Ordered groups and rigidity in dynamics and topology

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June 16, 2019–June 21, 2019

1 Overview of the Field

A left-ordering of a group G is a total (*i.e.* linear) order that is invariant under multiplication from the left. The study of orderings of groups has a long history among algebraists, including discoveries by prominent figures such as Hölder, Vinogradov and Rudin that go back as far as the early 1900's. Crucially, for a countable group G, admitting a left-invariant ordering is equivalent to G admitting a faithful action by order-preserving homeomorphisms on the real line. As a consequence, methods of dynamical systems and topology are often brought to bear in the study of orderable groups, with powerful algebraic results often arising through non-algebraic means. Conversely, results from the field of orderable groups have opened new directions of research in low-dimensional topology and dynamical systems. Analogously, a circular ordering on G is a circular ordering invariant under left-multiplication (i.e. a left-invariant "orientation cocycle" $c : G^3 \to \{0, \pm 1\}$); for countable groups admitting a circular ordering is equivalent to admitting a faithful action by orientation-preserving homeomorphisms on the circle.

There are many driving forces behind current work in the field of orderable groups. Some basic questions can be remarkably difficult with very deep answers–for example, a very basic-sounding problem is to understand, perhaps upon restriction to a certain class of groups, when a given group is left-orderable or not. Work on this seemingly simple problem has often brought to light surprisingly deep connections with other fields of mathematics.

For example, if we restrict to the class of linear groups, this problem is related to the so-called Zimmer conjecture (the C^0-version of the conjecture) which roughly states that large linear groups should not act on small connected manifolds (*e.g.* the circle or the line). As another example, upon restriction to the class of fundamental groups of closed, connected, compact, orientable and irreducible 3-manifolds, it is conjectured that a group is left-orderable if and only if is it the fundamental group of a 3-manifold supporting a co-orientable, codimension-one taut foliation, and that this holds if and only if the manifold is *not* a Heegaard-Floer homology L-space. Results in this area began first with an observed correlation between the years 2000–2010, and were formalized as a conjecture in 2012 by Boyer, Gordon and Watson in [4]. These days, their conjecture is commonly referred to as the *L-space conjecture*.

Another driving force in the field is to understand rigidity and flexibility phenomena for left-orderable groups. For instance, we would like to understand when a given action of a group in the line or the circle can be deformed so that the new action is not semiconjugate to the initial action. This type of question has long been of interest to dynamicists, yet most classical tools require some assumptions on the regularity of the action (i.e. smoothness). For the case of group actions by homeomorphisms of the line, there is the so called space of left-orders of G, $\mathrm{LO}(G)$, which is a compact space made of all the left-orders on the given group. This was introduced by Sikora in [29], and it was, for instance, crucial in Linnell's argument showing that the number of left-orders that can be defined on a group cannot be infinite and countable. Since its introduction, a general research theme has emerged to correlate topological properties of $\mathrm{LO}(G)$ with algebraic properties of G and dynamical properties of actions of G on the line. As an example, we now understand that there is a precise relationship between isolated points of $\mathrm{LO}(G)$ and actions of G on the line that are "rigid" in a certain dynamical sense [23].

A last area of research discussed at this conference is the question of "smoothness of left or circular orderings." This can be made precise in various ways. For instance, given a group that acts on the line, the closed interval, or the circle, it is an interesting question to ask what the maximum regularity of such an action can be. As examples of such results, Plante and Thurston showed that a non-Abelian nilpotent group can not act faithfully by C^2-diffeomorphisms of the closed interval, whereas Farb and Franks showed that any nilpotent group action on the closed interval can be conjugated into a C^1-action. A different approach is to ask for groups of a given *critical regularity*, that is, a group G_α such that G_α acts faithfully by C^α-diffeomorphisms but there is no faithful action of G_α by C^β-diffeomorphisms for $\beta > \alpha$. For each $\alpha > 0$, Kim and Koberda have recently constructed a groups G_α with this property, with a different construction later provided by Mann and Wolff.

