APPLIED FUNCTIONAL ANALYSIS

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1 Introduction

Meeting in Oaxaca concentrated on a discussion of some of those theoretical problems from functional analysis and approximation theory, which are important in numerical computation. The fundamental problem of approximation theory is to resolve a possibly complicated function, called the target function, by simpler, easier to compute functions called approximants. Increasing the resolution of the target function can generally only be achieved by increasing the complexity of the approximants. The understanding of this trade-off between resolution and complexity is the main goal of approximation theory. Thus the goals of approximation theory and numerical computation are similar, even though approximation theory is less concerned with computational issues. Approximation and computation are intertwined and it is impossible to understand fully the possibilities in numerical computation without a good understanding of the elements of approximation theory. In particular, good approximation methods (algorithms) from approximation theory find applications in image processing, statistical estimation, regularity for PDEs and other areas of computational mathematics. Also, theoretical analysis of contemporary algorithms is based on deep methods from functional analysis. This makes the combination of functional analysis, approximation theory, and numerical computation, which we call *applied functional analysis*, a very natural area of the interdisciplinary research.

It was understood in the beginning of the 20th century that smoothness properties of a univariate function determine the rate of approximation of this function by polynomials (trigonometric in the periodic case and algebraic in the non-periodic case). A fundamental question is: What is a natural multivariate analog of univariate smoothness classes? Different function classes were considered in the multivariate case: isotropic and unisiotropic Sobolev and Besov classes, classes of functions with bounded mixed derivative and others. The next fundamental question is: How to approximate functions from these classes? Kolmogorov introduced the concept of the n-width of a function class. This concept is very useful in answering the above question. The Kolmogorov n-width is a solution to an optimization problem where we optimize over n-dimensional linear subspaces. This concept allows us to understand which n-dimensional linear subspace is the best for approximating a given class of functions. The rates of decay of the Kolmogorov n-width are known for the univariate smoothness classes. In some cases even exact values of it are known. The problem of the rates of decay of the Kolmogorov n-width for the classes of multivariate functions with bounded mixed derivative is still an open problem. We note that the function classes with bounded mixed derivative are not only interesting and challenging object for approximation theory but they are important in numerical computations. This topic was discussed in detail at the workshop.

Numerical integration is one more challenging multivariate problem where approximation theory methods are very useful. For a given function class F we want to find m points $x^1,\dots,x^m$ in D such that $\sum_{j=1}^m \frac{1}{m} f(x^j)$ approximates well the integral $\int_D f d\mu$, where μ is the normalized Lebesgue measure on D. Classical geometric discrepancy theory is concerned with different versions of the following questions: What is the most uniform way to distribute finitely many points in various geometric settings (in particular, in high dimensions)? In other words, how well can one approximate continuous objects by discrete ones? And how large are the errors that inevitably arise in such approximations? Quite naturally this topic is directly related to approximation theory, more specifically – numerical integration: well-distributed point sets provide good cubature formulas and different notions of discrepancy yield integration error estimates for various function classes. This immediately brings up the connection between discrepancy and functional analysis and the function space theory. Classical discrepancy theory provides constructions of point sets that are good for numerical integration of characteristic functions of parallelepipeds of the form $P = \prod_{j=1}^d [a_j, b_j]$. The typical error bound is of the form $m^{-1}(\log m)^{d-1}$. Note that a regular grid for $m = n^d$ provides an error of the order $m^{-1/d}$. The above mentioned results of discrepancy theory are closely related to numerical integration of functions with bounded mixed derivative (the case of the first mixed derivative). *Sparse grids* play an important role in numerical integration of functions with bounded mixed derivative. In the case of rth bounded mixed derivative they provide an error of the order $m^{-r}(\log m)^{(d-1)(r+1)}$. Also, they provide the recovery error in the sampling problem of the same order. Note again that the regular grid from above provides an error of the order $m^{-r/d}$. The error bound $m^{-r}(\log m)^{(d-1)(r+1)}$ is reasonably good for moderate dimensions d, say, $d \le 40$. It turns out that there are practical computational problems with moderate dimensions where sparse grids work well. Sparse grids techniques have applications in quantum mechanics, numerical solutions of stochastic PDEs, data mining, finance. This topic, including numerical integration on the sphere was also discussed in detail at the workshop.

