Big Surfaces Here,
Big Surfaces There,
Big Surfaces Everywhere

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Opening Question:

Compact Surfaces

Question (Mann)

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RMK: Yes in C^1 case, Zimmer program

How does this relate to big surfaces?

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Let's focus on special case $S = \mathbb{D}^2$, giving an answer in the C^1-setting with additional hypotheses, but avoiding the Zimmer program.

Theorem (Calegari '04)

Let G be a countable group.
If G acts faithfully on $\mathbb{D}^2$ such that

- the action is by C^1-homeomorphisms on $\mathrm{int}(\mathbb{D}^2)$, and
- $\exists\, x \in \mathbb{D}^2$ s.t. $\overline{G \cdot x} \cap \partial \mathbb{D}^2 = \phi$,

then G is cyclically orderable.

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This yields an obstruction to a group acting on $\mathbb{D}^2$ in the above fashion.

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Proof idea: $\exists$ SES

$$\underline{1} \to K \to G \to MCG(\mathbb{D}^2 \setminus C),$$

where $C \subset \mathrm{int}(\mathbb{D}^2)$ is either finite or a Cantor set. Now, prove K and $MCG(\mathbb{D}^2 \setminus C)$ are cyclically orderable.

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requires C^1

Complex Dynamics

Def (Shift polynomial)

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Def (Shift locus of degree d)

$S_d = \{\text{degree } d \text{ shift polynomials}\} / \text{Affine Conjugation}$

Problem

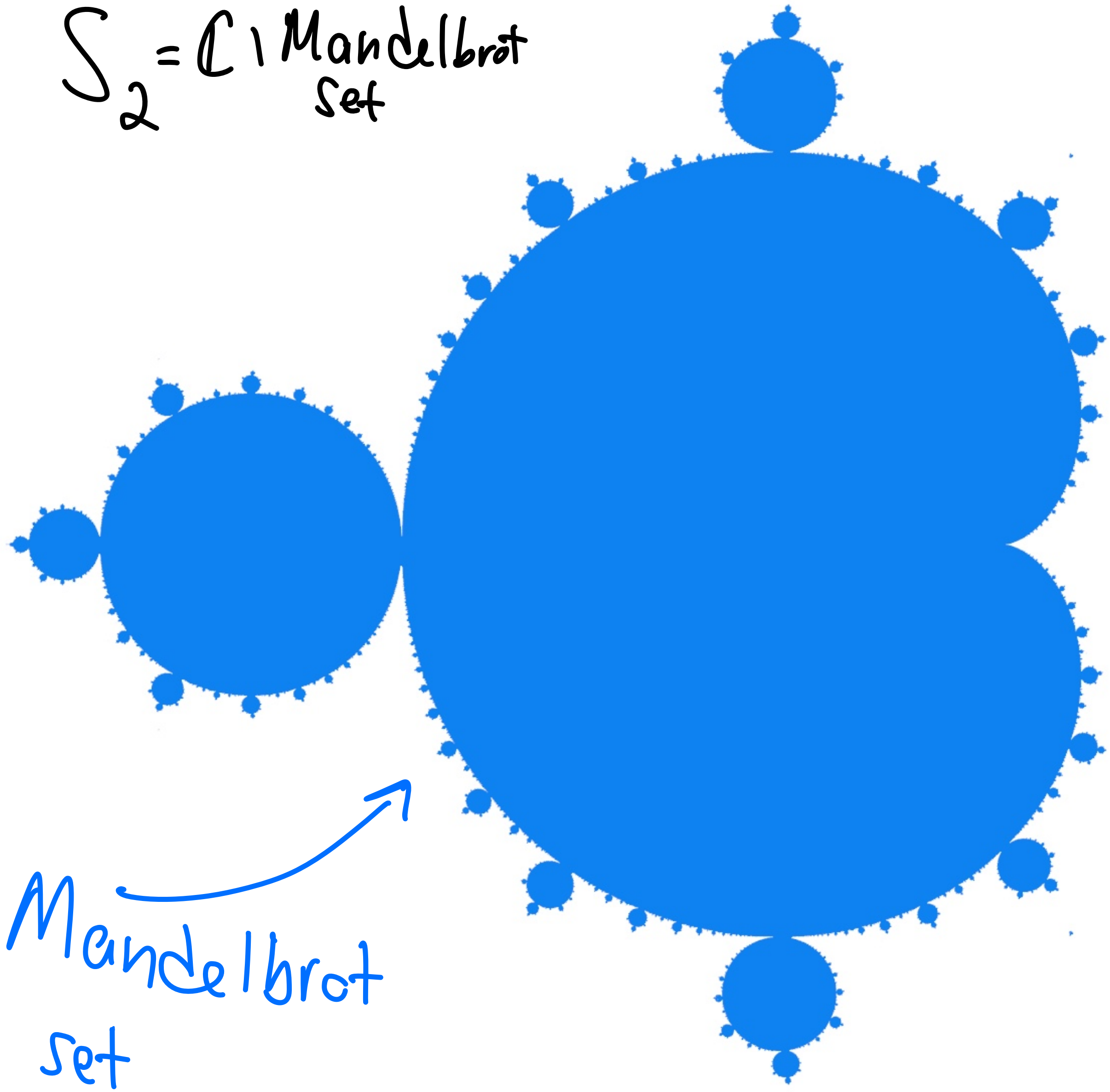
Understand the topology of S_d.