2 Recent Developments and Open Problems

Simple left-orderable groups and left-orderable monsters: Certainly one of the most fantastic recent discoveries in this area is that of a left-orderable group which is finitely generated and simple. This is a natural and old problem that can be traced back to over forty years to the work of Rhemtulla in 1980. The first example of such a group was recently constructed by Hyde and Lodha [17] by a very clever modification (and extension) of Thompson's group F and its action on the line. Soon after, Matte-Bon and Triestino delivered a much simpler and illuminating proof of the same fact [24]. The key idea behind Matte-Bon and Triestino is that the real line should not be thought off as a *straight* line, but rather as a leaf inside a *selenoid*, which is a compact space foliated by lines. This makes possible, for instance, to use classical arguments of Epstein to show simplicity of the group.

The simple group from Hyde and Lodha was latter shown, by Hyde, Lodha, Navas and Rivas [18], to be (the first example of) a *left-orderable monster* (in the terminology of [11]). This means that for every action of the group on the line and for every compact interval $I \subset \mathbb{R}$, there is a sequence of group elements $(g_n)_n$ such that $g_n(I)$ converges to a point . The conspicuous name comes from the fact that a potential counter-example to the C^0-Zimmer conjecture should act precisely in this way, see [11].

Property (T) and actions on the circle and real line: Recall that a countable group Γ has property (T) if every affine isometric action of Γ on a Hilbert space has a fixed point. We first observe that if Γ has property (T) then $H^1(\Gamma, \mathbb{R}) = \{0\}$, so there are no homomorphisms to the line. In fact, it will have no infinite amenable quotients.

Now if Γ is a lattice of finite co-volume inside a simple Lie group G with $rk(G) \geq 2$ then Γ has property (T), and there is a sequence of results showing that these groups do not admit homomorphisms into $\mathrm{Homeo}_+(S^1)$ with infinite image, or into into $\mathrm{Homeo}_+(\mathbb{R})$. For example:

1. Dave Witte showed that finite index subgroups of $\mathrm{SL}(n, \mathbb{Z})$ is not left-orderable [30].
2. Ghys-Burger-Monod showed that if Γ is a high rank lattice, then every action of Γ on S^1 has a finite orbit (e.g. [5]). From Thurston stability, this means that every $\Gamma \hookrightarrow \mathrm{Diff}^1(S^1)$ has finite image.
3. If Γ has (T) then every action of T on a real tree by isometries has a fixed point.
4. If Γ has (T), then Navas showed every homomorphism $\Gamma \hookrightarrow \mathrm{Diff}^{1+\frac{1}{2}}(S^1)$ has finite image [25].
5. (Cornulier, Triestino, many others) The previous result extends to actions that are piecewise as above, that is they are piecewise $\mathrm{Diff}^{1+1/2}(S^1)$.

So from here, we arrive at:

Question 2.1 *(Andrés Navas) Does there exist a group with property (T) that acts on the real line or an infinite group with property (T) that acts on the circle?*

Again, a group with Kazhdan property (T) that acts on the line must necessarily be a left-orderable monster, see [11].

Spaces of orderings: The space of left-orderings of a group G, denoted $\mathrm{LO}(G)$, is the set of all left-orders topologized so that two orderings of G are "close" if they agree on large finite sets. Similarly, one can define the space of bi-orderings $\mathrm{BO}(G)$ (i.e. total orders invariant under left and right multiplication) and also the spaces $\mathrm{CO}(G)$ and $\mathrm{BCO}(G)$ of left-invariant circular orders and bi-invariant circular orders respectively. In every case, the resulting space is compact and totally disconnected and the groups G and $Aut(G)$ naturally acts on them by homeomorphisms. A natural problem is to relate the topology of these spaces to the algebraic structure of G.

For instance it is known that, no matter the group G, $\mathrm{LO}(G)$ is either finite or uncountable [21] and the same holds for $\mathrm{CO}(G)$ [8]. Further, there are examples of groups having left-orders that are isolated yet they admit uncountably many left-orders, and we now understand that isolated (left or circular) orders of countable groups correspond precisely to actions of the group (on the real line or the circle) that are locally rigid, meaning, roughly, that any small perturbation of the action produces an action that is semi-conjugate to the initial one, see [23].

