Spectral Properties of Quasicrystals via Analysis, Dynamics, and Geometric Measure Theory

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September 27 – October 2, 2015

The 2011 Nobel Prize in Chemistry was awarded to Dan Shechtman for "the discovery of quasicrystals" — materials with unusual structure, interesting from the point of view of chemistry, physics, and mathematics. In order to study electronic properties of quasicrystals, one considers Hamiltonians where the aperiodic order features are reflected either through the configuration of position space or the arrangement of the potential values. The spectral and quantum dynamical analysis of these Hamiltonians is mathematically very challenging. On the other hand, investigations of this nature are fascinating as one is invariably led to employ methods from a wide variety of mathematical subdisciplines. At present one understands very well the key quasicrystal models in one space dimension and the community is finally on the verge of making serious progress in the much more challenging higher-dimensional case by drawing on a new connection to yet another mathematical subdiscipline.

At this meeting, mathematicians from various areas and also some physicists came together to push the boundaries of our understanding of mathematical quasicrystal models and other closely related topics. While it is impossible to give a comprehensive summary of the numerous talks and the results that were discussed during the meeting in a short note, we will provide here an overview of four different topics to show how the different approaches interconnect and influence each other. In the first section we briefly discuss some recent results on spectral properties of one dimensional quasicrystals obtained using the dynamical properties of the corresponding polynomial trace maps, and the way these results can be applied to study spectral properties of the Square and Cubic Fibonacci Hamiltonians. Then in the section "Some Aspects of Quantum Walks in a Fibonacci Environment" (written by Jake Fillman) the properties of Fibonacci quantum walks and related CMV matrices are discussed. The section "Spectral Calculations for Discrete Schrödinger Operators with Quasiperiodic Potentials" (written by Mark Embree) describes some numerical methods that can be used to study spectral properties of quasicrystals. Finally, the section "Polynomial Dynamics and Trace Maps" (written by Eric Bedford) consists of a very short survey of the modern theory of polynomial dynamical systems that can be applied to study the properties of the trace maps that appeared in the previous sections.

Spectral properties of Square and Cubic Fibonacci Hamiltonian

The spectral properties of the Fibonacci Hamiltonian $H_{\lambda,\omega} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$,

$$[H_{\lambda,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + \lambda\chi_{[1-\alpha,1)}(n\alpha + \omega \bmod 1)\psi(n),$$

where the parameter λ is a coupling constant, $\alpha = \frac{\sqrt{5}-1}{2}$ is the frequency, and $\omega \in S^1$ is the phase, have recently been studied in detail [13, 14, 15, 16, 37, 38]. In particular, it is known that its spectrum Σ_λ is a dynamically defined Cantor set for all $\lambda > 0$ (see [17]). Many of the methods that were used to study the Fibonacci Hamiltonian on the one dimensional lattice cannot be directly extended to two- (or higher-) dimensional quasicrystals. For example, the results on existence of the integrated density of states [28, 29, 31] are the only rigorous results on spectral properties of Laplacian on Penrose tilings (see the section ”Spectral Calculations for Discrete Schrödinger Operators with Quasiperiodic Potentials” below for a discussion of some numerical experiments related to this operator). One of the ways to gain some intuition on spectral properties of two-dimensional quasicrystals is to consider the Square and Cubic Fibonacci Hamiltonians. Namely, one can consider the bounded self-adjoint operator

$$\begin{aligned}[H^{(2)}_{\lambda_1,\lambda_2,\omega_1,\omega_2}\psi](m,n) = \psi(m+1,n) + \psi(m-1,n) + \psi(m,n+1) + \psi(m,n-1) + \\ + \Big(\lambda_1\chi_{[1-\alpha,1)}(m\alpha + \omega_1 \mathrm{mod}\ 1) + \lambda_2\chi_{[1-\alpha,1)}(n\alpha+\omega_2 \mathrm{mod}\ 1)\Big)\psi(m,n) \quad (1)\end{aligned}$$

in $l^2(\mathbb{Z}^2)$, with $\alpha = (\sqrt{5}-1)/2$, coupling constants $\lambda_1, \lambda_2 > 0$ and phases $\omega_1, \omega_2 \in S^1$. The theory of separable operators (cf. [34]) quickly implies that

$$\Sigma_{\lambda_1,\lambda_2} = \Sigma_{\lambda_1} + \Sigma_{\lambda_2}, \quad \text{and} \quad \nu_{\lambda_1,\lambda_2} = \nu_{\lambda_1} * \nu_{\lambda_2}, \quad (2)$$

where the set sum and the convolution of measures are defined by

$$A + B = \{a+b : a \in A,\ b \in B\}, \quad \int_{\mathbb{R}} g(E)\, d(\mu * \nu)(E) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(E_1+E_2)\, d\mu(E_1)\, d\nu(E_2).$$