(Bavard - Calegari - He - Koch - Walker)

Def The **(filled) Julia set** of a polynomial p consists of the points $x \in \mathbb{C}$ such that $p^{\circ n}(x) \not\to \infty$.

Thm The Julia set of a shift polynomial is a Cantor set.

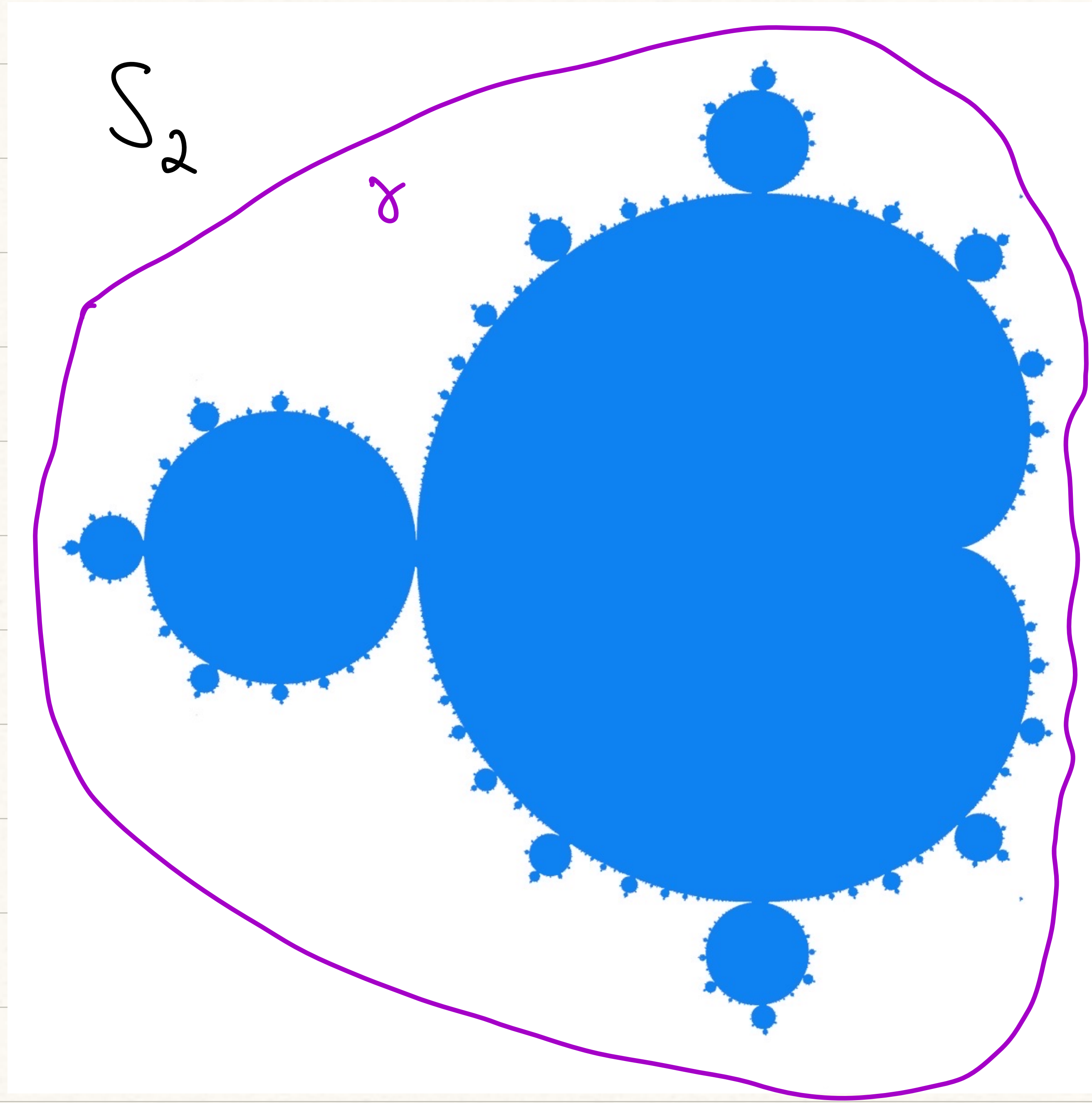
$S_2 = \mathbb{C} \setminus \text{Mandelbrot set}$



Theorem (Douady-Hubbard)

S_2 is an annulus.

$\Rightarrow \pi_1(S_2) = \langle \gamma \rangle \cong \mathbb{Z}$



The Julia sets of the $\gamma(t)$ vary continuously in t.