The first example of a group having infinitely many left-orderings and at the same time supporting isolated left-orderings are braid groups (we will say that these groups support *genuine* isolated ordering). See [10] for a nice survey about orderability of braids. In fact, one of the starting points for the connection between orderings and topology comes from the Dehornoy ordering of the braid groups. There are deformations of the Dehornoy ordering for braids, in the sense that this ordering is not isolated. However, if one "flips" the convex subgroups of the Dehornoy ordering in an alternating manner to produce the Dubrovina-Dubrovin ordering, it becomes an isolated point in $\mathrm{LO}(G)$ [12]. This construction, however, gives little insight into what is special about the construction of the Dubrovina-Dubrovin ordering and how analogous orderings might arise on other groups. So we ask

Question 2.2 *(Tetsuya Ito) Generalize the Dubrovina and Dubrovin ordering for $n \geq 4$ to produce new isolated orderings on groups analogous to braid groups.*

For example, the Dubrovina and Dubrovin ordering of B_3 has been generalized to produce isolated orderings of groups of the form $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$, or groups admitting presentations satisfying certain combinatorial conditions ([26, 19, 9]). In particular, what groups analogous to B_4 (Artin groups, mapping class groups, or something different) admit isolated orderings analogous to the Dubrovina-Dubrovin ordering? This would presumably follow if one were able to address the following challenge: Show that B_4 has an isolated ordering without appealing to the technology developed by Dehornoy (e.g. i-positivity, etc.).

On the other hand, there are many groups which do not admit genuine isolated left-orderings. This is the case of virtually solvable groups [27], free products of groups as well as free products of non-Abelian free groups amalgamated over $\mathbb{Z}$ [1], in particular, fundamental groups if closed hyperbolic surfaces do not admit isolated left-orders. So we ask

Question 2.3 *(Cristóbal Rivas) Is there a left-orderable and (Gromov-)hyperbolic group supporting a genuine isolated ordering? What about funmdantal groups of closed hyperbolic 3-manifolds?*

For particular classes of groups, determining the structure of $\mathrm{LO}(G)$ is a particularly appealing problem. This seems to be the case for Right Angled Artin Groups (RAAGs, for short) since they allow both types of behavior: $\mathbb{Z}^n$ and F_n have no genuine isolated left-orders yet $F_n \times \mathbb{Z}$ supports a genuine isolated left-order if and only if n is even [22]. Could it be the case that there is a way of determining, say from the graph of the RAAG G, whether or not G admits a positive cone having a particularly nice structure? More explicitly:

Question 2.4 *(Kathryn Mann) Can you classify which RAAGs have isolated left-orders?*

In general, we can ask for more than just about isolated left-orders. For instance we can ask about the Cantor-Bendixson rank of the space $\mathrm{LO}(G)$. It is unknown if there is a group whose space of left-orders has rank different from 0 or 1, so an interesting challenge is:

Question 2.5 *(Kathryn Mann) Give an example of a finitely generated group G such that $\mathrm{LO}(G)$ (or $\mathrm{CO}(G)$) is not one of the known examples, i.e. it is not:*

1. *finite;*
2. *a Cantor set;*
3. *a union of isolated points and a Cantor set.*

As mentioned before, $\mathrm{LO}(G)$ is finite or uncountable no matter the group G. On the other hand, the space of bi-orderings can behave very differently than the space of left or circular orderings: for example $\mathrm{BO}(G)$ can be countably infinite, as there is an example of a group G which has this property [6]. Not much is known, however, about the possible structure of $\mathrm{BO}(G)$ for a given group. For instance, the following is still unknown:

Question 2.6 *For which groups is $\mathrm{BO}(G)$ finite?*

In contrast, there is a classification of the groups G for which $\mathrm{LO}(G)$ and $\mathrm{CO}(G)$ are finite. Another step towards understanding the space of bi-orderings would be to address the space of bi-orderings for a free group.

Question 2.7 *If $n > 1$, is $\mathrm{BO}(F_n)$ homeomorphic to the Cantor set?*

Advances in the L space conjecture: As mentioned above, the L-space conjecture relates left-orderability of the fundamental group of a compact, connected, orientable and irreducible 3-manifold M to whether or not M admits a taut foliation, or is an L-space. Known results from the world of Heegaard-Floer homology naturally give rise to questions involving left-orderability of fundamental groups (for example, by more or less replacing the word "L-space" with "non-left-orderable fundamental group"). From this we arrive at several appealing directions for future research:

Conjecture 2.8 *(Special case of the L-space conjecture) Suppose that K is a knot in S^3, and denote the result of $\frac{p}{q}$-surgery on S^3 along K by $S_{\frac{p}{q}}(K)$. If $\frac{p}{q} > 0$ and $\pi_1(S_{\frac{p}{q}}(K))$ is not left-orderable, then $\pi_1(S_{\frac{r}{s}}(K))$ is not left-orderable for all $\frac{r}{s} > \frac{p}{q}$.*

Based upon recent results concerning homologically thin links, the following would also be true if the L-space conjecture holds.

