Beyond Toric Geometry

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1 Overview of the Field

Over the past several decades, toric geometry has become an increasingly important area of algebraic geometry. On the one hand, many deep theorems in algebraic geometry can be reduced to statements about toric varieties, for example, the stable reduction theorem and the weak factorization theorem. On the other hand, toric geometry has been used in numerous applications, ranging from coding theory and geometric modelling to mirror symmetry and algebraic statistics.

One of the most appealing aspects of toric geometry is that the objects of study, toric varieties, can be understood completely in combinatorial terms. This connection between combinatorics and algebraic geometry often makes the study of toric varieties considerably easier than that of arbitrary varieties. Additionally, this connection is also useful in the opposite direction, allowing for algebro-geometric tools to be used to solve certain problems in combinatorics, for example, McMullen's conjecture on the f-vectors of simplicial polytopes.

Recently, there has been a growing attempt to apply tools similar to those used in toric geometry to approach a larger class of varieties and problems. Such topics with especially exciting new developments include the varieties with lower-dimensional torus action or T-varieties, Cox rings and Mori Dream Spaces, and equivariant cohomology theory.

1.1 T-Varieties

Varieties with torus actions by lower-dimensional tori, known as T-varieties, provide a natural generalization of toric varieties. By work of Altmann-Hausen [4] and Altmann-Hausen-Süß [5], an n-dimensional variety X with an effective action by a k-dimensional torus can be encoded by an $n-k$-dimensional quotient variety Y endowed with some additional combinatorial structure. In the toric case, the quotient variety is simply a point and the combinatorial structure is a rational polyhedral fan.

This quasi-combinatorial description has led to a very fruitful study of the geometry of T-varieties over the past several years. Many of the classical results concerning toric varieties have been at least partially generalized to this setting: descriptions of divisors and intersection theory [63]; classification of singularities [57]; description of automorphism groups [10]; and a characterization of the properties of being Frobenius split [3], just to name a few. These results, and the techniques used, have been applied to a number of other areas, including coding theory [45], existence of Kähler-Einstein metrics [66, 44], and the study of the affine Cremona group [56].

One particularly well-studied class of T-varieties are equivariant toric vector bundles. They have been described in toric language by Klyachko [52] using collections of filtered vectored spaces. The recent approach via the machinery of T-varieties has answered questions concerning their Cox rings and Frobenius-splitting properties. Even more recent work of Di Rocco-Jabbusch-Smith [64] approaches toric vector bundles via so-called ”parliaments” of polytopes, and promises to further illuminate their geometry.

1.2 Cox rings and Mori Dream Spaces

Cox rings of general algebraic varieties were defined by Hu and Keel [43], generalizing D. Cox's construction of homogeneous coordinate rings of toric varieties. Cox rings have important applications to birational geometry and the minimal model program. The main problem in the theory of Cox rings is to determine which varieties are Mori Dream Spaces (MDS), which means that they have finitely generated Cox rings. Toric varieties are examples of MDS by the work of Cox [26]. Other examples of MDS include Fano varieties, in particular homogeneous spaces, and proper images of MDS [61]. On the negative side, Mukai [59] has shown that blowups of projective spaces at sufficiently many points are not MDS.

There has been a great amount of recent work studying Cox rings of varieties closely related to toric varieties. These include Cox rings of toric vector bundles by Gonzalez-Hering-Payne-Süß [31], complexity one T-varieties by Altmann-Petersen [6], general varieties with torus action by Hausen-Süß [39] and many others.

The blowups of weighted projective planes $\mathbb{P}(a, b, c)$ at a general point provide some of the simplest examples of non-MDS. These blowups have a long history in the commutative algebra under the name ”symbolic Rees algebras of monomial prime ideals” (see for example the CMO workshop following this one). Cutkosky [27] and Srinivasan [65] have connected the commutative algebra of the symbolic powers to Cox rings and the birational geometry of the varieties. In particular, they have shown that the blowup of $\mathbb{P}(a, b, c)$ is a MDS if $a \leq 4$ or if $a = 6$.

Goto, Nishida, and Watanabe [35] gave infinite sequences of weighted projective planes $\mathbb{P}(a, b, c)$ whose blowup at a point is not a MDS. These sequences were generalized by Gonzalez-Karu [32] to a larger set of examples. The construction was further sharpened by He [40], giving new examples of both MDS and non-MDS. Despite an abundance of examples and non-examples, we still do not know which weighted projective planes $\mathbb{P}(a, b, c)$ blown up at a point are MDS. For example, it is not known if the blowups of $\mathbb{P}(a, b, c)$ with $a = 5$ are always MDS.