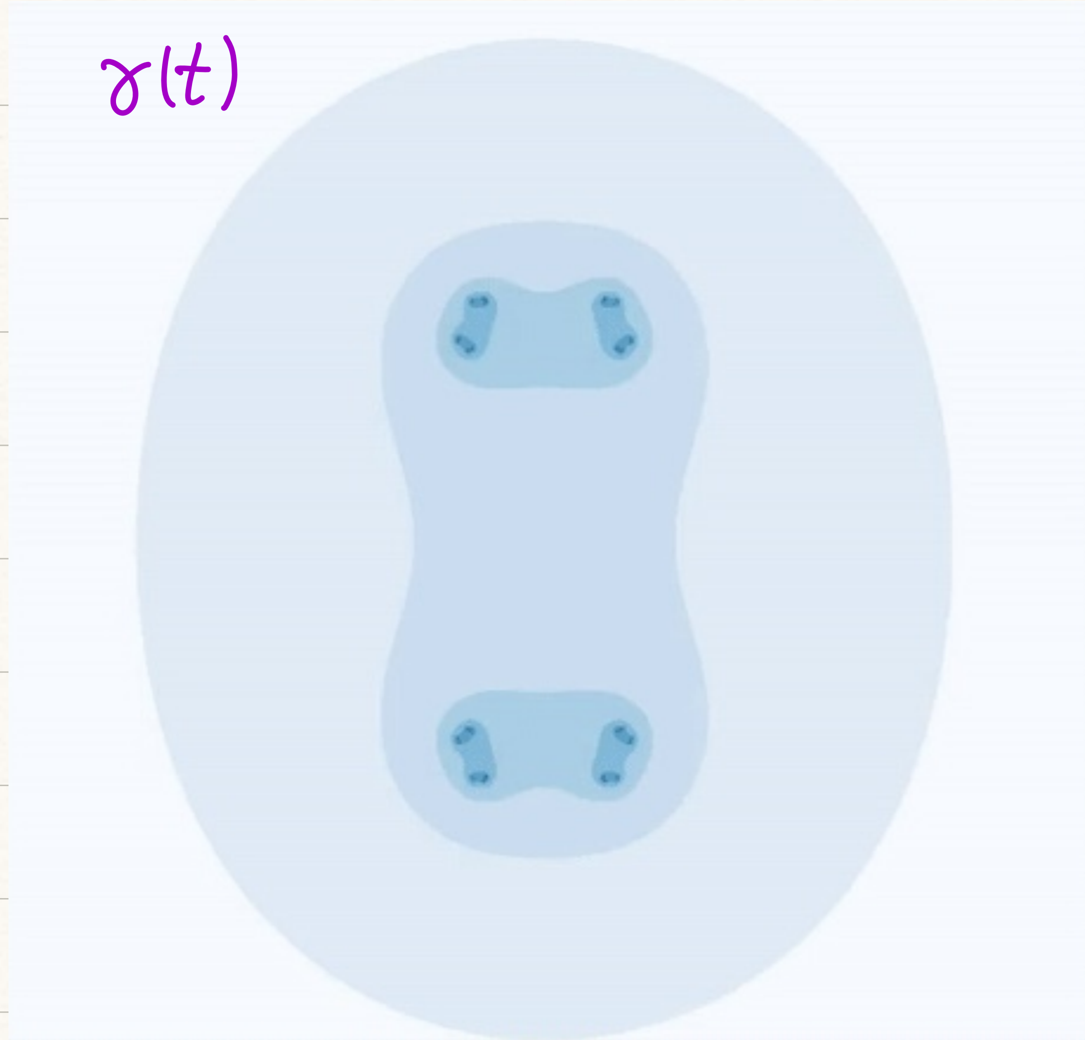
$\Rightarrow \exists$ homom

$\varphi_d : \pi_1(S_d) \to MCG(\mathbb{R}^2 \setminus C)$

where C is a Cantor set.

<u>Conj</u> (Bavard - Calegari - He)

φ_d is injective.



(Juhun Baik)

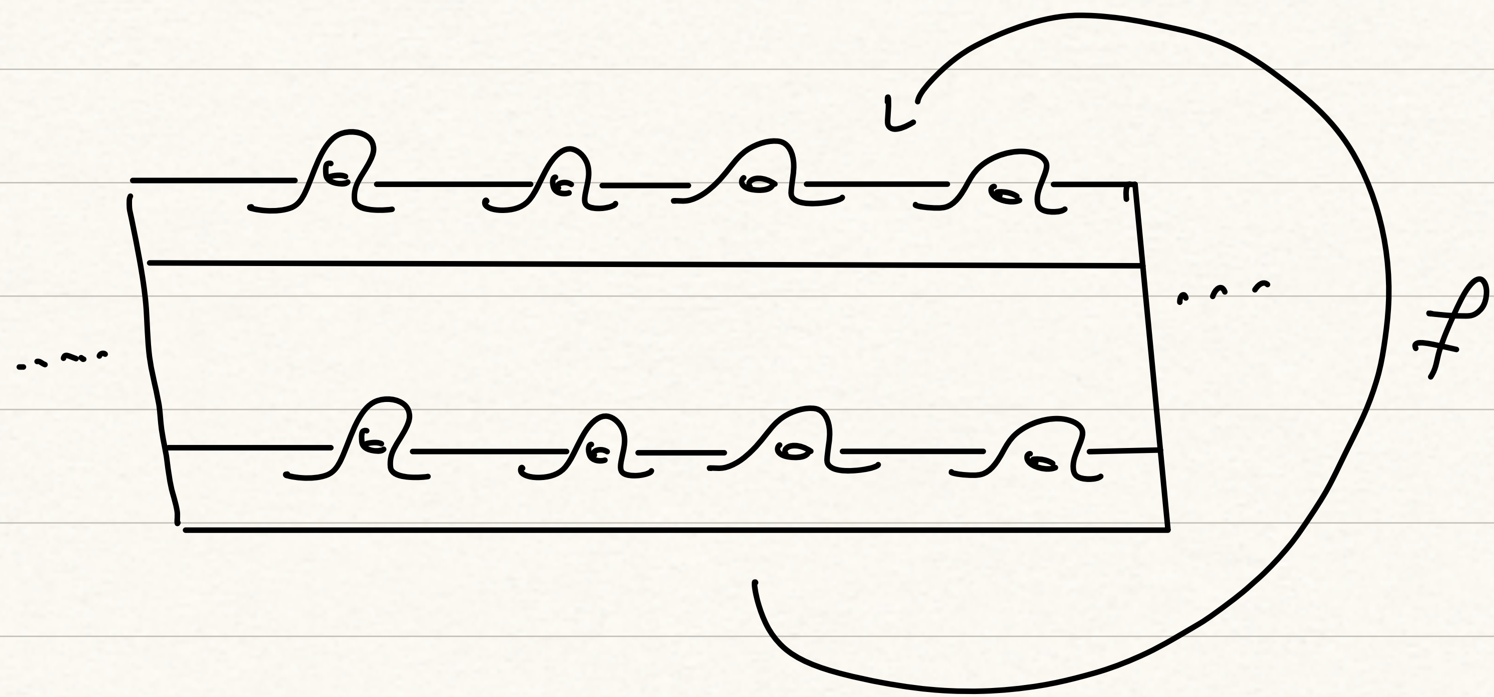
<u>Question</u>: What is the image of φ_d?