Conjecture 2.9 *(Liam Watson–a special case of the L-space conjecture) If L is a thin link, then the twofold branched cover of L has a non-left-orderable fundamental group.*

Recent progress has come in fits and starts, owing partly to the fact that some of the most common methods of producing left-orderings of fundamental groups are known to be inadequate to address general case. For example, of the several computational tricks that yield left-orderings of $\pi_1(M)$, one of these methods is to find a representation of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{R})$ which lifts to a representation into $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. While this method works in many cases, Xinghua Gao proved that there exists a 3-manifold M with left-orderable fundamental group $\pi_1(M)$ admitting no representations into $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$—in other words, this technique is inadequate in general [13]. There are other common methods of left-ordering $\pi_1(M)$ when $H_1(M)$ is finite, but most of these methods (all of them, potentially), produce left-orderings arising from representations $\pi_1(M) \to \widetilde{\mathrm{Homeo}_+(S^1)}$ (note that $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is subgroup of $\widetilde{\mathrm{Homeo}_+(S^1)}$, so the method mentioned above is an example of this). Thus we arrive at:

Question 2.10 *(Nathan Dunfield, Xinghua Gao) Does there exist a 3-manifold M such that $\pi_1(M)$ is left-orderable, yet there is no interesting representation $\pi_1(M) \to \widetilde{\mathrm{Homeo}_+}(S^1)$?*

This is equivalent to asking whether or not there exists a 3-manifold group which is also a "left-orderable monster."

Similarly, there are computational methods for showing that a group is *not* left-orderable, for example by trying to create "potential positive cones" on each ball B_n of radius n in the Cayley graph of G and stopping when one finds $n > 0$ such that no such subset exists. Related to this, we ask the following question:

Question 2.11 *(Nathan Dunfield) Let G be a 3-manifold group. Is left-orderability of G algorithmically decidable?*

It is known that for general groups, the question of left-orderability is not decidible ([2, Theorem 3.3]). Yet, since it is algorithmically decidable whether or not a given 3-manifold M is an L-space, the truth of the L-space conjecture would imply that the answer to Question 2.11 is "yes" for 3-manifold groups.

Bi-orders and generalized torsion: A group G is said to admit *generalized torsion* if there exists $g \in G$ such that $\prod_{i=1}^{n} h_i g h_i^{-1} = id$ for some collection of elements $h_i \in G$. If no such element exists, G is called *generalized torsion free*. It is not hard to see that bi-orderable groups are generalized torsion free, though the converse does not hold (see, for example, [3]). Again, it is conjectured that 3-manifold groups might behave in a special way with respect to this algebraic property.

Conjecture 2.12 *(Ito-Motegi-Teragaito [20]) Suppose that G is the fundamental group of a 3-manifold. Then G is bi-orderable if and only if G is generalized torsion free.*

This conjecture seems plausible as Motegi, Teragaito and Ito have shown that the conjecture is true if and only if it holds for all irreducible 3-manifolds, and that it holds for all Seifert fibred manifolds and various hyperbolic 3-manifolds [20].

On the other hand, it is known that torsion-free one-relator groups are left-orderable, while Question 16.48 of the Kourovka notebook asks if a generalized-torsion free group is necessarily left-orderable. Related to these notions, we arrive at a possibly simpler question:

Question 2.13 *(Yago Antolín) If G is a one-relator group that admits no generalized torsion, then is it true that G admits a bi-ordering?*

Complexity of positive languages: Given a finitely generated and left-ordered group G, a *positive language* is a set of words in the generators that represents the positive cone of the underlying left-order (*i.e.* the set of elements bigger than the identity element). It was recently observed by Hermiller and Sunic [15] that on the free group (or more generally on a free product of groups) there is no positive language that can be described by an automaton (in other words, for free groups there is no positive language which is regular in the sense of Chomsky). This generalized the already known fact that free groups (and more generally, free products of groups) do not admit positive cones which are finitely generated as semi-groups, and raises the following question.

