher-dimensional generalizations of expanders for a long time, but it was not really clear in which direction to go, and Gromov's result has given the theory of high-dimensional expanders a major boost.

During this workshop, in a joint work with D. Dotterrer and T. Kaufman, Uli Wagner presented a simple and fairly elementary proof of Gromov's Topological Overlap Theorem.

Let X is a finite d-dimensional simplicial complex. Informally, the theorem states that if X has sufficiently strong higher-dimensional expansion properties (which generalize edge expansion of graphs and are defined in terms of cellular cochains of X) then X has the following topological overlap property: for every continuous map from X to d-dimensional Euclidean space, there exists an image point p contained in the images of a positive fraction $\mu > 0$ of the d-simplices of X. More generally, the conclusion holds if d is replaced by any d-dimensional piecewise-linear manifold M, with a constant μ that depends only on d and on the expansion properties of X, but not on M.

According to Roman Karasev's interpretation of Gromov's ideas, he presented in this workshop the dependence of the heavily covered points on parameters. That is, he examines Gromov's method of selecting a point "heavily covered" by simplices chosen from a given finite point sets, in order to understand the dependence of the heavily covered point on parameters. He explains us, that there is no continuous dependence in this problem, but it is possible to utilize the "homological continuous dependence" of the heavily covered point, if we follow Gromov's approach to the problem. This allows us to infer some corollaries in a usual way. He also gave an elementary argument to prove the simplest of these corollaries in the planar case and discussed other approaches and some open problems in the area.

In this talk, Florian Frick also showed us why the combinatorics of missing faces of a simplicial complex give nontrivial information about whether it is embeddable into d-space, and more generally whether every continuous map to d-space exhibits a point of r-fold intersection. This can be used to relate intersection patterns of finite sets as in Kneser's conjecture to intersection patterns of convex sets as in Tverberg's theorem and its continuous generalizations. He presented a theorem that is a common generalization of results from Tverberg-type theory and lower bounds for chromatic numbers of uniform intersection hypergraphs, extending work of Sarkaria.

Usually, the topological methods introduced by Gromov are valuable tools for improving the bounds for certain selection theorems. In a joint work with Bárány, Mushulam and Nevo, Martin Tancer gave a talk about why the Pachs Selection theorem does not admit a topological extension. Pach's selection theorem asserts that for any positive integer d there exists a constant $c_d > 0$ such that for any positive integer n and any finite sets $X_1, ..., X_{d+1}$ in $\mathbb{R}^d$ each with n points there exist disjoint subsets $Z_1, ..., Z_{d+1}$, where Z_i is a subset of X_i with $|Z_i| \geq c_d n$ and a point z such that z belongs to any rainbow $(Z_1, ..., Z_{d+1})$-simplex; that is, a convex hull of points $z_1, ..., z_{d+1}$ where z_i belongs to Z_i.

4 Tolerated Tverberg and some other variations

The Tolerated Tverberg theorem generalizes Tverberg's theorem by introducing a new parameter t called *tolerance*. It states that there is a minimal number $N = N(d,t,r)$ so that any set X of at least N points in $\mathbb{R}^d$ can be partitioned into r disjoint sets $A_1,\dots,A_r$ such that $\bigcap_{i=1}^{r}\operatorname{conv}(A_i\setminus Y)\neq\emptyset$ for any $Y\subset X$ with at most t points.

In contrast with the classical Tverberg theorem, the best known bounds for $N(d,t,r)$ are not tight. In [11], Larman proved that $N(d,1,2)\le 2d+3$, García-Colín showed that $N(d,t,2)\le(t+1)(d+1)+1$. This was later generalized by Strausz and Soberón who gave the general bound $N(d,t,r)\le(r-1)(t+1)(d+1)+1$ [16]. Later, Mulzer and Stein gave the bound $N(d,t,r)\le 2^{d-1}(r(t+2)-1)$ which improves the previous bound for $d\le 2$ and is tight for $d=1$.

As for lower bounds, Ramírez-Alfonsín [14] and García-Colín, using oriented matroids, proved that $\lceil\frac{5d}{3}\rceil+3\le N(d,1,2)$ and $2d+t+1\le N(d,t,2)$, respectively. Furthermore, Larman's upper bound is known to be sharp for $d=1,2,3$ and 4 [11]. Lastly, Soberón gave the bound $r(\lfloor\frac{d}{2}\rfloor+t+1)\le N(d,t,r)$.

Later on García-Colín, Roldán-Pensado and Raggi [8] proved that for d and r, the correct value for $N(d,t,r)$ is asymptotically equal to rt. To be precise, they prove the following theorem.