An important class of varieties with interesting Cox rings are the moduli spaces $\overline{M}_{0,n}$. The minimal model program of these varieties is expected to have a moduli theoretic interpretation. Castravet [23] showed that for $n \leq 6$ these spaces are MDS, with explicitly given generators of the Cox ring. Castravet and Tevelev [24] constructed a large class of hypertree divisors on the moduli spaces. It was hypothesized that these divisors generate the Cox ring. However, Doran-Giansiracusa-Jensen [28] showed that these divisors are not enough, even in degree 2, and finally Castravet and Tevelev [25] proved that the moduli spaces are not MDS when $n \geq 134$. This bound was reduced to $n \geq 13$ by Gonzalez-Karu [32] and to $n \geq 10$ by Hausen-Keicher-Laface [38]. The cases $n = 7, 8, 9$ are still open.

1.3 Equivariant Cohomology Theories

The T-equivariant cohomology and K-theory of a nonsingular toric variety have a simple description as the rings of piecewise polynomial and piecewise exponential functions on the fan [19]. A great deal of effort has been spent in recent years to extend these results to singular toric varieties and to other cohomology theories.

The combinatorial description of the cohomology of a nonsingular toric variety is a special case of the localization formula of Atiyah-Bott and Berline-Vergne. The localization property applies to more general varieties. For example, there is a long history of using localization to compute the equivariant cohomology of such classical varieties as Grassmannians and Flag varieties. Many other theories satisfy the localization property and hence can be described combinatorially by restricting to fixed points.

Goreski-Kottwitz-MacPherson [34] gave a general method of computing the intersection cohomology of a T-variety with finitely many fixed points and 1-dimensional orbits (such spaces are since called GKM spaces). Combinatorial intersection cohomology of toric varieties [18, 12] can be viewed as a special case of this construction. Braden and MacPherson [17] specialized this construction to Schubert varieties in flag varieties, giving a combinatorial description of the equivariant intersection cohomology from the moment graph of the variety. Elias and Williamson [30] used a slightly different but related combinatorial description of the intersection cohomology in terms of Soergel bimodules to prove non-negativity of Kazhdan-Lusztig polynomials for general Coxeter groups.

Generalizing Chow theory and K-Theory, Levine-Morel [55] gave the general definition of an oriented homology theory. They also constructed algebraic cobordism as a universal such homology theory. It is not clear if the equivariant versions of these general theories satisfy the localization to fixed points property. For example, the equivariant algebraic cobordism of nonsingular toric varieties was described combinatorially by Krishna-Uma [53] and it indeed satisfies the localization property. The algebraic cobordism ring of flag varieties and the Schubert calculus in the cobordism ring were studied by Hornbostel-Kiritchenko [42]. General oriented cohomology theories on flag varieties were studied by Calmés-Zainoulline-Zhong [22], again by localizing to fixed points.

A class of cohomology theories that can be constructed from homology theories are the Fulton-MacPherson operational theories. The known examples of these theories on toric varieties all satisfy the localization property. It is not known if the localization property holds on general T-varieties and for general operational theories. (However, see the talk by Payne where this is discussed for K-theory.)

1.4 Interactions

The three topics described above complement each other and provide ample opportunity for interaction. T-varieties are a natural class of varieties whose Cox rings are more easily understood than in general. In particular, the above-mentioned recent work of Gonzalez-Karu (2015) could be used to construct new examples of T-varieties and toric vector bundles with non-finitely generated Cox rings. Likewise, T-varieties give a natural setting for studying equivariant cohomology theories. On the other hand, the spectrum of the Cox ring of an MDS X is itself a T-variety, with action by the torus whose character lattice is the Picard group of X. Furthermore, any cohomology theory on X can be understood as the equivariant cohomology of the universal torsor over X, which is an open subvariety of the spectrum of its Cox ring.

A unifying theme among these three areas is that all concern varieties endowed with group actions sharing many of the features found in the toric setting. In the equivariant cohomology and the localization formula, the torus usually acts with finitely many fixed points, as is the case for toric varieties. In the case of T-varieties, the simplest non-trivial and best understood case is that where the torus has codimension one, very close to the codimension-zero case of the toric setting. Finally, typical examples of varieties for which Cox rings are computed are quotients of the affine space by a product of the torus and several factors of the additive group G_a. This mirrors the situation for toric varieties, which are quotients of open subsets of affine space by a torus.