Question 2.14 *Let G be your favourite left-orderable group. What is the simplest (in the language theoretical sense) left-order that can be defined on G?*

For instance, for non-Abelian free groups Sunic has built positive languages that are *context free*, which is the level of complexity just above regular languages in Chomsky hierarchy. On the other hand, every left-orderable, polycyclic group supports a left-ordering whose positive cone can be described by a regular language.

Question 2.15 *(Cristóbal Rivas) Is there a (Gromov)-hyperbolic group supporting a regular positive cone?*

Regularity of group actions: A *resilient pair* for the action of a group G on the line is a tuple $(f, g; x, y)$ with $f, g \in G$ and $x, y \in \mathbb{R}$ such that

$$x < f(x) < f(y) < g(x) < g(y) < y.$$

It it known that a left-ordering on a countable group G has the Conradian property (that is, $f > id$ and $g > id$ imply that $fg^2 > g$) if and only if the corresponding action on the line has no resilient pairs [**?**]. It is not hard to see that elements f and g in a resilient pair generate a free-semigroup, and so many left-orderable groups turn out to admit only Conradian orderings as they contain no free subsemigroups. This is the case for nilpotent groups and, more generally, for finitely generated groups of subexponential growth.

It turns out that for actions by C^2-diffeomorphisms [14], as well C^1-actions on the closed interval (this is in an unpublished work of Hurder), the presence of resilient pairs turn out to be equivalent to the action having positive entropy. By entropy, we mean the following.

Recall that if G is a finitely generated group acting on $\mathbb{R}$, we say that $x, y \in \mathbb{R}$ are (n, ϵ)-separated if there is $f \in B_G(n)$, the ball of radius n in G with respect to a word metric, such that $|f(x) - f(y)| \geq \epsilon$. In this way the entropy for the action is defined by $n(G) = \lim_{\epsilon \to 0} n_\epsilon(G)$, where

$$n_\epsilon(G) = \limsup_{n \to \infty} \frac{\log(\max \#\{S \mid \text{every } x, y \in S \text{ is } (n, \epsilon)\text{-separated})\})}{n}.$$

This number is not invariant under conjugation but its triviality, namely if $n(G) = 0$ or not, is a conjugation invariant. Moreover, when the G-action is by C^1-diffeomorphisms, the entropy of G is bounded by a constant that depends on the C^1-norm of the generators of G. So we ask

Question 2.16 *(Andrés Navas) Let G be a finitely generated group of C^1 diffeomorphisms of the interval. Assume that G has no resilient pair. Does there exist a sequence of coordinate changes ϕ_n such that for all $g \in G$, the conjugates $\phi_n g \phi_n^{-1}$ converge to the identity in the C^1 topology?*

A positive answer to the above question would imply that for C^1-actions, the positivity of entropy is equivalent to the presence of resilient pairs.

The Tarski Monster: Left-orderability of a group carries a fair amount of information about its algebraic structure, and as such it is often of interest to determine whether or not groups of mathematical significance can be left-ordered. For instance, Rivas and Triestino recently showed that the Higman group $H = \langle a_i (i \in \mathbb{Z}/4\mathbb{Z}) \mid a_i a_{i+1} a_i^{-1} = a_{i+1}^2 \rangle$ is left-orderable [28]. In line with the theme of investigating left-orderability of mathematically significant torsion-free groups, we have:

Question 2.17 *(Michel Boileau) Can a torsion-free Tarski monster be left-orderable?*

Recall a torsion-free Tarski Monster is a non-cyclic group such that any proper subgroup is isomorphic to $\mathbb{Z}$. Groups with these properties where built by Ol'shanski. Tarski Monsters monsters are candidates for being left-orderable since, in some sense, they marginally fail to satisfy known criteria for left-orderability. As en example, a famous criterion due to Conrad is that a group is left-orderable (in fact, Conradian left-orderable) if and only if every finitely generated subgroup of a group G admits a surjection onto $\mathbb{Z}$.

Another question related to Tarski Monster is the following.

Question 2.18 *(Michel Boileau) Let M be a compact, connected, orientable and irreducible 3-manifold. Can you show that $\pi_1(M)$ is not a Tarski monster, without appealing to geometrization?*

Dynamics in dimension two: The focus of most of the discussion above has been on one-dimensional dynamics, but for the moment let us consider dimension two.