<u>3 - manifolds</u>

Mapping Torus

Σ a surface, $f : \Sigma \to \Sigma$ homeo

$$M_f = \Sigma \times [0,1] \Big/ (x,0) \sim (f(x),1)$$



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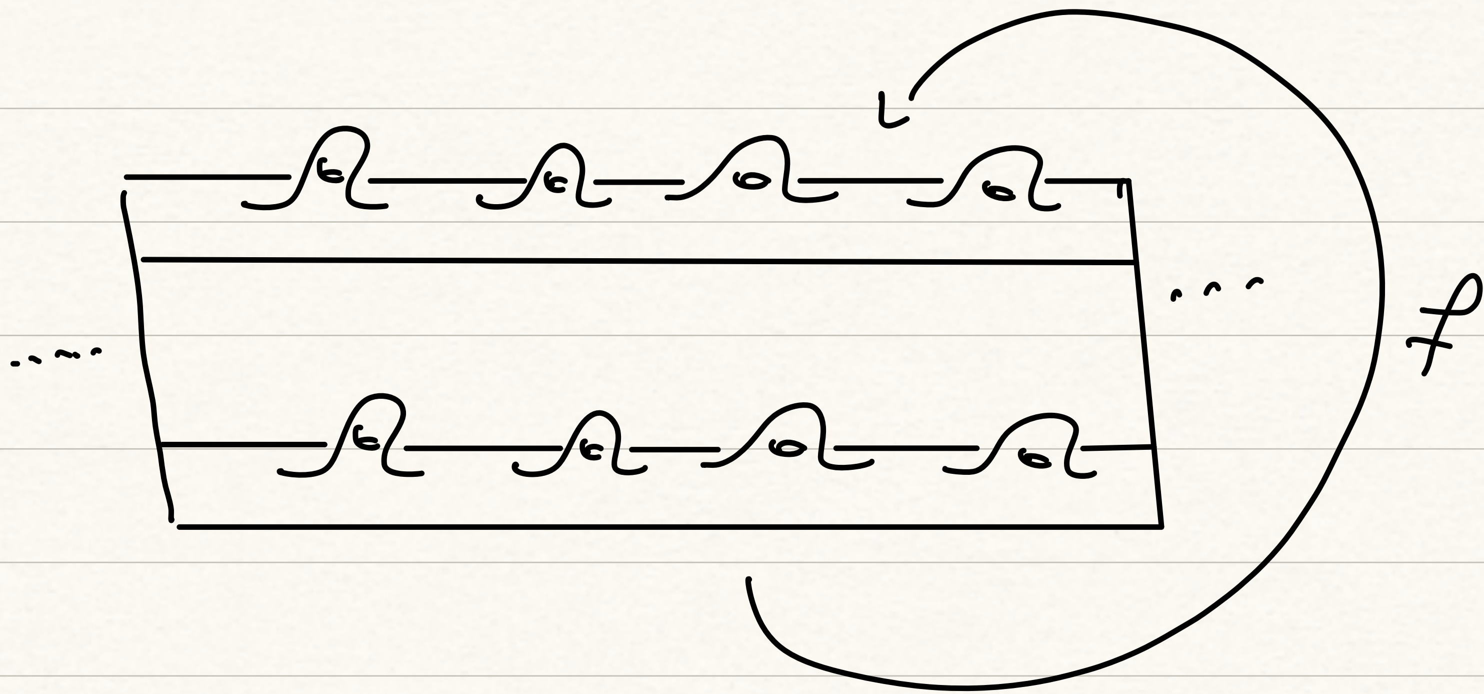


Rmk Up to finite covers, this is what closed hyperbolic 3-manifolds look like (w/ Σ closed, f pA).