Square operators (sometimes called *operators with separable potentials*) were suggested by physicists [23, 24, 25] as a reasonably approachable model for higher dimensional quasicrystals. Cubic models follow analogously, though now on a three-dimensional lattice, giving $\Sigma_{\lambda_1,\lambda_2,\lambda_3} = \Sigma_{\lambda_1} + \Sigma_{\lambda_2} + \Sigma_{\lambda_3}$. The closely related *labyrinth model* (where the spectrum is the product of the spectra of the one-dimensional models instead of the sum) was considered in [35, 36]. Combining results from [15] and [7] we get the following result.

Theorem 1. *For all sufficiently small $\lambda_1, \lambda_2 > 0$ the spectrum $\Sigma_{\lambda_1,\lambda_2}$ of the operator (1) is an interval. For all sufficiently large $\lambda_1, \lambda_2 > 0$ it is a Cantor set of zero measure.*

In [18] the following result regarding the density of states of the operator (1) is proven.

Theorem 2. *For almost all (with respect to the Lebesgue measure) sufficiently small $\lambda_1, \lambda_2 > 0$ the density of states measure $\nu_{\lambda_1,\lambda_2}$ of the operator (1) is absolutely continuous. For all sufficiently large $\lambda_1, \lambda_2 > 0$ it is singular.*

These results leave open the question on the structure of the spectrum of (1) in the intermediate regimes of couplings. Due to (2), this question is directly related to the notoriously hard questions on sums of dynamically defined Cantor sets that also have applications in dynamical systems and number theory, and were heavily studied. In particular, the results from [32] motivate the following conjecture.

Conjecture. *There exists a domain $U \subset \mathbb{R}^2$ such that for almost all (with respect to the Lebesgue measure) $(\lambda_1, \lambda_2) \in U$ the spectrum $\Sigma_{\lambda_1,\lambda_2}$ of the operator (1) is a Cantorval (i.e. it is a compact subset of $\mathbb{R}^1$ that has a dense interior, has a continuum of connected components, and none of them is isolated).*

This suggests a completely new topological type of spectrum of a ”natural” operator.

Some Aspects of Quantum Walks in a Fibonacci Environment

Jake Fillman (Virginia Tech, USA)

A quantum walk is described by a unitary operator on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$, which models a state space in which a wave packet comes equipped with a “spin” at each integer site. To specify a quantum walk, one chooses *quantum coins*, i.e., 2×2 unitaries:

$$C_n = \begin{pmatrix} c_n^{11} & c_n^{12} \\ c_n^{21} & c_n^{22} \end{pmatrix} \in \mathbb{U}(2), \quad n \in \mathbb{Z}. \tag{3}$$

As one passes from time t to time $t+1$, the update rule of the quantum walk is:

$$\delta_n \otimes e_\uparrow \mapsto c_n^{11} \delta_{n+1} \otimes e_\uparrow + c_n^{21} \delta_{n-1} \otimes e_\downarrow, \tag{4}$$

$$\delta_n \otimes e_\downarrow \mapsto c_n^{12} \delta_{n+1} \otimes e_\uparrow + c_n^{22} \delta_{n-1} \otimes e_\downarrow. \tag{5}$$

If we extend (4) and (5) by linearity and continuity to general elements of $\mathcal{H}$, this defines a unitary operator U on $\mathcal{H}$. Now, if we order the basis of $\mathcal{H}$ via $\varphi_{2m-1} = \delta_m \otimes e_\uparrow$, $\varphi_{2m} = \delta_m \otimes e_\downarrow$ for $m \in \mathbb{Z}$, then U is represented by the matrix

$$U = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & & \\ & 0 & 0 & c_1^{21} & c_1^{22} & & & & & \\ & c_0^{11} & c_0^{12} & 0 & 0 & & & & & \\ & & & 0 & 0 & c_2^{21} & c_2^{22} & & & \\ & & & c_1^{11} & c_1^{12} & 0 & 0 & & & \\ & & & & & 0 & 0 & c_3^{21} & c_3^{22} & \\ & & & & & c_2^{11} & c_2^{12} & 0 & 0 & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{6}$$