Very recently, James Hyde showed that the group $\mathrm{Homeo}(D^2, \partial D^2)$ of homeomorphisms of the disk that pointwise fix the boundary is not left-orderable [16]. This was a a bit of a surprise since $\mathrm{Homeo}(D^2, \partial D^2)$ is torsion free and also, by Thurston Stability Theorem, we know that $\mathrm{Diff}^1(D^2, \partial D^2)$ admits a Conradian order, and by Rolfsen and Calegari $\mathrm{PL}(D^2, \partial D^2)$ is similarly Conradian left-orderable [7]. Amazingly we still cannot answer the following question:

Question 2.19 *(Andrés Navas) Does there exist a finitely-generated torsion-free group G that does not embed in* $\mathrm{Homeo}(D^2, \partial D^2)$*?*

Another question is whether admitting an action on an object of dimension two, say the sphere S^2, is a closed property. Indeed, one of the most significant properties of left-orderability—and hence of group acting on one-manifolds—is that it is a local property. By this we mean that if a group G is not left-orderable, then G contains a finite set S such that no compatible and total order can be defined on $\langle S \rangle$ such that the order is invariant under left-multiplication. One consequence of this fact is that, inside the space of finitely generated groups (some times also called the space of marked groups, see [**?**] for a survey), the set of left-orderable group is closed, that is if $\Gamma_n \to \Gamma$ and Γ_n is left-orderable for all n then so is Γ. The question, then, is whether or not this extends to higher dimensions? For instance we can ask:

Question 2.20 *(Andrés Navas) Does there exist a sequence of groups $\Gamma_n \to \Gamma$ with $\Gamma_n \subset \mathrm{Homeo}_+(S^2)$ with Γ NOT embedding in* $\mathrm{Homeo}_+(S^2)$*?*

3 Scientific Progress Made

The meeting brought together several diverse communities of mathematicians that might not otherwise have had an opportunity to meet and discuss the connections between their different fields of mathematics. In particular, with expertise ranging from dynamical systems, knot theory and low dimensional topology, and to group theory, the meeting was an excellent opportunity for cross-pollination between fields that have become more interconnected in recent years.

For example, as a result of this conference Dave Gabai has resumed a project from over 30 years ago with other participants of the conference. The outcome of the project will be, roughly, a correspondence between actions of fundamental groups of 3-manifolds on 1-dimensional objects and a certain type of essential lamination supported by the manifold. Such a result would constitute significant progress towards proving the L-space conjecture.

In another instance, Tyrone Ghaswala, motivated by Liam Watson's talk and use of the Montesinos trick in investigating two-fold cyclic branched covers, has developed a generalized Montesinos trick used for analyzing n-fold cyclic branched covers for arbitrary $n \geq 2$. The hope is that this result will open the door to a new approach to analyzing the fundamental groups of the higher cyclic branched covers of knots that uses Tyrone's results from the field of mapping class groups as a cornerstone.

Further connections between the algebra of left-ordered groups and topology arose in the form of a new collaboration between Zoran Sunic, Yago Antolín and Cristóbal Rivas, who began investigating a new connection between the regularity of positive cones in groups and Bieri-Neuman-Strebel invariants (these are powerful invariants arising from geometry, and encode significant topological information in the case that the group is the fundamental group of a 3-manifold).

4 Outcome of the Meeting

The conference featured many informal breakout meetings to discuss new mathematical ideas or to advance active projects. As such, we found the best way to capture the outcome of the meeting would be to present the stories and individual feedback of the participants, as each had a different mathematical experience. Some of their feedback is below, slightly edited for clarity and typesetting purposes.

Ty Ghaswala: Liam Watson's talk started out with a review of the well known Montesinos trick for studying double branched covers of knots, however this was my first exposure to said trick! There are potential generalizations to higher degree branched covers, which would be applications of previous projects of mine with Becca Winarski (The liftable mapping class group of balanced superelliptic covers) and Alan McLeay (Mapping class groups of covers with boundary and braid group embeddings). I am currently splashing around in this general direction to hopefully find some useful results I can prove. I had many conversations with Hannah Turner while at the conference, and if anything comes of my splashing, the plan is to get in contact with her and see if there are any collaboration opportunities.