Theorem 4.1. *For fixed* r *and* d *we have that*

$$N(d,t,r)=rt+o(t).$$

This improved all previously known upper bounds whenever t is large compared to r and d, and comes with a matching lower bound. The proof follows from studying the behavior of t with respect to N and using the Erdös-Szekeres theorem for cyclic polytopes in $\mathbb{R}^d$. However, the $o(t)$ term hides a $\operatorname{twr}_d(0(r2d2))$ factor, where the tower functions $\operatorname{twr}_i(\alpha)$ are defined by $\operatorname{twr}_1(\alpha)=\alpha$ and $twr_{i+1}(\alpha)=2^{\operatorname{twr}_i(\alpha)}$. The tower function is unavoidable with this method, as it relies on geometric Ramsey-type results.

Improving on the above bounds Soberón [17] proved that $N(t,d,r)=rt+\tilde{o}(r^2\sqrt{td}+r^3d)$, where the $\tilde{o}$ notation hides only polylogarithmic factors in t,d and r. The proof uses the probabilistic method and Sarkaria's trick to obtain versions of the colorful Carathéodory theorem and Tverbergs theorem that allow for small sets of points to be removed without breaking the conclusion of the result.

Concerning a different variation of Tverberg's Theorem, Attila Pór consider in this workshop what he called: "Canonical Tverberg partitions". He showed that for every d,r,N positive integers there exists $n=n(d,r,N)$ such that any sequence p in R^d of length n has a subsequence p of length N such that every subsequence of p of length $T(d,r)=(r-1)(d+1)+1$ has identical Tverberg partitions, namely the "rainbow"–partitions. A partition (or coloring) of the first $T(d,r)$ integers into r parts (with r colors) is called rainbow if every color appears exactly once in each of the following r-tuples: $(1,.,r),(r,,2r-1),(2r-1,,3r-2),.,((d-1)r-(d-2),,dr-(d-1))$.

Finally, Pavel Paták talked about tight colorful Tverbereg partitions for matroids. The Colorful Tverberg theorem states that given a set of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ divided into m color classes of size at most $(r-1)$, there exist r rainbow simplices whose intersection is non-empty. A simplex is called rainbow, if all its vertices are points of different colors. Thus Paták shows for the matroidal version of the problem the same bounds . Since a simplex is the convex hull of its vertices the conclusion of the original colorful Tverberg can be restated as "The intersection of convex hulls of some r rainbow sets is non-empty". In the matroidal version, he replaces convex hulls with any (matroidal) closure operator (e.g. affine hulls). The advantage of the 'affine closure" version is that it is valid even for fields for which convex hulls are not defined, we may weaken the assumptions and assume that one of the color classes has size at most r and the remaining have size at most $r-1$, and that the rainbow sets can be found algorithmically. On the other hand, the conclusions of the theorem are weaker. He also showed that the theorem is tight and present some application of it.

As we have seen, many generalizations, variants and extensions of Tverberg's theorem, including colorful, fractional, and topological versions, have been developed over the years. Jesus de Loera discussed yet another way to interpret Tverberg's theorem, now with a view toward number theory, lattices, integer programming, all things discrete not continuous nor topological.

Given a discrete set S of $\mathbb{R}^d$ (e.g., a lattice, or the Cartesian product of the prime numbers), he studies the number of points of S needed to guarantee the existence of an m-partition of the points $A_1,\dots,A_m$ such that the intersection of the m convex hulls of the parts contains at least k points of S. The proofs of the main results require new quantitative integer versions of Helly's and Carathéodory's theorems.

This was a joint work with subsets of the following: Reuben La Haye, David Rolnick, and Pablo Soberon, Frederic Meunier and Nabil Mustafa.

5 More about Topology in Discrete Geometry

Andreas Holmsen presented a survey about the topology of geometric joins. This is joint work with Bárány and Karasev, and deals with the topology of the union of rainbow simplices spanned by a collection of colored points in $\mathbb{R}^d$. A geometric join is the union of all colorful simplices spanned by a colored point set in the d-dimensional space. For a given dimension, how many colors are needed to guarantee that the geometric join will be contractible? He explained some of the bounds and conjectures.