We can connect quantum walks to CMV matrices using the following observation. If $\mathcal{E}$ is an extended CMV matrix for which all Verblunsky coefficients with even index vanish, then $\mathcal{E}$ takes the form.

$$\mathcal{E} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & & & & & & \\ & 0 & 0 & \overline{\alpha_1} & \rho_1 & & & & & \\ & \rho_{-1} & -\alpha_{-1} & 0 & 0 & & & & & \\ & & & 0 & 0 & \overline{\alpha_3} & \rho_3 & & & \\ & & & \rho_1 & -\alpha_1 & 0 & 0 & & & \\ & & & & & 0 & 0 & \overline{\alpha_5} & \rho_5 & \\ & & & & & \rho_3 & -\alpha_3 & 0 & 0 & \\ & & & & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{7}$$

Since $\rho_n \geq 0$ for CMV matrices, one has to tweak this a bit (but not much); namely, it is easy to inductively defined a diagonal unitary matrix Λ so that $\Lambda U \Lambda^*$ is a CMV matrix of the form (7).

Naturally, one can consider the family of quantum walks generated by the Fibonacci sequence. Concretely, let $\omega \in \{0,1\}^{\mathbb{Z}}$ be a Fibonacci sequence, choose quantum coins $\theta_0 \neq \theta_1 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and define a quantum walk operator $U = U_\omega$ via

$$Q_0 = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad C_n = Q_{\omega(n)}.$$

One has a transfer matrix formalism for this operator family much like the transfer matrix formalism for the Fibonacci Hamiltonian. Specifically, define

$$M_{-1}(z) = \sec\theta_1 \begin{pmatrix} z & -\sin\theta_1 \\ -\sin\theta_1 & z^{-1} \end{pmatrix}, \quad M_0(z) = \sec\theta_0 \begin{pmatrix} z & -\sin\theta_0 \\ -\sin\theta_0 & z^{-1} \end{pmatrix}$$

and $M_n = M_{n-1} M_{n-2}$ for $n \geq 1$. One then considers $x_n(z) = \frac{1}{2}\mathrm{tr}(M_n(z)$, which obeys the usual recursion

$$x_{n+1} = 2x_n x_{n-1} - x_{n-2}.$$

As in the Schrödinger case, the spectrum of U_ω is characterized as the set of parameters at which the trace-map orbit is bounded:

$$\sigma(U_\omega) = \{z \in \mathbb{C} : \{x_n(z)\}_{n=1}^{\infty} \text{ is a bounded sequence}\}.$$

One distinction here is that the Fricke–Vogt character is not constant on the spectrum. Instead, one has

$$I(z) := I(x_{n-1}(z), x_n(z), x_{n+1}(z)) = \sec^2\theta_0 \sec^2\theta_1 (\sin\theta_0 - \sin\theta_1)^2 (\mathrm{Im}(z))^2.$$

Using the transfer matrix formalism, one can show (much as in the Fibonacci case) that $\sigma(U_\omega)$ is a Cantor set of zero Lebesgue measure and that the spectral type of U_ω is purely singular continuous for all ω and all choices of $\theta_0 \neq \theta_1$.

In this setting, the role of the coupling constant is (roughly) played by

$$\mu = \mu(\theta_1, \theta_b) := \inf_{k \geq -1} \min_{z \in \sigma_k} I(z). \tag{8}$$

Let us also define

$$\kappa = \kappa(\theta_1, \theta_b) = |\sec\theta_1 \tan\theta_b - \tan\theta_1 \sec\theta_b| = |\sec\theta_1||\sec\theta_b||\sin\theta_1 - \sin\theta_b|. \tag{9}$$

The main result on the unitary dynamics of U_ω is the following theorem:

