Metric measure Spaces with Symmetry and Lower Ricci Curvature Bounds

Diego Corro (Karlsruher Institut für Technologie),
Andrea Mondino (University of Oxford),
Jesús Ángel Núñez Zimbrón (Universidad Nacional Autónoma de México),
Masoumeh Zarei (Westfälische Wilhelms-Universität Münster)

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1 Overview of the Field

The problem of finding an optimal way to transport a given mass has been studied in Mathematics since the pioneering work of G. Monge in 1781 and L. Kantorovich in the 1950'ies, and it is known as the optimal transport problem. Lott-Villani [26], and Sturm [38, 39], using optimal transport techniques, defined the class of $\mathsf{CD}(K, N)$ metric measure spaces, $K \in \mathbb{R}, N \in [1, \infty]$.

A metric measure space is a triple (X, d, m) where (X, d) is a complete and separable metric space, and m is a Borel non-negative measure playing the role of "reference volume measure". The parameters K and N in the class $\mathsf{CD}(K, N)$ play the role of lower bound on the Ricci curvature and upper on the dimension, respectively.

A refinement by Ambrosio-Gigli-Savaré [1] of the class of CD-spaces led to the notion of RCD-spaces: these are CD-spaces whose Sobolev space $W^{1,2}(X, m)$ is a Hilbert space. The basic intuition is that $\mathsf{CD}(K, N)$-spaces are "possibly non-smooth *Finsler* spaces with Ricci curvature bounded below by K and dimension bounded above by N", while $\mathsf{RCD}(K, N)$-spaces are "possibly non-smooth *Riemannian* spaces with Ricci curvature bounded below by K and dimension bounded above by N".

One can also see such families as the analog for Ricci curvature, of the class of Alexandrov spaces. Indeed Alexandrov spaces are examples of RCD-spaces [34], [42] and, in dimension two, the two classes coincide [27].

In fact RCD-spaces also arise as limits of sequences of Riemannian manifolds with a uniform lower bound on the Ricci curvature with respect to the pointed measured Gromov-

Hausdorff convergence (see e.g. [18]). The class of $\mathsf{RCD}(K, N)$-spaces enjoys several important geometric and analytic properties: Splitting Theorem for $\mathsf{RCD}(0, N)$-spaces [16], Laplacian comparison [17], Bochner inequality [1, 12, 2], isoperimetric inequalities [8], local-to-global property [7], rectifiability [30], constancy of dimension [6], to mention a few. The theory flourished in the last ten years and is a very active ongoing area of research.

Recent works [21, 37], show that the group of measure preserving isometries of an RCD-space is a Lie group. Also, when we consider a Lie group G acting by measure preserving isometries on an RCD-space X, the orbit space X/G is also an RCD-space [15].

Still many directions remain open, specially in looking for relationships between the local analytic properties and the global topology and geometry of the space. In particular, advancements have been made in the description of the universal covers of RCD-spaces, how geodesics behave (non-branching), and the description of the symmetries and quotient spaces under symmetries of these spaces. Moreover there are some techniques on how to construct new spaces from old ones, in particular taking cones.

2 Open Problems

During discussions between participants of the workshop, the following problems where highlighted as relevant for the advancement of the field:

1. Given a Lie group action by measure preserving isometries, prove a suitable version of the Slice Theorem for RCD-spaces (see for example [14] for Alexandrov spaces).

2. Show that the universal cover of an RCD-space must be simply-connected. In the Alexandrov case, this follows by Perelman's conical neighborhood theorem and, in the Ricci-limit case, by the recent work of Pan and Wei [33]. At the time of writing of this report, a new manuscript [40] appeared on arxiv.org on which the author Jikang Wang shows that RCD-spaces are indeed semi-locally simply connected, directly solving this problem

3. *Yau's conjecture:* A Ricci curvature lower bound implies an L^1-universal bound on scalar curvature. The Alexandrov case was settled by Petrunin in the affirmative.

4. *The boundary conjecture*: The boundary of an Alexandrov space is also an Alexandrov space with the same lower curvature bound.

   Relaxed conjecture: The boundary of an Alexandrov space is an RCD-space with the appropriate curvature bound. In dimension 3, the relaxed conjecture implies the full conjecture by the work of Lytchak and Stadler [27]. This problem is also interesting for Ricci limit spaces, trying to use localization techniques.

Even more relaxed conjecture: Any totally geodesic subspace of an RCD-space with the induced intrinsic metric and an "enlarged" measure is again an RCD-space with the same curvature bound.

5. Address the "Fundamental gap" problem in the RCD-setting. Work in this direction has been obtained, in the manifold setting, by Andrews, Clutterbuck, Dai, Wei, Seto, Wang, et. al (see for example [3], [11] [36], and references therein). Note that localization techniques do not work very well here.

6. *Collapsed vs. Non-collapsed:* Given a metric measure space $(X, d, \mathcal{H}^m)$ which satisfies the $\mathsf{RCD}(K, N)$ condition for some K and $N > m$, is it true that the space also satisfies the $\mathsf{RCD}(K, m)$ condition? This is true in the smooth case by writing the Bakry-Emery estimate. Note that here we do not assume the space is non-collapsed. It is sufficient to check that the trace of the Hessian is the Laplacian as this is equivalent to non-collapsing.