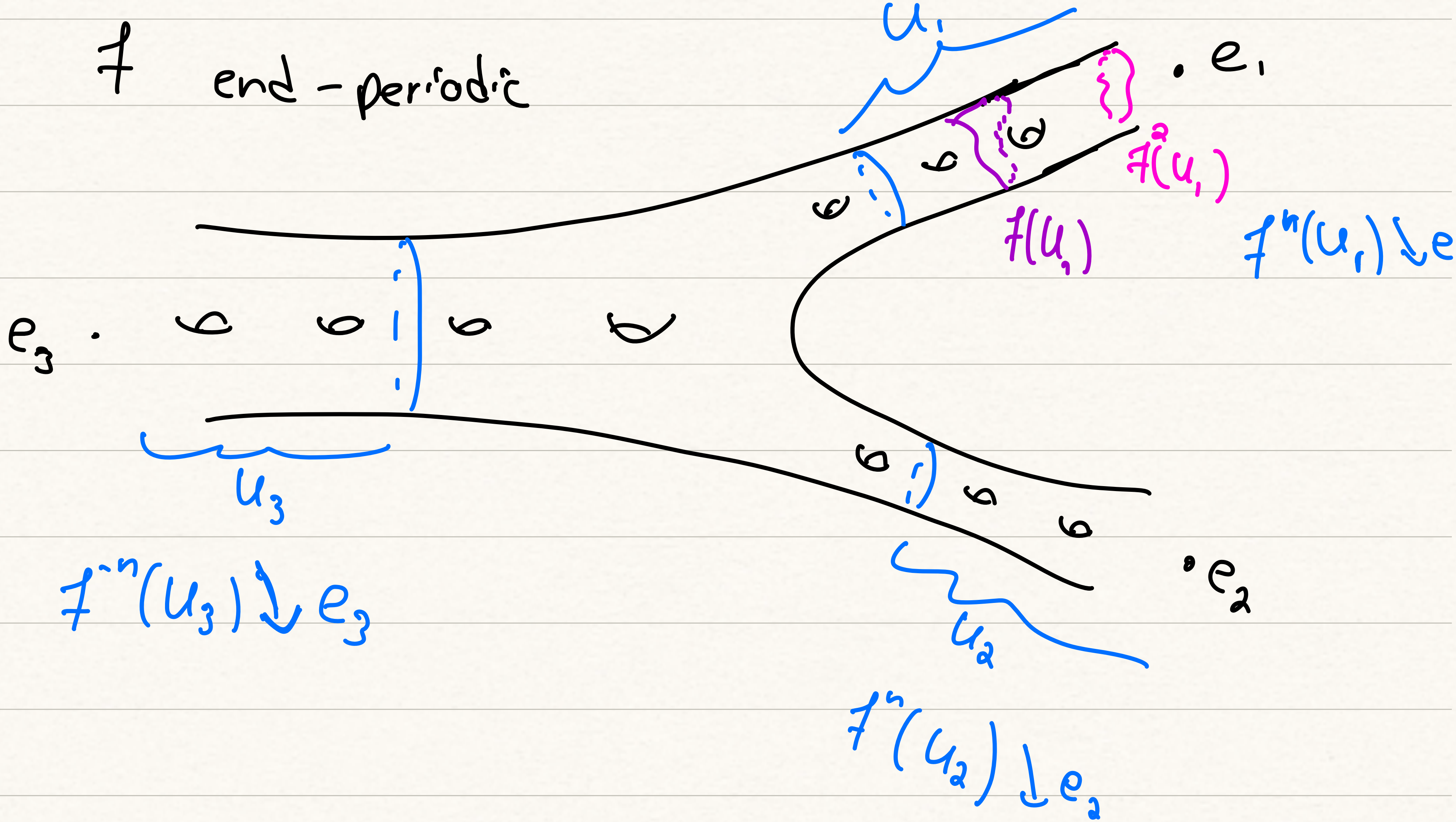
End-periodic homeomorphisms

Σ : ∞-type surface, finitely many ends,
no planar ends

Def $f: \Sigma \to \Sigma$ is end-periodic if for every end e of Σ there exists an open neighborhood U of e in Σ such that either $\{f^n(U) \mid n \in \mathbb{N}\}$ or $\{f^{-n}(U) \mid n \in \mathbb{N}\}$ is a neighborhood basis for e.

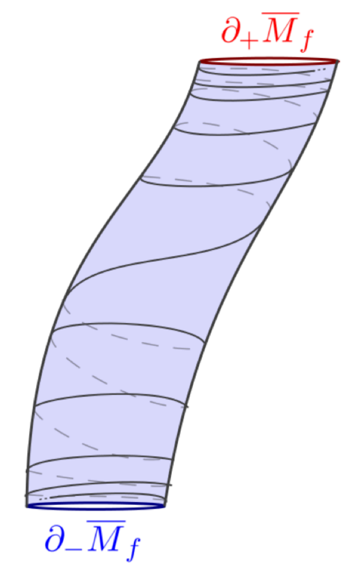
End-periodic homeomorphisms

f end-periodic



Proposition (Fenley)

If Σ is ∞-type and f is endperiodic, then M_f is homeomorphic to the interior of a compact 3-mfld $\overline{M}_f$.