2 Recent Developments and Presentation Highlights

2.1 Characterizations of Combinatorial Varieties

There are several different classes of varieties which may be described in combinatorial or quasi-combinatorial terms: toric varieties, rational homogeneous spaces, and more generally T-varieties and spherical varieties, to name a few. As J. Wisniewski pointed out during his talk, it is both interesting and useful to find intrinsic characterizations for such classes of varieties. In particular, if one wishes to prove that some given variety has a particular property known to hold for a class of combinatorial varieties, a fruitful approach can be to use such an intrinsic characterization to show that the given variety belongs to this class of varieties.

There are numerous characterizations of toric varieties: those varieties admitting a generalized Euler sequence [47], those varieties whose Cox ring is a polynomial ring [15], and those varieties whose structure sheaf splits as a direct sum of line bundles after Frobenius pushforward [2], among other characterizations.

In his talk, J. Wisniewski described a new characterization of flag varieties. Namely, if a projective variety X admitting sufficiently many structures as a $\mathbb{P}^1$-fibration such that the classes of the fibers span $H_2(X, \mathbb{Q})$, then X is a flag variety. Not only is this an interesting characterization of flag varieties, it can be applied to show the global rigidity of flag varieties in smooth families of Fano varieties.

In the talk of P. Achinger, another question concerning characterization of combinatorial varieties was addressed: is the smooth image Y of a smooth projective toric variety X again a toric variety? This was originally conjectured in [60], wherein an affirmative answer was given in the case that Y has Picard number one. The question remains open for the case when the Picard rank of Y exceeds one.

Achinger and his collaborators (J. Witaszek and M. Zdanowicz) have related this question to the liftability of the Frobenius morphism in characteristic p. There are some obvious classes of varieties for which the Frobenius morphism is liftable modulo p^2, such as toric varieties, ordinary abelian varieties, and toric fibrations over ordinary abelian varieties. Achinger and collaborators conjecture that the only varieties with liftable Frobenius morphism are, up to a finite Galois cover, such toric fibrations over ordinary abelian varieties. In addition to proving this conjecture in several special cases (e.g. for surfaces, Fano threefolds, and for homogeneous spaces), they show that this conjecture implies an affirmative answer to the question concerning smooth images of toric varieties.

2.2 Cox Rings and Mori Dream Spaces

Just as morphisms of projective varieties can be described via maps between their homogeneous coordinate rings, one might hope that morphisms between Mori Dream Spaces can be described via maps between the corresponding Cox rings. Indeed, in the smooth case, this follows from the functoriality of the Cox ring [14].

The singular case is a bit more subtle. A first step was taken by G. Brown and J. Buczyński, who proved that rational maps between toric varieties lift to *multi-valued* maps between their Cox rings. In his talk, A. Hochenegger described joint work with E. Martinengo in which they show that a morphism between Mori Dream Spaces X and Y can be described via a map between the Cox rings of certain Mori Dream Stacks associated to X and Y. This result is similar to recent work of J. Buczyński and O. Kedzierski [20].

The talk of J. Hausen started with an introducing to the Cox ring $R(X)$ of a variety X, emphasizing the Cox space $\operatorname{Spec} R(X)$. For a Mori Dreams Space X with free class group, its Cox ring is always a unique factorization domain, but as soon as the class group has torsion, this may fail. Hausen instead asks for the weaker property of *iteration of Cox rings*: X has iteration of Cox rings is the Cox ring R of X is factorial, or (inductively) if the spectrum of R is a Mori Dream Space having iteration of Cox rings. In joint work with M. Wrobel, he characterizes exactly which complexity-one T-varieties have iteration of Cox rings [36].

Even more fundamental and challenging is the question of finite generation of the Cox ring of a projective variety, as described in §1.2. Z. He gave a talk about among blowups of weighted projective planes $\mathbb{P}(a, b, c)$ at a general point and MDS. Examples of such blowups that are not MDS were constructed in [32]. He [40] generalized and sharpened these examples to a necessary and sufficient criterion: given a weighted projective plane $\mathbb{P}(a, b, c)$ whose blowup contains a special curve of negative self-intersection, determining if the blowup is a MDS becomes a problem of finding plane curves through a set of lattice points. This gives new examples of MDS and non-MDS among blowups of weighted projective planes.