Dale Rolfsen: Nearly all the talks were excellent, but one by Sunic stood out for me, as it described a startlingly simple proof of a classical theorem of Vinogradov, that free products of orderable groups are orderable. I had shown a few years ago that the category of orderable groups is a tensor category, by a rather complicated argument. This new approach gives a much clearer understanding of that. Another highlight of the conference for me was meeting James Hyde and discussing with him his remarkable construction (with Lodha) of a finitely generated simple left orderable group, answering a fifty-year old question of Rhemtulla. I also worked several evenings with Clay and Ghaswala on the open question (which I learned at the conference) of whether a group without generalized torsion must be left-orderable. Â Alas, without success other than showing a suspected counterexample indeed was not.

Dave Gabai: Being invited to the workshop motivated me to rethink a project that stalled 30+ years ago. While the motivations were different then, a special case has relevance to the conjecture that LO [fundamental group of a 3-manifold] implies existence of a taut foliation. "If" (and it's a big if) the project succeeds as stated, then it would give LO implies the existence of a T.O. essential lamination. I few weeks before the workshop Rachel agreed to join the project, and during long conversations before and during the workshop, we might have made some progress. I should add that there is a very long way to anything definitive.

Masakazu Teragaito: It was absolutely a great experience to me as a speaker and an attendant. Especially for me, Cameron [Gordon]'s presentation always inspires me. In fact, his talk at Melbourne in 2011 about the L-space conjecture was the exact start for me to study orderings. The experience encourages me and my coworkers to keep our research project pushing on. In addition to many stimulating talks, all daily experience at BIRS-CMO, such as mathematical and non-mathematical conversation with participants, are also very impressive for me. I am sure that it would continue to encourage me.

Sam Kim: The workshop was a fantastic chance to develop highly focused discussions on the interactions between one-dimensional dynamics and three-manifold topology. I was particularly intrigued by a talk and a following discussion by Hélène Eynard-Bontemps, which certainly shed a new number theoretic light on my research on circle diffeomorphism groups.

Michele Triestino:

1. With Nicolas Matte Bon we had the possibility to finish working on our paper "Groups of piecewise linear homeomorphisms of minimal flows", and we realized that we are unable to do anything better.
2. Sam Kim and Thomas Koberda explained to me more about special groups, and we realized that for any $n \geq 2$ there are fundamental groups of hyperbolic n-manifolds which are left-orderable (many experts, including me, were not aware of that).
3. The free discussion time on Friday was a great opportunity to work informally with people. With Hélène Eynard-Bontemps, Adolfo Guillot and Andrés Navas we tried to understand how to extend the recent results by Eynard and Navas to higher regularity. We found a potential obstruction to the solution of a cohomological problem, which seems very interesting.

Ying Hu: During the workshop, I had a productive discussion with Steve Boyer and Cameron Gordon on a project we are currently working on. Rachel Roberts's talk and many conversations with her were also extremely useful in pushing the project forward.

Andrés Navas: This was a very fruitful meeting on a subject that has been rapidly evolving over the last years. I benefited from the ambience as well as the presence in the conference of some of my collaborators, with whom I pursued joint research. In summary, this was an excellent experience in a very nice place!

Hélène Eynard-Bontemps: The BIRS-CMO Workshop on orderable groups enabled Andrés Navas and I to work intensely and make huge progress on our recent joint work about (conjugacy classes of) interval diffeomorphisms, a subject we were able to discuss further with other specialists of group actions on one-dimensional manifolds gathered for the occasion.

Cristóbal Rivas: This workshop was an incredible opportunity to engage on many fruitful conversations. Of particular interest was a discussion with Zoran Sunic and Yago Antolín about regularity of positive cones in solvable groups, and its relation to the Bieri-Neuman-Strebel invariant. This is potentially an excellent project and I hope we can finish it soon.

Adam Clay: After Ty Ghaswala presented our recent work on promoting circular orderability to left-orderability, we received excellent feedback on our project in two forms. First, a group of dynamicists offered up a dynamical approach to proving some of the main results of our paper, which would represent an excellent new perspective on our ideas. Second, Dave Witte-Morris approached us a day or two later with a technique for calculating some new numerical obstructions to left-orderability of a group that we outlined in our paper. His contribution will be included in future drafts of our work as an appendix.

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