The multivariate approximation theory in the classical setting has close connections with other areas of mathematics and has many applications in numerical computations. However, as we mentioned above, classical methods do not work for really high dimensions. High-dimensional approximation is a hot rapidly developing area of mathematics and numerical analysis where researchers try to understand which new approaches may work. A promising contemporary approach is based on the concept of sparsity and nonlinear m-term approximation. The last decade has seen great successes in studying nonlinear approximation which was motivated by numerous applications. The fundamental question of nonlinear approximation is how to devise good constructive methods (algorithms) of nonlinear approximation. This problem has two levels of nonlinearity. The first level of nonlinearity is m-term approximation (sparse approximation) with regard to bases. In this problem one can use the unique function expansion with regard to a given basis to build an approximant. Nonlinearity enters by looking for m-term approximants with terms (i.e. basis elements in approximant) allowed to depend on a given function. Since the elements of the basis used in the m-term approximation are allowed to depend on the function being approximated, this type of approximation is very efficient. On the second level of nonlinearity, we replace a basis by a more general system which is not necessarily minimal (for example, redundant system, dictionary). This setting is much more complicated than the first one (bases case), however, there is a solid justification of importance of redundant systems in both theoretical questions and in practical applications. Recent results have established that greedy type algorithms are suitable methods of nonlinear approximation in both m-term approximation with regard to bases and m-term approximation with regard to redundant systems. It turns out that there is one fundamental principal that allows us to build good algorithms both for arbitrary redundant systems and for very simple well structured bases like the Haar basis. This principal is the use of a greedy step in searching for a new element to be added to a given m-term approximant. By a *greedy step*, we mean one which maximizes a certain functional determined by information from the previous steps of the algorithm. We obtain different types of greedy algorithms by varying the above mentioned functional and also by using different ways of constructing (choosing coefficients of the linear combination) the m-term approximant from the already found m elements of the dictionary. We payed a special attention to this new promising area of research and invited several experts on greedy approximation to speak at the workshop.

In the next section we will discuss presentation highlights of several invited talks which were given during our meeting.

2 Presentation Highlights (including Recent Developments and Open Problems)

2.1 Greedy approximation

A number of talks were devoted to greedy approximation in Banach spaces and its applications to different problems:

Denka Kutzarova: An X-Greedy Algorithm with Weakness Parameters,

Thomas Schlumprecht: Greedy Bases and Renormings of Banach spaces which have them,

Volodya Temlyakov: Greedy algorithms in numerical integration.

In order to address the contemporary needs of data managing, a very general model of approximation with regard to a redundant system (dictionary) has been considered in many recent papers and some of these results were presented at the workshop. As such a model, we choose a Banach space X with elements as target functions and an arbitrary system $\mathcal{D}$ of elements of this space such that the closure of span$\mathcal{D}$ coincides with X as a representation system. We would like to have an algorithm of constructing m-term approximants that adds at each step only one new element from $\mathcal{D}$ and keeps elements of $\mathcal{D}$ obtained at the previous steps. This requirement is an analogue of *on-line* computation that is very desirable in practical algorithms. Clearly, we are looking for good algorithms which converge for each target function. It is not obvious that such an algorithm exists in a setting at the above level of generality (X, $\mathcal{D}$ are arbitrary).

The approximate sparse representation problem was studied in the following way.

1. General convergence results were discussed in Kutzarova's and Temlyakov's talks. They proved convergence results for a given greedy-type algorithm for all $f \in X$ with respect to an arbitrary dictionary $\mathcal{D}$.

2. Rate of convergence results were discussed in all three above mentioned talks. (2a) First, the authors prove that a given greedy-type algorithm guarantees some rate of convergence for f from a specific class (typically, it is the closure of the convex hull of a symmetrized dictionary) with no extra assumptions on the dictionary. (2b) Second, they prove some better rate of convergence results under additional assumptions on the dictionary. For instance, strong results can be obtained for greedy bases (Schlumprecht).