C. Casagrande gave a talk concerning one very specific family of Mori Dream Spaces: the Fano scheme parametrizing $(m-1)$-planes in a smooth complete of two quadric hypersurfaces in $\mathbb{P}^{2m+2}$. The study of this variety goes back to the PhD thesis of M. Reid. By making use of the moduli space of parabolic bundles, Casagrande and C. Araujo are able to show that this Fano scheme is isomorphic in codimension-one to $\mathbb{P}^{2m}$ blown up at $2m+3$ points , and use this to compute cones of nef, movable, and effective divisors [9].

2.3 Toric Bundles

Klyachko's equivalence of categories between equivariant vector bundles on a toric variety X and compatible collections of filtered vector spaces has been instrumental in making the study of toric vector bundles accessible via combinatorial tools. Recent work of Di Rocco-Jabbusch-Smith has pushed the work of Klyachko further, associating to any toric vector bundle a matroid along with a rational polytope for each element of the ground set of the matroid. Among other things, this data can be used to check for global generation of toric vector bundles.

In his talk, G. Smith outlined this construction, and showed how the matroid associated to a toric vector bundle actually determines a finite resolution of the bundle by equivariant line bundles. Of particular interest is the case when the equivariant bundle under consideration is the kernel bundle M_L for the embedding of a smooth toric variety X in projective space. Using the associated resolution by line bundles, Smith hopes to prove vanishing of $H^1(X, M_L \otimes L^m)$ for $m \geq 1$, which would imply Oda's long-standing conjecture (see §3 below).

A. Dey talked about T-equivariant principal G-bundles on toric varieties. These are principal G-bundles with an action of $T \times G$, compatible with the T-action on the toric variety. In a joint work with Biswas and Poddar [16], they describe a G-bundle in terms of the local data of a morphism $T \to G$ and a 1-cocycle. This construction generalizes Kaneyama's description of toric vector bundles to principal bundles and to non-compact toric varieties. An interesting feature is that isomorphism classes of algebraic and holomorphic bundles turn out to be the same.

2.4 Toric Degenerations and Okounkov Bodies

A common tool used to apply toric techniques to non-toric varieties is that of *toric degenerations*. In order to prove something about a non-toric variety X, one degenerates it to a toric variety Y, proves some desired properties about Y, then shows that these properties lift back to X. Such techniques have been used in a variety of settings, ranging from mirror symmetry (e.g. [13]) to bounds on Seshadri constants (e.g. [46]).

Two common frameworks for constructing toric degenerations are tropical geometry (prime maximal cones in the tropicalization of X give toric generations) and the theory of Newton-Okounkov bodies (full rank valuations on X with finitely generated value semigroups give toric degenerations [7]). Recent work of K. Kaveh and C. Manon [49] connects these two approaches, showing how prime cones give rise to special full rank valuations.

C. Manon gave a talk pursuing these ideas further in the setting of complexity-one T-varieties. He discussed joint work with N. Ilten [44], in which they show that every affine rational complexity-one T-variety X has an embedding in affine space for which all maximal cones in the tropicalization of X correspond to prime ideals. Furthermore, every torus homogeneous full rank valuation has a finitely generated value semigroup, which can be explicitly described. As a corollary, one obtains that every $T \times \mathbb{K}^*$-equivariant one-parameter degeneration of a projective rational complexity-one T-variety with integral special fiber has a very particular form, generalizing a result of Ilten and Süß [44].

One of the first examples of a Newton-Okounkov body are the Gelfand-Zetlin polytopes, which arise as Newton-Okounkov bodies of flag manifolds. It was shown by K. Kaveh [48] that the Khovanskii-Pukhlikov ring for such a polytope is isomorphic to the cohomology ring of the corresponding flag manifold. In her talk, V. Kiritchenko discussed the problem of representing the classes of Schubert cycles by facets of such polytopes. This has been done by Kiritchenko, V. Timorin, and E. Smirnov for type A flag manifolds [51]. Kiritchenko reported on work in progress extending these results to type C. The attractiveness of such a representation is that the structure coefficients for multiplication would be manifestly positive, without any cancellation appearing as is the case when using Schubert polynomials.