Theorem 1. *Let $\pi/4 < \theta_0 < \pi/2$ be given. There exist constants $m = m(\theta_0)$, $M = M(\theta_0)$, and $\lambda = \lambda(\theta_0)$ such that if $\theta_0 < \theta_1 < \pi/2$ with $\mu \geq \lambda$, one has*

$$\frac{1}{1+\tau} - \frac{3\tau+\eta}{p(1+\tau)} = \frac{p - 3\tau - \eta}{p(1+\tau)} \leq \widetilde{\beta}^-_{\delta_0}(p) \leq \beta^+_{\delta_0}(p) \leq \frac{2\log\varphi}{\log\xi}, \tag{10}$$

where

$$\xi = m\sqrt{\mu}, \quad \Xi = M\sqrt{\mu}, \quad \eta = \frac{\log\Xi}{\log\varphi} - 1, \quad \tau = \frac{2\log\left((\kappa+2)(2\kappa+5)^2\right)}{\log\varphi}, \tag{11}$$

and κ is as in (9).

One can roughly summarize the theorem by saying that the exponents $\beta^{\pm}_{\delta_0}(p)$ converge to zero roughly like constant$/\log\kappa$ as $\theta_1 \uparrow \pi/2$, which constitutes a natural analog of the large-coupling asymptotics of the dynamics of the Schrödinger group associated with the Fibonacci Hamiltonian. The key to thes dynamical analysis is the following consequence of the Parseval formula, which enables one to estimate dynamical quantities via estimates on the matrix elements of the resolvent: Given any unitary operator U on a Hilbert space $\mathcal{H}$, and any two elements $\varphi, \psi \in \mathcal{H}$, one has

$$\sum_{\ell=0}^{\infty} e^{-2\ell/L} |\langle\varphi, U^\ell\psi\rangle|^2 = e^{2/L} \int_0^{2\pi} |\langle\varphi, (U - e^{i\theta+1/L})^{-1}\psi\rangle|^2 \, \frac{d\theta}{2\pi}. \tag{12}$$

The proof of Theorem 1 then proceeds by proving careful estimates on the trace map orbit at energies just off the unit circle, which provides estimates on transfer matrices, and hence on the resolvent of U_ω. Using (12), the resolvent estimates yield estimates on the dynamics.

Of course, this sketch is painting with a rather broad brush – the analysis is fairly delicate, and many further things come into play. In particular, one also needs a careful combinatorial analysis of the band structure of periodic approximants, as well as good estimates on the Lebesgue measure of the spectra of such approximants.

The connection between 1D coined quantum walks and CMV matrices was first observed in [5], and has since led to some interesting discoveries in that setting; see [6] and references therein for general quantum walks and [9, 10, 19] for the Fibonacci quantum walk.

The characterization of the spectrum of U_ω via bounded trace orbits and the zero-measure Cantor structure thereof comes from [19]. However, they do not characterize the spectral type as purely continuous. In fact, I can't find that statement in any of the usual papers on this topic. In any case, it follows easily from Gordon-type arguments, e.g. [26].

The formula (12) was first observed in [10], in the hopes of working out general resolvent estimates schemes in the spirit of [20, 21, 22]; this vision was realized in [9] by using the key formula of [11] which relates the two families of $\mathrm{GL}(2, \mathbb{C})$ cocycles associated to a CMV matrix.

The trace-map estimates used in the proof of Theorem 1 come from [12]. One really needs this, and not earlier results, because [12] is able to deal with complex orbits.

The combinatorial analysis of the band structure of the periodic approximants is spiritually related to [33], but follows the arguments of [30] (in fact, this is the main place where the large coupling assumption is necessary).

Spectral Calculations for Discrete Schrödinger Operators with Quasiperiodic Potentials
Mark Embree (Virginia Tech, USA)

The spectra for Schrödinger operators with quasiperiodic potentials exhibit a variety of inscrutable properties. Numerical computations can help one develop insight and intuition, and eventually formulate conjectures. However, many of the same factors that make these spectral problems mathematically challenging also tax conventional algorithms for computing eigenvalues. For example, to approximate a spectrum that is a Cantor set, one seeks *all* the eigenvalues of *large* symmetric matrices; given the Cantor structure, these eigenvalues are often *quite close together*, implying that they should computed to high relative accuracy, and this challenge becomes particularly acute when the potential is scaled by a large coupling constant. Furthermore, one often seeks to cover the spectrum with a union of intervals: imprecise knowledge of the ends of these intervals complicates the calculation of their unions (for covers) and sums (for higher dimensional models), as well as the numerical estimation of the fractal dimension of the spectrum.