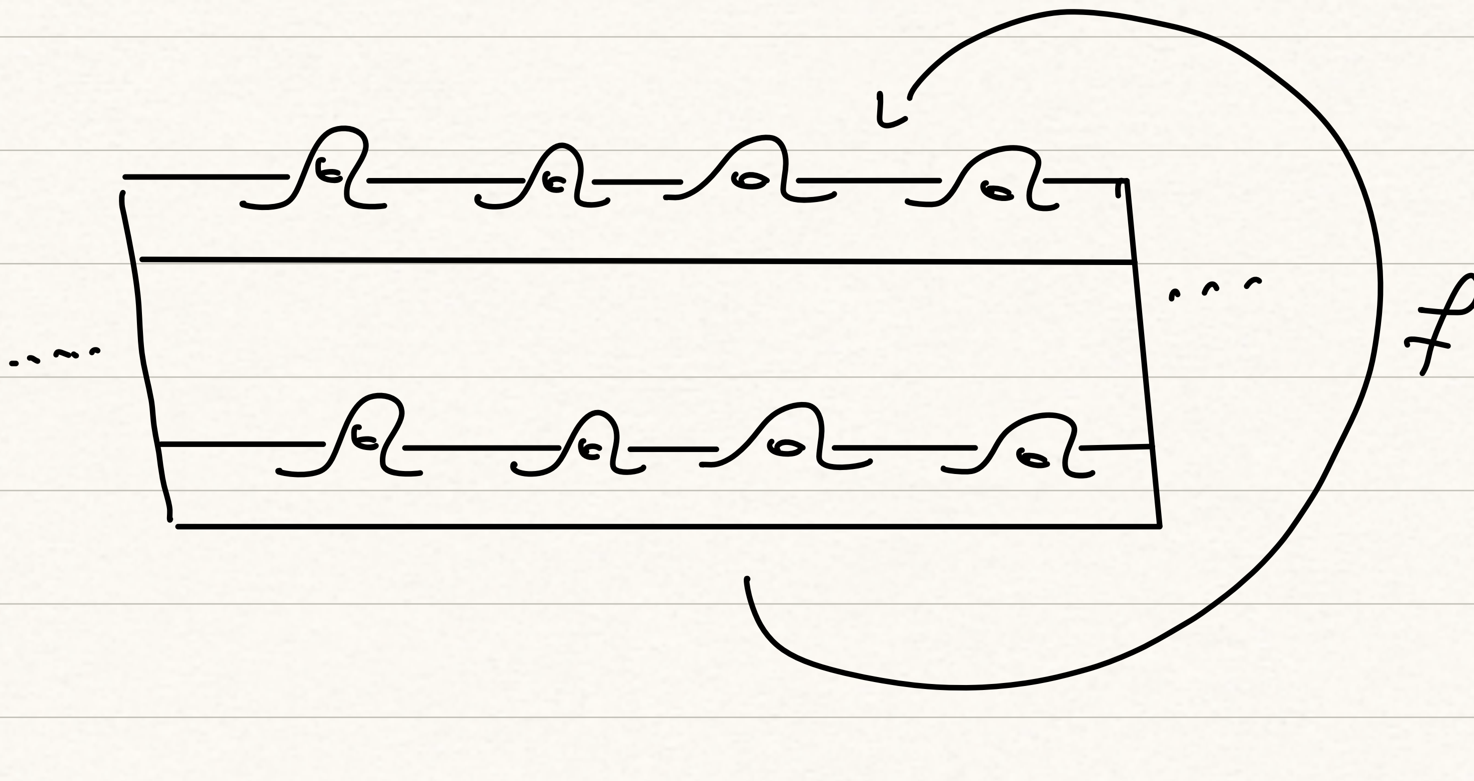


$\overline{M}_f$

(Brandis Whitfield)