2.5 Equivariant Cohomology and K-Theory

J. Rajchgot gave a talk about Quiver varieties. The representation space of a quiver is the product of matrix spaces. It has the action of a group G by coordinate change at each node. The closures of the G-orbits are called quiver varieties. Rajchgot addressed the problem of finding the equivariant Chow and K-theory classes of these varieties. In a joint work with Kinser and Knutson [50], in the type A case they find three different formulas for these classes. The formulas in the K-theory (resp. Chow theory) are in terms of double Grothendieck (resp. double Schubert) polynomials, sums of pipe dreams, and sums of Grothendieck (resp. sums of Schubert) polynomials.

S. Payne talked about equivariant operational Chow theory and K-theory. In earlier work Payne [62] and Anderson-Payne [8] constructed these equivariant theories for toric varieties. For arbitrary, possibly singular, toric varieties these theories have the same form as the ordinary equivariant Chow and K theories for smooth toric varieties. In joint work with Anderson and Gonzalez, they construct a Riemann-Roch transformation between the two theories and show that the theories satisfy the localization to fixed points property. Another application of their research is that the equivariant K-theory of a toric variety does not in general surject onto the non-equivariant K-theory.

K. Zainoulline talked about motives defined via correspondences and idempotent completion. These motives can be defined in the G-equivariant setting. Further, in the definition of correspondences one can replace Chow classes with classes in an arbitrary oriented G-equivariant cohomology theory. In a joint work with Calmés and Neshitov [21], they study the endomorphism ring of the motive of a homogeneous G-variety X. One reason that the endomorphism ring is interesting is because its decomposition corresponds to a decomposition of the motive. In the case where X is a flag variety, the endomorphism ring can be given explicit generators in terms of push-pull operators. Depending on the oriented cohomology theory used, one can identify the endomorphism ring of a flag variety with several types of Hecke algebras.

2.6 Other Topics

A number of other highly interesting talks were also given. A. Liendo gave a nice survey on the Altmann-Hausen theory of T-varieties, providing a comprehensive picture of the current state of the art of the theory. He then discussed smoothness criteria for T-varieties. Surprisingly, until now a characterization of smooth T-varieties in terms of their Altmann-Hausen descriptions had only previously been known in the complexity one case. Liendo's characterization, which is joint work with C. Petitjean, is quite natural: a T-variety is smooth if and only if its Altmann-Hausen description is étal locally isomorphic to the description for affine space equipped with the action of a subtorus. This generalizes the well-known characterization of smooth varieties (with no torus action).

D. Maclagan reported on recent advances in tropical geometry. Building on work of the Giansiracusa brothers, she and F. Rincon have developed the notion of tropical ideals [58]. In order to move from the world of tropical varieties to that of tropical schemes, one would like to work in the polynomial semiring over the tropical semiring, but this ring has many pathological properties, including being highly non-noetherian. By restricting one's attention to the special class of tropical ideals, many of these pathologies are resolved, and an attractive theory emerges which promises to resolve a number of issues in tropical geometry, for example, the existence of too many lines on tropical cubic surfaces.

D. Duarte talked about Nash blowups of toric varieties. It is an open problem whether these blowups can be used to resolve singularities of toric varieties [11, 33]. The Nash blowup of a variety X is a projective morphism, hence can be given as the blowup of an ideal sheaf J on X. On a toric variety there is an explicit construction of J determined by a presentation of the coordinate ring of X with binomial equations. The question explored in this talk was whether the zero locus of J equals the singular locus of X. This has been known for toric varieties that are complete intersections. Duarte and Martinez [29] extend this result to possibly non-normal general toric surfaces.

3 Open Problems

During the meeting, many of the talks and ensuing discussions centered around known open problems, as well as posing new ones. One of the best known open problems in toric geometry is Oda's conjecture:

Conjecture 1 (Oda's Conjecture)**.** *Let X be a smooth toric variety with very ample line bundle $\mathcal{L}$. Then $\mathcal{L}$ defines a projectively normal embedding of X.*

There have been numerous (as of yet unsuccessful) attempts at proving this conjecture, see e.g. [1]. Recent work of G. Smith involving resolutions of toric vector bundles by line bundles described in §2.3 gives hope of a new mode of attack on this conjecture. More generally, it has been conjectured that all ample line bundles on a smooth projective toric variety satisfy property N_p for all p. This problem is still open in almost all cases, although it is known that a tensor power of the ample bundle satisfies property N_p [41].