Often upper bounds on the spectrum can be obtained by approximating the quasiperiodic potential with a related potential having period p, whose spectrum (the union of p intervals) can be computed using tools from Floquet theory. To determine the endpoints of these p intervals, one must compute all the eigenvalues of two $p \times p$ symmetric matrices $J_\pm$ that are zero everywhere but the main diagonal, first super- and sub-diagonals, and the corner entries. The first step in a symmetric eigenvalue computation reduces the matrix to tridiagonal form via a unitary similarity transformation, which, given the structure of $J_\pm$, will require $O(p^3)$ operations.

This talk (based on joint work with Charles Puelz and Jake Fillman) described several techniques to improve the computation of spectra for periodic potentials with large p. By reordering the rows and columns of $J_\pm$, we can reduce the complexity of the eigenvalue computations to $O(p^2)$ operations. This acceleration makes the computation of eigenvalues in extended (quadruple precision) arithmetic more tractable, thus mitigating accuracy limitations and enabling numerical calculations in parameter regimes (approximation lengths p and coupling constants λ) for which the conventional approach would be too slow or inaccurate.

To illustrate the utility of the algorithm, we showed numerically computed covers of the spectrum of the period doubling and Thue–Morse potentials, and discussed attempts to estimate the Hausdorff dimension of the spectra of these operators as a function of the coupling constant These potentials conventionally act on a one-dimensional lattice, but it is natural to extend them to two- and three-dimensional lattices. The spectra of these higher dimensional models – sums of Cantor sets – can be approximated by summing covers of the one-dimensional models. We presented numerically computed covers for the period doubling and Thue–Morse examples on a two-dimensional lattice, as illustrated in Figure 1.

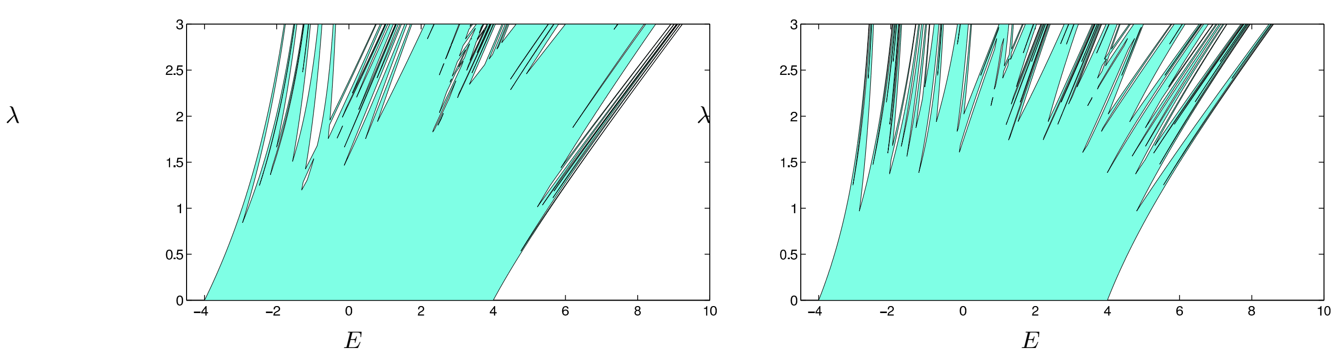

Figure 1: Numerically computed covers of the spectra of the period doubling (left) and Thue–Morse (right) model on a two-dimensional lattice, as a function of the coupling constant λ.