Complete Knesser Transversals were introduced by Jorge Ramirez Alfonsin. Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$. Let $m(k, d, \lambda)$ be the maximum positive integer n such that every set X of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hulls of all k-sets have a common transversal $(d-\lambda)$-plane (called *Kneser Transversal*). It turns out that $m(k, d, \lambda)$ is strongly connected with other interesting problems, for instance, the chromatic number of Kneser hypergraphs and a discrete version of Rado's center point theorem. In the same spirit, He introduced a natural discrete version m^* of m by considering the existence of *complete Kneser transversals* (in which he ask, in addition, that the transversal $(d-\lambda)$-plane contains $(d-\lambda)+1$ points of X). In this Talk, he presented results concerning the relation between m and m^* and gave a number of lower and upper bounds of m^* as well as the exact value in some cases. After introducing the notions of *stability* and *instability* for (complete) Kneser transversals he gave a stability result that leads to a nice geometric properties for the existence of (complete) Kneser transversals. He ended by presenting some computational results (closely related to the stability and unstability notions). This was a joint work with J. Chappelon, L. Martinez, L. Montejano and L.P. Montejano.

The KKM (Knaster - Kuratowski - Mazurkiewicz) theorem is a topological theorem that has many applications in combinatorics, algorithms, game theory and mathematical economics. In this talk, Oleg Musin considered generalizations of Gale's colored KKM lemma and Shapley's KKMS theorem. It was shown that space and covers can be much more general and the boundary KKM rules can be substituted by more weaker boundary assumptions.

6 Transversal and Helly Theorems geometry and Combinatorics

Leonardo Martinez-Sandoval introduced the notion of *intersection depth* for a finite family of convex sets $\mathcal{F}$ in $\mathbb{R}^d$. Specifically, he say that a point p has *intersection depth* m *with respect to* $\mathcal{F}$ if every hyperplane that contains p intersects at least m sets of $\mathcal{F}$. He study some nice properties of intersection depth and he related it to other notions of depth in the literature.

By imposing additional intersection hypothesis to the family $\mathcal{F}$, he showed how to prove sharp center point theorems for intersection depth. These results can be thought of as a refinement that interpolates between the classical Rado's center point theorem and Helly's theorem. Finally, he used this result to get a new Helly-type theorem for fractional transversal hyperplanes that cannot be obtained from the well-studied $T(k)$ hypothesis.

In 1982, Bárány, Katchalski and Pach proved the following quantitative form of Helly's theorem: If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. They gave the bound $v(d) \leq d^{2d^2}$, and conjectured that $v(d) \leq d^{cd}$. In his talk Marton Naszodi confirmed this conjecture and discussed the proof and further results. The techniques of the proof of this quantitative form of Helly's theorem are extremely interesting. He uses the theory of the existence of minimal ellipsoids and the Dvoretzky-Roger lemma, that states the existence of a simplex of relatively large volume in a John decomposition of the identity. Furthermore, in his talk Ferenc Fodor presented a new measure theoretic version of the Dvoretzky-Rogers lemma which provides information about the distribution of isotropic measures. This result is joint with K.J. Böröczky (Budapest) and D. Hug (Karlsruhe).

During the work shop a joint work with Bárány and Grinberg about small subset sums was presented by Gergely Ambrus. In his work he considers a finite dimensional, real normed space, and for a given set of vectors of norm at most 1, which sum to 0, he tries to select a k-element subset with small norm. In his talk he gave some sharp estimates and proved some consequences regarding Steinitz's theorem as well.

A brief review of classical transversal results for finite families of ovals in the plane, and the recent developments that focus on weakening the disjoint condition in these results were presented in this workshop by Ted Bisztriczky. In his talk he presented a joint work with Heppes Aladar under a weaker than usual condition (namely ϕ-disjointness), and concluded with a list of research problems that follow from this generalization.

Antoine Deza presented an interesting new upper bounds on the diameter of lattice polytopes. Let $D(d, k)$ denote the largest possible diameter over all polytopes which vertices are drawn from $\{0, 1, ..., k\}^d$. In 1989, Naddef showed that $D(d, 1) = d$. This result was generalized in 1992 by Kleinschmidt and Onn who proved that $D(d, k) \leq kd$, before Del Pia and Michini tightened in 2016 the inequality to $D(d, k) \leq (k - 1/2)d$ for $k \geq 2$. Deza shows with coauthors that $D(d, k) \leq (k - 2/3)d$ for $k \geq 3$. In addition, he shows that $D(4, 3) = 8$, which substantiates the conjecture stating that $D(d, k) \leq (k + 1)d/2$, and is achieved by a Minkowski sum of lattice vectors. This work is based on collaboration with Lionel Pournin, Université Paris 13.