Another open conjecture fully within the realm of toric geometry is the conjecture of Occhetta-Wisniewski on smooth images of toric varieties:

Conjecture 2 (Occhetta-Wisniewski)**.** *Let X, Y be smooth projective varieties, $f : X \to Y$ a surjective morphism, and X toric. Then Y is also toric.*

As mentioned in §2.1, this conjecture would follow from the following conjecture of Achinger, Witaszek, and Zdanowicz:

Conjecture 3 (Achinger-Witaszek-Zdanowicz)**.** *Let X be a smooth projective variety over a field of characteristic p, with Frobenius morphism liftable modulo p^2. Then up to a finite étale Galois cover, X is a toric fibration over an ordinary abelian variety.*

Just as there are only finitely many deformation families of smooth Fano varieties in any given dimension, there are also only finitely many toric Gorenstein Fano varieties in any fixed dimension [54]. It is expected that a similar statement holds true for complexity-one T-varieties:

Conjecture 4 (Hausen-Ilten-Süß)**.** *For a fixed dimension d, there exist only finitely many equivariant deformation classes of non-toric complexity-one Fano T-varieties with at worst canonical Gorenstein singularities.*

There is an effective approach to classifying Fano varieties with complexity-one torus action by making use of homogeneous coordinate rings [37]. However, this approach requires a bound on the Picard number. Alternatively, such varieties may be described by a quasi-combinatorial gadget known as a Fano divisorial polytope, see [44]. To produce an effective algorithm classifying such gadgets, one requires a bound on the volume of the moment polytope.

Throughout the meeting, complexity-one T-varieties were used to illustrate many phenomena going beyond toric geometry, for example, iteration of Cox rings (§2.2), and toric degenerations (§2.4). The prevalence of complexity-one T-varieties as a good test bed for examples stems from the fact that many aspects concerning their geometry are almost as well understood as in the toric case. A big next step in the theory of T-varieties would be to push many of the results for complexity-one T-varieties to complexity-two and beyond:

Question 5. *What results about complexity-one T-varieties can be extended to higher complexity?*

For example, one might hope that combining the approach of [44] regarding K-stability of complexity-one T-varieties with Tian's work on the α-invariant for surfaces could lead to a criterion for K-stability of complexity-two Fano T-varieties.

As discussed in §2.2, it remains open exactly which blowups of weighted projective planes (or more general toric varieties) are Mori Dream Spaces. Srinivasan has shown that the blowup of $\mathbb{P}(a, b, c)$ is a Mori Dream Space if $a \leq 4$ or $a = 6$. This leads to:

Question 6. *Is the blowup of $\mathbb{P}(5, b, c)$ at a general point always a Mori Dream Space?*

Among other applications, a more complete understanding of the toric case could help narrow the range of n for which it is unknown if $\bar{M}_{0,n}$ is MDS.

The talk of J. Hausen on iteration of Cox rings for complexity-one T-varieties gives rise to the following natural question:

Question 7. *For what classes of Mori Dream Spaces is iteration of Cox rings possible?*

For instance, for affine complexity-one T-varieties with non-trivial invariant functions, log terminal singularities guarantees iteration of Cox rings. Is it true in general that sufficiently mild singularities will guarantee iteration of Cox rings?

Inspired by the talk of J. Wisniewski, one may ask:

Question 8. *What classes of combinatorial varieties have useful intrinsic characterizations?*

For instance, is there an effective method for recognizing that a variety is (or is not) a complexity-one T-variety? Likewise, what about recognizing if a variety is spherical?

4 Outcome of the Meeting

This workshop brought together experts using toric tools in non-toric settings, resulting in extensive sharing of state-of-the art techniques and a discussion of how to further develop the connection between combinatorics and algebraic geometry. Many of the participants had been present at a semester-long program on the broad area of combinatorial algebraic geometry at the Fields Institute in Toronto in Fall 2016. This workshop provided a welcome opportunity for these participants to reconnect and discuss progress made since then.

The workshop allowed mathematicians with expertise in torus actions, Mori Dream Spaces, and equivariant cohomology to discuss their techniques and share their recent results with one another. Many of the results presented were quite new, and had yet to be thoroughly discussed. Furthermore, workshop provided the opportunity for non-experts in these areas to acquire a working knowledge in this field. Many talks began at a very introductory level, and provided welcome overviews of important problems and techniques in these areas.

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