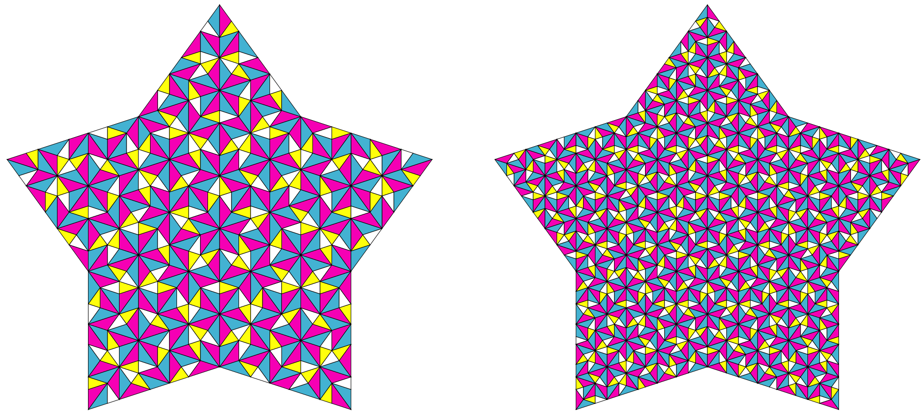

Figure 2: Levels 5 and 6 of the substitution rule applied to Robinson triangular tiles.

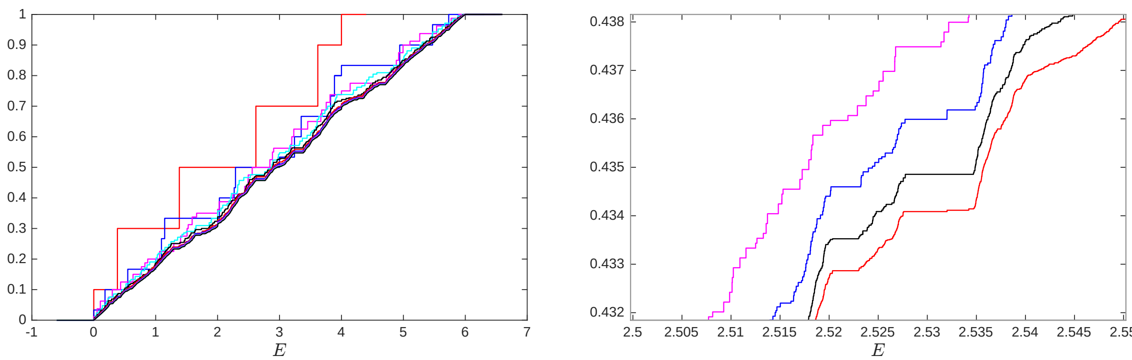

Figure 3: Left: integrated density of states for the graph Laplacian for the Robinson stone construction of the Penrose aperiodic tiling, levels 1 through 10. Right: a close-up of levels 7 (magneta), 8, 9 and 10 (black), suggesting a potential gap in the spectrum of the infinite tiling.

The talk closed with a discussion of model with more challenging two-dimensional structure, the graph Laplacian on the Penrose aperiodic tiling of the plane. Following earlier spectral calculations conducted by May Mei and her students, we use a substitution rule applied to Robinson's triangular tiles to recursively construct increasingly large portions of the Penrose tiling. Figure 2 shows levels 5 and 6 of this construction. At the end of the workshop we computed all eigenvalues of the graph Laplacian for the tenth level of this tiling (comprising 177,110 tiles, generated by nine iterations of the level 1 base case that starts with 10 triangular tiles), made possible by applying a row/column reordering scheme similar to the one described above for one-dimensional problems. (Equivalently, index the tiles in manner that gives particularly efficient numerical calculations.) Figure 3 shows a superposition of the integrated density of states for the first ten levels, and zooms in around one region in the spectrum for which the numerical evidence suggests there might be gaps. It is too early to shape these calculations into a robust conjecture, but these results show the potential for numerical calculations to yield some insight into these challenging higher dimensional models.

Polynomial Dynamics and Trace Maps

Eric Bedford (Indiana University, USA)

We will discuss some aspects of complex dynamics that should be relevant for the study of trace maps, which are polynomial self-maps of $\mathbb{R}^k$. Here we will extend the trace maps to all of $\mathbb{C}^k$. A basic characteristic of a polynomial $p(x_1, \ldots, x_k) = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} x_1^{i_1} \cdots x_k^{i_k}$ is its degree: $\deg(p) = \max(i_1 + \cdots + i_k)$, where the maximum is taken over all k-tuples $(i_1, \ldots, i_k)$ such that $a_{i_1,\ldots,i_k} \neq 0$.