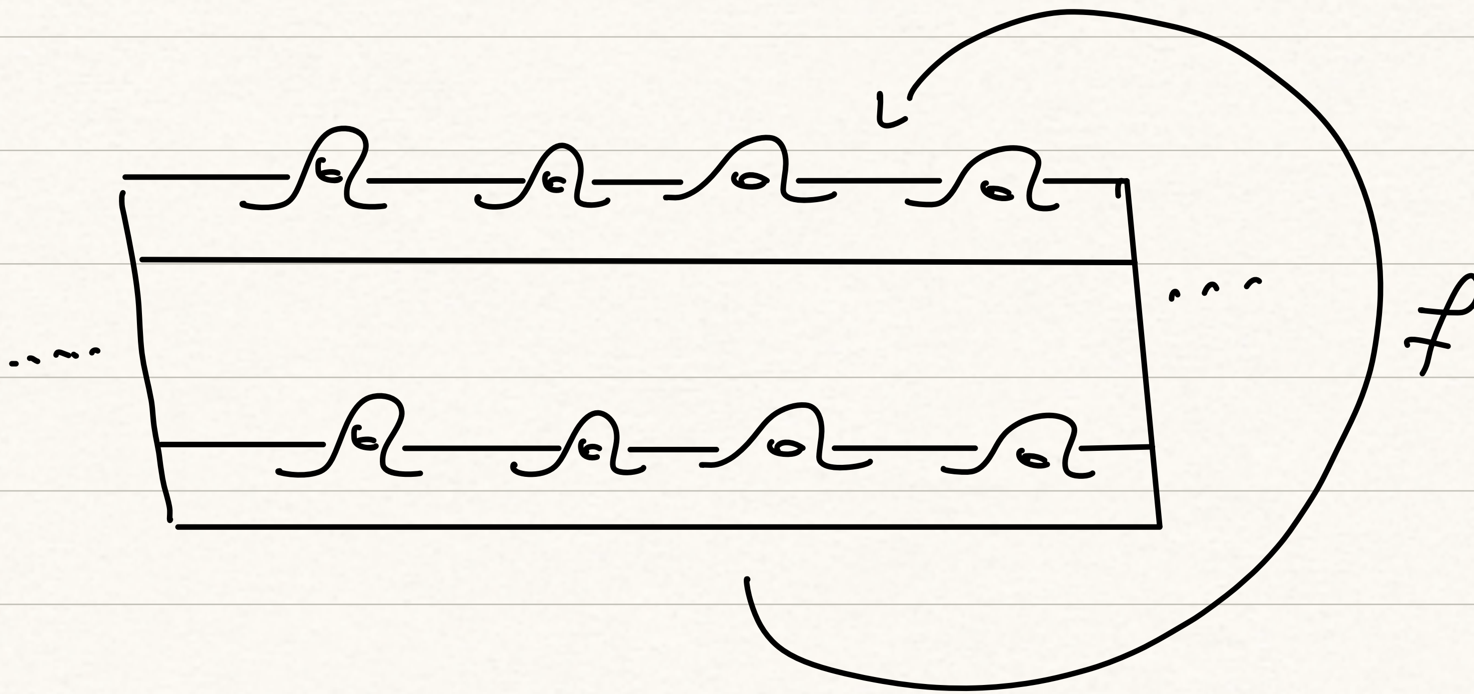
There are several natural structures associated to the compactification $\overline{M}_f$.

- A foliation: $\Sigma \times \{t\}$, $\partial \overline{M}_f$



There are several natural structures associated to the compactification $\bar{M}_f$.

- A foliation: $\Sigma \times \{t\}$, $\partial \bar{M}_f$
- A flow transverse to the $\Sigma \times \{t\}$



(Modulo adjectives)

end-periodic homeomorphisms of ∞-type surfaces

taut depth-one foliations of 3-manifolds

pA flows on 3-manifolds

Cantwell - Conlon - Fenley, Landry - Minsky - Taylor, Field - Kim - Leininger - Loving

Problem (Cremaschi)

For a general ∞-type surface Σ and a homeomorphism $f : \Sigma \to \Sigma$.

When is M_f hyperbolic?

Logic and Geometric Group Theory

S = orientable surface

$MCG(S)$ has a natural topology:

Equip $\mathrm{Homeo}^+(S)$ with the compact-open topology and give $MCG(S) = \mathrm{Homeo}(S) / \mathrm{Homeo}_0(S)$ the quotient topology.

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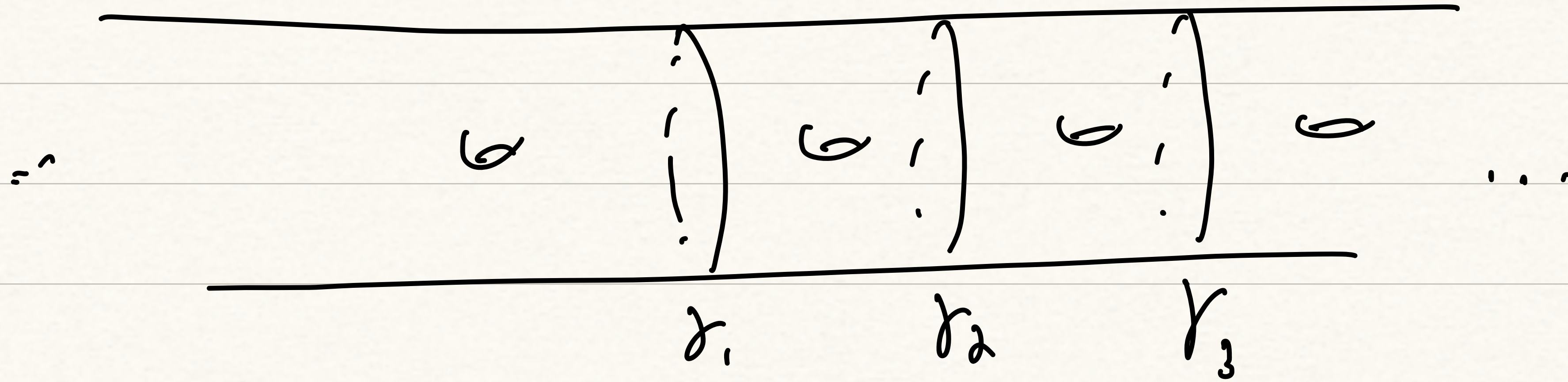
Maps are close if they agree on a large compact set.

$\Rightarrow$ Sets of the form $U_K = \{ f \mid f|_{U_K} = id \}$ with $K \subset S$ compact give a neighborhood basis at the identity.