7. Given a principal fiber bundle over an RCD-space, give sufficient conditions on the fibers or total space to lift the RCD-condition to the total space.

8. *Stratification of* RCD-*spaces vs. Stratification of Alexandrov spaces:* Stratification of limit spaces is nicer, by Cheeger-Jiang-Naber ([10]).

9. *Topology of* 3-*dimensional* RCD-*spaces:* Mondino conjectures that they are homeomorphic to orbifolds, possibly with boundary. At the level of tangent cones this is true because 2-dimensional spaces are Alexandrov, however it is not clear how to pass to the space itself. The only tool right now is Reifenberg's Theorem (see [9]).

10. Obtain the analogues of Finsler geometry results on CD-spaces; For example, show that there exists at least **one** tangent cone which is isomorphic to $\mathbf{R}^N$ equipped with a norm.

11. Find a condition P for a metric measure space $(X, d, \mathfrak{m})$, which is weaker than the property "X is infinitesimal Hilbertian", to ensure that if a metric measure space X satisfies the CD-condition and P, then the space is essentially non-branching.

3 Presentation Highlights

- Fernando Galaz-Garcia and Catherine Searle gave an overview of the main techniques used in the presence of group actions by isometries on both manifolds and Alexandrov spaces, and different invariants that translate into a classification results for Riemannian manifolds with positive sectional curvature.

- Fabio Cavalletti and Daniele Semola gave an introduction to RCD-spaces, and general geometric, analytic and topological properties these spaces have. Results of Ricci limit spaces and non-collpased RCD-spaces where highlighted.

- Jaime Santos gave a presentation on the structure of isometries of RCD-spaces, showing that it is a Lie group G. Moreover, in the case when the group is compact, we can modify our reference measure to obtain a measure invariant under the action of the group.

- Dimitri Navarro spoke about the moduli spaces of compact $\mathsf{RCD}(0, N)$-structures. This space describes all RCD structures (i.e. distance function d and measure m) on a space X satisfying the $\mathsf{RCD}(0, N)$ condition. He presented the Albanese and soul maps on moduli spaces of $\mathsf{RCD}(0, N)$ structures (which reflect how structures on the universal cover split), and showed that these maps are continuous. As an application of his result he showed that that the moduli spaces of compact $\mathsf{RCD}(0, 2)$ structures are contractible.

- Elia Bruè talked about the notion of metric measure boundary on RCD-spaces. The concept of metric measure boundary was introduced by Kapovitch, Lytchak, and Petrunin in an attempt to solve a long-standing conjecture of Perelman-Petrunin about the existence of infinite geodesics on Alexandrov spaces. They proved that the conjecture is satisfied provided the metric measure boundary of any Alexandrov space without boundary is vanishing. However, the latter turned out to be a difficult question and remained open. The talk presented a joint work of the speaker with Mondino and Semola, showing that the metric measure boundary is vanishing on any RCD-space without boundary; this settles the Perelman-Petrunin conjecture.

- Raquel Perales presented a rigidity result for RCD-spaces generalizing a results by Gromov and Gallot, and Cheeger and Colding for Riemmannian manifolds. She presented a revised first Betti number and showed that under sufficient conditions related to the curvature and diameter of the space, this number is bounded by the dimension of the space. Then she showed that this bound results in a rigidity statement: when the upper bound is attained then the space if finitely covered by a space close in the measured Gromov-Hausdorff topology to a flat tours.

- Goufang Wei gave a talk showing that for Ricci limit spaces the universal cover of this spaces are simply connected.

- Jiayin Pan presented examples of open manifolds with positive Ricci curvature. With these examples, questions related to the properness of Busemann functions at a point and to the singular set of Ricci limit spaces where answered in the negative.

- John Harvey gave an overview of circle actions on Alexandrov spaces. In particular he presented details on the classification of 4-dimensional Alexandrov spaces with circle actions.

- Chiara Rigoni gave a discussion of the CD condition for $N < 0$, by allowing for metric measure structures in which the reference measure is quasi-Radon. She presented a notion of distance between metric measure spaces where the CD condition passes to the limit.

- Qin Deng talked about regularity results on Regular Lagrangian Flows in the $\mathsf{RCD}(K,N)$ setting. Regular Lagrangian Flows are generalizations of flows via vector field to the non-smooth setting and an understanding of its spatial regularity has applications in generalizing results from the Riemannian setting where smooth flows are used.

4 Scientific Progress Made and Outcome of the Meeting

During the workshop there have been several discussions and question sessions where the participants explored strategies to describe the global topology of low dimensional RCD-spaces using different techniques. One of the prominent avenues of discussion was on how to extend the symmetry-reliant techniques which are already successful in the Alexandrov and manifold cases.

Another important field of discussion was about studying closer the universal cover of an RCD-space due to the results presented by Goufang Wei and Raquel Perales.

There was a heated debate on how to expand the use of group actions and fibrations to study the geometry or topology of RCD-spaces. In particular several ideas where discussed in the settings of group actions, pointing to the direction of studying the existence of a slice theorem in the RCD setting.

There was also a strong interest of the participants on considering the boundary of spaces in the RCD setting, in particular if the techniques developed so far for RCD-spaces may shine some light on a conjecture about the boundary of Alexandrov spaces.

Several mentorship sessions were conducted, focusing on many aspects of career orientation, as well as mathematical guidance from senior members of the community.

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