On the other hand, if $f = (f_1, \ldots, f_k)$ is a polynomial mapping, i.e., all the coordinate functions f_j are given by polynomials, there is more than one way to define the degree. First, there is the *algebraic degree* $\deg_{\text{alg}}(f) := \max_{1\le j\le k} \deg(f_j)$. Under composition, we have $\deg_{\text{lag}}(f \circ g) \le \deg_{\text{lag}}(f)\deg_{\text{lag}}(g)$, and in general this inequality is strict.

There is also the *mapping degree* or *topological degree*, which is the number or preimages $\deg_{\text{top}}(f) := \#f^{-1}(y)$, where y is a generic point. This has the property that $\deg_{\text{top}}(f \circ g) = \deg_{\text{top}}(f)\deg_{\text{top}}(g)$.

For example, we consider two maps:

$$s(x, y) = (x, y + x^2), \quad h(x, y) = (x^2 - y, x)$$

These maps have polynomial inverses. In both cases, the algebraic degree is 2, and the topological degree is 1. However, we are interested in the iterates $f^n := f \circ \cdots \circ f$, and we see that $\deg(s^n) = 2$, whereas $\deg(h^n) = 2^n$. This leads us to define the *dynamical degree*

$$\delta(f) = \lim_{n\to\infty} (\deg(f^n))^{1/n}$$

which has the essential propery of being invariant under conjugation: $\delta(f) = \delta(g^{-1} \circ f \circ g)$. The dynamical degree is also known as *degree complexity* of the map.

Examples. In the case of the Fibonacci trace map, we have $f_{\text{fib}}(x, y, z) = (y, xy - z, x)$, which is an automorphism of $\mathbb{C}^3$, and the dynamical degree is $\delta(f_{\text{fib}}) = \phi$, the golden ratio.

In the case of Period Doubling, the trace map is $f_{\text{PD}} : (x, y) \mapsto (xy - 2, x^2 - 2)$. In this case we have $\deg_{\text{top}}(f_{\text{PD}}) = 2$ and $\delta(f_{\text{PD}}) = 2$.

In the case of Thue-Morse, the trace map is $f_{\text{TM}} : (x, v) \mapsto (x^2 - 2 - v, v(x + 4 - x^2)$. We have $\deg_{\text{lag}}(f^n_{\text{TM}}) = 2^{n+1} - 1$, so $\delta(f_{\text{TM}}) = 2$, and $\deg_{\text{top}}(f_{\text{TM}}) = 5$.

Intermediate degrees. Another way to think of degrees is as follows. Let $H = \{\sum a_j x_j = 0\} \subset \mathbb{C}^k$ be a linear hypersurface. Then for generic H, the degree of $f^{-1}(H) = \{\sum a_j f_j(x) = 0\}$ is just $\deg_{\text{lag}}(f)$. And a linear subspace P of codimension k is just a point, so $\deg_{\text{top}}(f)$ is just the degree (= number of points) of $f^{-1}(P)$. Motivated by this, we can define a degree $\deg_\ell(f)$ to be the degree of $f^{-1}(L)$ for a generic linear subspace $L \subset \mathbb{C}^k$ of codimension ℓ. The degree in this case is the number of points $\#(V \cap f^{-1}(L))$, for generic linear subspaces V and L where V has dimension ℓ, and L has codimension ℓ. A number of the properties of the degrees δ_ℓ, $1 \le \ell \le k$, are given in the survey paper [27]. However, it is currently challenge to actually evaluate the dynamical degrees; at present, there are very few nontrivial examples f for which δ_ℓ has been computed for $1 < \ell < k$.

Approaches to understanding the dynamics of a polynomial (or rational) map: large topological degree. A k dimensional mapping is said to be of *large topological degree* if $\delta_k > \delta_{k-1}$. The Thue-Morse map is an example of a map with large topological degree. In this case, the dynamical degrees are dominated by the topological degree: $\delta_\ell < \delta_{\text{top}}$ for all $1 \le \ell < k$, and the dynamics is described by the pullbacks of zero-dimensional objects (points) in the following way: There is an *exceptional set* $\mathcal{E}$ and a measure μ such that for $x_0 \notin \mathcal{E}$,

$$\lim_{n\to\infty} (\delta_{\text{top}})^{-n} \sum_{a \in f^{-n}(x_0)} \delta_a = \mu$$

There are several theorems to say that $\mathcal{E}$ is "small" when f satisfies various hypotheses (see the survey article [27]).