Example $\quad T_{\gamma_n} \to \mathbb{1}$



If S is ∞-type, then $MCG(S)$ is a

non-locally compact non-Archimedean Polish group.

basis for identity consisting of subgroups

its topology is seperable and completely metrizable

If S is ∞-type, $MCG(S)$ is a non-locally compact non-Archimedean Polish group.

Question

Are there other topologies on $MCG(S)$ that make it a Polish group?

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- No, whenever $MCG(S)$ have ACP (Bestvina-Domat-Rafi)
- Homeo(S) has a unique Polish group structure (Kallman)

Def Let G be a topological group

- $A \subset G$ is <u>coarsely bounded</u> if the diameter of A is bounded in every continuous left-invariant metric on G.

- G is <u>CB-generated</u> if it is generated by a coarsely bounded subset.

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- $A \subset G$ is <u>coarsely bounded</u> if the diameter of A is bounded in every continuous left-invariant metric on G.

- G is <u>CB-generated</u> if it is generated by a coarsely bounded subset.

<u>Prop</u> (Rosendal) A CB-generated topological group has a "canonical" left-invariant metric compatible with its topology and it is quasi-isometric to the word metric associated to any CB generating set.

Takeaway: CB-generated groups can be studied via the methods of geometric group theory.

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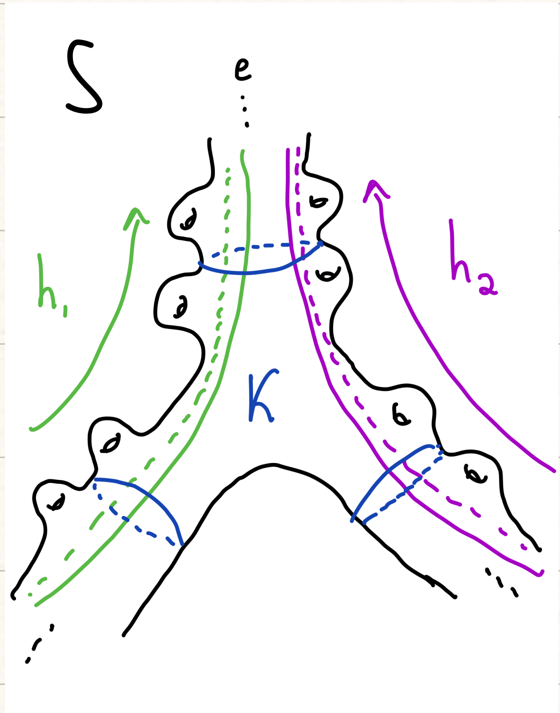
Thm (Mann-Rafi) Classification of CB-generated big mapping class groups

Example

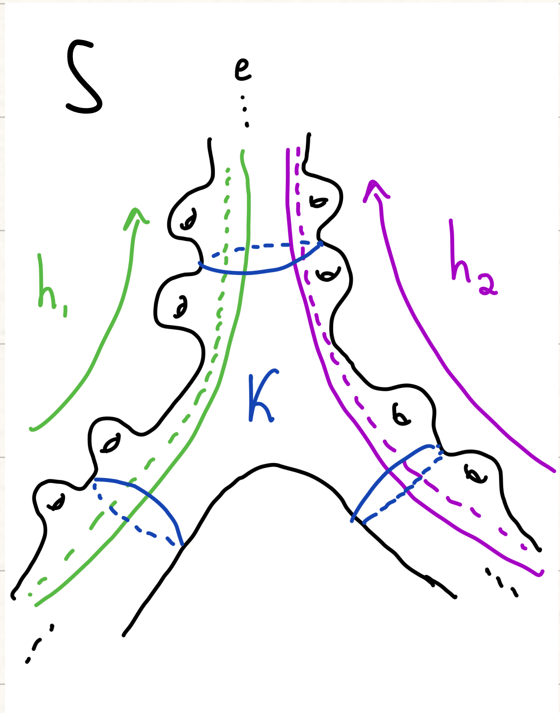


Lemma U_K is CB.

$U_K \cup \{h_1, h_2\} \cup \{\text{fin gen. set for } MCG(K)\} \cup \{\sigma_1, \sigma_2\}$

is a CB generating set for $MCG(S)$.

Example



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Problem Relate geometric invariants of $MCG(S)$ to topological properties of S and/or (top.) group properties of $MCG(S)$.

<u>Thm</u> (Bavard + Mann-Rafi + Schaffer-Cohen)

$MCG(\mathbb{R}^2 \setminus \text{Cantor set})$ is Gromov hyperbolic.

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<u>Question</u>. What properties come from hyperbolicity in the non-locally compact setting?

- Can hyperbolicity obstruct groups acting on $\mathbb{R}^2$?

Closing Question:

Teichmüller Theory

<u>Question</u> Is there/What should be the analog of a pseudo-Anosov homeomorphism in the ∞-type setting?

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<u>Question</u> How do existing constructions overlap / interact?

- Loxodromic isometries of Teichmüller space
- Loxodromic isometries of various curve graphs
- Hooper-Thurston-Veech constructions
- Atoroidal end-periodic homeomorphisms
- f w/ M_f hyperbolic

¡Gracias!