In the workshop the theory of one-sided epsilon-approximants was also present. Boris Bukh in his talk (which was a joint work with Gariel Nivasch) discussed the definition of a set A in $\mathbb{R}^d$ to be a one-sided ϵ-approximant to a set P in $\mathbb{R}^d$. if every convex set containing α-fraction of points P contains at least $(\alpha - \epsilon)$-fraction of points of A, for every α. He showed that every P admits a one-sided ϵ-approximant of size depending only on ϵ and on d.

presented a joint work with Edgardo Roldan and Gelasio Salazar about arrangements of pseudocircles on surfaces. Here a *pseudocircle* is an oriented closed Jordan curve on some surface, and a finite collection of pseudocircles that pairwise cross in exactly two points is an *arrangement of pseudocircles*, and it is *strict* if each pseudocircle is a separating curve on the host surface. Following Linhart and Ortner, the combinatorial properties of an arrangement of pseudocircles are encoded in an *intersection matrix*, in which each row corresponds to a pseudocircle, and the entries of the row give the cyclic order of its (signed) intersections with the other pseudocircles in the arrangement. Ortner proved that an arrangement of pseudocircles (given as an intersection matrix) can be embedded into the sphere if and only if each of its subarrangements of size four can be embedded in the sphere. In this workshop, Carolina Medina Graciano presented an extended result, where it was shown that an arrangement of pseudocircles (given as an intersection matrix) is embeddable into the compact orientable surface S_g of genus g if and only if each of its subarrangements of size $4(g + 1)$ can be embedded in S_g.

Finally and to give a different and refreshing topic on the workshop Wlodzimierz Kuperberg talked about the recent developments of the notion of extensive parallelograms and double-lattice packings along with various questions and conjectures that arise naturally. For instance a parallelogram inscribed in a given convex disk K in the plane is called *extensive* if each of its sides is at least as long as one-half of the affine diameter of K parallel to the side. Such a parallelogram generates a double-lattice packing of the plane that mixes translates of K with translates of $-K$. The smaller the area of the parallelogram, the denser the packing. The densest double-lattice packing of the the regular pentagon has density $(5 - \sqrt{5})/3 = 0.92131\ldots$, which is conjectured to be the highest density among all packings with the pentagon's congruent copies.

7 Problems

During the workshop some interesting new problems came up very naturally. Next we enumerate some of this interesting problems:

Given a proper subset $S \subset \mathbb{R}^d$, consider the space of all possible sets which are the intersection of a standard convex sets in $\mathbb{R}^d$ with S. In the literature these are called *convexity spaces*. What can we say about a Helly theorem over S?. For example, there is a Helly-type theorem that talks about the existence of intersections over the integer lattice $\mathbb{Z}^d$, proved by Doignon [6] (later rediscovered in [5, 9, 15]); it states that a finite family of convex sets in $\mathbb{R}^d$ intersects at a point of $S = \mathbb{Z}^d$ if every $h(S) = 2^d$ members of the family intersect at a point of $\mathbb{Z}^d$. A second example is the work of Averkov and Weismantel [1] who show that $h(\mathbb{Z}^{d-k} \times \mathbb{R}^k) = 2^{d-k}(k + 1)$ that is, they gave a *mixed* version of Helly's and Doignon's theorems which includes them both.

An interesting open porblem is the following. Given a general set $S \subset \mathbb{R}^d$, give bounds on the Helly number $h(S)$ if it is finite. For instance, note that when S is finite then the bound $h(S) \leq \#(S)$ is trivial. The original Helly number is $h(\mathbb{R}^d) = d+1$ and, interestingly, if $\mathbb{F}$ is any subfield of $\mathbb{R}^d$, then Radon's proof of Helly's theorem directly shows that the S-Helly number of $S = \mathbb{F}^d$ is still $d+1$. However, if S is not necessarily a lattice but a general additive subgroup (e.g., $S = \{(\alpha\pi + \beta, \gamma) \in \mathbb{R}^2 : \alpha, \beta, \gamma \in \mathbb{Z}\}$), then the S-Helly number has a more complex answer given by De Loera, La Haye, Oliveros, and Roldán-Pensado. But for general sets $S \in \mathbb{R}^d$ the problem remains open. Thus in the problem session of the workshop, Jesus De Loera proposed the problem of studying the case when $S = (Primes)^d \in \mathbb{R}^d$ and see if a Helly number is possible.

On the open problem session several interesting problems over finitely many points in general position in the plane where treated, For example Imre Bárány, proposed the following interesting question: Let X be a set of finitely many points in general position in the plane. Distinct points $x, y, z \in X$ forms an empty triangle if the triangle with vertices x, y, z contains no further point from X. Let $h(X)$ be the number of empty triangles in X, and define

$$h(n) = \min\{h(X) \mid X \subset \mathbb{R}^2 \text{ in general position and } |X| = n\}.$$

It is known that $n^2 - 5n < h(n) < 1.7n^2$, (see for instance [4]) Then the question is to improve the lower bound, even to $h(n) > 1.0001n^2$.