Dynamics of surface maps with small topological degree. If we consider the case of dimension $k = 2$, then there are only two dynamical degrees of interest: $\delta_1 = \delta$ and $\delta_2 = \deg_{\text{top}}$. The case of *low topological degree* is $\delta_1 > \delta_2$, which includes all surface diffeomorphisms for which $\delta_1 > 1$. For instance, the Fibonacci trace map is a diffeomorphism of $\mathbb{C}^3$, and $\mathbb{C}^3$ is filled out by a family $\{S_t : t \in \mathbb{C}\}$ of invariant surfaces. The restriction $f_{\text{fib}}|_{S_t}$ is such a surface diffeomorphism.

In this case, the dynamics is described by the pullback of an object of codimension 1, say a (complex) line L. In this case, we pull back the current of integration $[L]$ and obtain an invariant current:

$$\mu^+ := \lim_{n\to\infty} \delta_1^{-n} [f^{-n} L]$$

A special feature of a surface diffeomorphism is that $\delta_1(f) = \delta_1(f^{-1})$. We may pull back a line under f^{-1} and obtain an invariant current μ^-. The wedge (intersection) product $\mu := \mu^+ \wedge \mu^-$ is an invariant measure, which has a number of dynamically interesting properties. These are described in [3] and subsequent papers. For a recent survey of the subject, we recommend [4].

Enumerative dynamics: the study of the complexifications of real maps. There is another approach which may be applicable to the study of trace maps. A real polynomial (or rational) map be considered simultaneously as a map of $\mathbb{R}^2$ and a map of $\mathbb{C}^2$. In [1] and [2] this approach was applied to the family:

$$f_a(x, y) = \left(y\frac{x+a}{x-1}, x+a-1\right)$$

As we mentioned in the previous paragraph, the complex approach is to consider the current μ^+ which given by the limit of the curves which are preimages $f_a^{-n}(L)$. The invariant measure μ is the limit of the average of point masses over the discrete set $f_a^{-n}(L_1) \cap f_a^n(L_2)$ for general real lines L_1 and L_2. Here we can consider L_j as real lines, and we also consider their complexifications. When we consider f_a as a real map of real lines, we use the geometry of how $f_a^{-n}(L)$ lies in $\mathbb{R}^2$, and we can say, for instance, that $f_a^{-n}(L)$ must cross an arbitrary vertical line V in at least d' points. Thus we have $f_a^{-n}(L) \cap V \geq d'$.

On the other hand, we may compactify $\mathbb{C}$ as the Riemann sphere $\mathbb{P}^1$, which is the same as projective space in complex dimension 1. For the map f_a, it is natural to compactify $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$ as $\mathbb{P}^1 \times \mathbb{P}^1$. A natural basis for the cohomology group $H^2(\mathbb{P}^1 \times \mathbb{P}^1)$ is given by the classes H (resp. V), corresponding to a horizontal (resp. vertical) complex line. If we write F_a for the extension of f_a to $\mathbb{P}^1 \times \mathbb{P}^1$, then the action of F_a on cohomology is given by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Thus if H is a horizontal line, its class will be given in this basis as $(1, 0)$, and the image under F_a^{*n} will be given in this basis as $\begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. In other words, this is the cohomology class of $F_a^{-n}(H)$. Now the intersection of cohomology classes corresponds to the intersection of curves according to the rule:

$$H \cdot H = 0 \quad V \cdot V = 0 \quad H \cdot V = 1 \tag{$*$}$$

Here "intersection" refers to all intersections inside $\mathbb{P}^1 \times \mathbb{P}^1$, counted with multiplicity. And in the complex domain, all intersection multiplicities are ≥ 1. Thus, the number of complex intersections (counted with multiplicity) between V and $F_a^{-n}(L)$ is given by the rule $(*)$: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \phi_n \\ \phi_{n-1} \end{pmatrix} = \phi_n$.

If we return to the previous paragraph, then we know that $\phi_n \geq d'$. If, in our situation, we find sufficiently many real intersections that $d' = \phi_n$, then we know that there can be no further intersections. This technique can be versatile. It has been used to study the family $\{f_a : a < -1\}$, which is a family of real maps with (maximal) entropy equal to $\log \phi$ (see [2]), and it was also used to show that the real map f_a has zero entropy when $a = 3$ (see [2]).

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