Boris Bukh proposed the following problem: Given a set S of n points in $\mathbb{R}^2$ in general position, and let $\chi : S \to \{-1, +1\}$. For a line L and point p, let $s_L(p) = +1$ if p lies on the left of L, and $s_L(p) = -1$ if p lies on the right of L. Define *disbalance* by

$$d(L) = \sum_{p \in P} s_L(p)\chi(p)L.$$

Let $0 < a \leq 1$, and suppose $d(L) \leq an$ for every line L. Does it follow that there is a point p such that every line L through p satisfies $d(L) \leq bn$ for some $b = b(a)$ that depends only on a and for which $b < a$?

And Frederic Meunier proposed the following open problem: Consider a half disk of diameter 1, containing n points denoted $p_1, \ldots, p_n$. Is it true that there always exists a permutation σ such that $\sum_{i=0}^{n+1} ||p_{\sigma(i+1)} - p_{\sigma(i)}||^2$ is at most 1? Here, p_0 and p_{n+1} are the endpoints of the diameter.

With a more topological flavor Luis Montejano proposed the following Nerve-type theorem that has interesting geometric applications. Let $X = A_1, \ldots, A_m$ be a polyhedron cover of X and let N be its nerve. Suppose that for every $\sigma \in N^{(k)}$, $H_{|\sigma|-2}(\bigcup_\sigma) = 0$. Then, $\text{Rank} H_k(N) \leq \text{Rank} H_k(X)$, where we consider reduced homology with coefficients in a field. The case $k = 0$ is trivial and it is true for $k = 1$, and it is open for $k \geq 2$. When the nerve N is the boundary of the $k+1$-simplex, then this claim take the form of a special topological Helly type theorem proved independently by Kalai and Meshulam, Montejano and by Karasev in terms of sheaf cohomology.

Very closely related to his talk on the notion of extensive parallelograms and double-lattice packings Wlodzimierz Kuperberg proposed the following question:

Characterize convex polyhera P in $\mathbb{R}^3$ that can tile space by translates of P combined with translates of $-P$. Generalize, if possible, to every dimension $d > 3$. It is important to observe that a characterization of convex polyhedra that can tile $\mathbb{R}^n$ by translates only is known for every n (Venkov, 1954; McMullen, 1980). In two dimensions, it is known and not hard to prove that a convex polygon P tiles the plane by translates of P combined with translates of $-P$ if and only if P is a *p-hexagon* by Wlodzimierz Kuperberg himself in 1982), that is, a triangle, a parallelogram, a pentagon with a pair of parallel sides, or a hexagon with a pair of parallel opposite sides of equal length.

The following problem was proposed by Wlodzimierz Kuperberg and Andras Bezdek. A *right convex prism* with base P is a subset of $\mathbb{R}^3$ congruent to the Cartesian product of a convex polygon P with the interval $[0, 1]$. If space can be tiled with congruent right convex prisms, must the base tile the plane by its congruent copies? Observing that If the base in not convex, then there is a counterexample and furthermore by assuming convexity of the base, but some relaxation on the requirement of rightness of the prism, allowing for instance the prism's lateral edges to deviate from perpendicularity to the base by an arbitrarily small angle $\varepsilon > 0$, then, again, there is a counterexample given by A. Bezdek and W.Kuperberg in 1990).

8 Conclusions

The workshop was successful in many ways, it broughth together old and new colleagues from all over the world. We had participants from many countries including Russia, Germany, France, USA, Mexico, Korea, Canada, Hungary, among others. The talks were far from being the only academic activity of the workshop. We had many formal and informal mathematical discussions and all these activities have given rise to many new research projects and new collaborations.

We appreciate and would like to thank the support we have received from BIRS and CMO. The excellent facilities and environment that was giving us by CMO and the beautiful city of Oaxaca, provide us with a perfect atmosphere for creative interaction that allow us to exchange ideas, knowledge, and methods within the Mathematical Sciences. We would like to thank program coordinator Chee Chow, CMO-BIRS coordinator Silvia Millán López and conference assistant Claudia Arias for all their support in the organization of the conference . We would like to thank as well all the participants of the Transversal, Helly and Tverberg type Theorems in Geometry, Combinatorics and Topology III for all their enthusiasm and the productive, enjoyable environment that was created.

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