These results give us the following picture. For a given greedy-type algorithm we guarantee its convergence in any situation (f and $\mathcal{D}$ are arbitrary). If f has some properties then we guarantee that the algorithm converges with a certain rate. Even stronger guaranties can be given if $\mathcal{D}$ has certain properties.

Application of general greedy algorithms for construction of good deterministic cubature formulas was discussed in Temlyakov's talk. He presented results on a relation between construction of an optimal cubature formula with m knots for a given function class and best nonlinear m-term approximation of a special function determined by the function class. The nonlinear m-term approximation is taken with regard to a redundant dictionary also determined by the function class. He demonstrated how greedy algorithms can be used for constructing such m-term approximations and the corresponding Quasi-Monte Carlo methods for numerical integration.

2.2 Dynamical Sampling, cyclical sets, cyclical frames and the spectral theory by Akram Aldroubi

Let f be a signal at time $t = 0$ of a dynamical process controlled by an operator A that produces the signals $Af, A^2f, \dots$ at times $t = 1, 2, \dots$. Let M be a measurements operator applied to the series $Af, A^2f, \dots$ at times $t = 1, 2, \dots$. The problem is to recover f from the measurements $Y = \{Mf, MAf, MA^2f, \dots, MA^Lf\}$. This is the so called Dynamical Sampling Problem. A prototypical example is when $f \in \ell^2(\mathbb{Z})$, X a proper subset of $\mathbb{Z}$ and $Y = \{f(X), Af(X), A^2f(X), \dots, A^Lf(X)\}$. The problem is to find conditions on A, X, L, that are sufficient for the recovery of f [1, 2]. This problem has connection to many areas of mathematics including frames, and Banach algebras, and the recently solved Kadison-Singer/Feichtinger conjecture. Some of the recent results in collaboration obtained with Carlos Cabrelli, Ilya Krishtal, Jacqueline Davis, Ursula Molter, Armenak Petrosyan, Ahmed Cakmak, and Sui Tang.

Akram Aldroubi presents the following result.

Theorem 2.1 *Let A be a normal operator in $\mathcal{B}(\mathcal{H})$ with spectrum $\sigma(A)$ s.t. $int(\sigma(A)) = \emptyset$ and $\mathbf{C} - \sigma(A)$ connected. Let $UAU^{-1} = N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \cdots$, be the spectral decomposition of A, where U is an an isometric isomorphism from $\mathcal{H}$ to $(L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$ and measures μ_i on $\mathbf{C}$ that are mutually singular Borel measures. Let $\Omega \subset \mathcal{H}$ be a countable set. Then the following are equivalent*

1. $\{A^{*j}g : \ g \in \Omega, \ j = 0, 1, \ldots, \}$ *is complete*
2. *for μ_j-a.e. x, $\{(P_j Ug)(x)\}_{g\in\Omega}$ is complete in $\mathbf{C}^j \cong l^2\{1, 2, \ldots, j\}$, $1 \leq j \leq \infty$, where P_j is the projections on $(L^2(\mu_j))^{(j)}$.*

2.3 Polynomial Approximation on Compact sets in the Plane by Vladimir Andrievskii

We presented some results and open problems concerning the following topics.

The Vasiliev-Totik's extension of the classical Bernstein theorem on polynomial approximation of piecewise analytic functions on a closed interval. The error of the best uniform approximation of such functions on a compact subset of the real line is studied.

A conjecture on the rate of polynomial approximation on the compact set of the plane to a complex extension of the absolute value function. The conjecture was stated by Grothmann and Saff in 1988. Related to this is another conjecture, Gaier's conjecture, on the polynomial approximation of piecewise analytic functions on a compact set consisting of two touching discs.

The estimates of the uniform norm of the Chebyshev polynomial associated with a compact set $K \subset \mathbf{C}$ consisting of a finite number of continua in the complex. These estimates are exact (up to a constant factor) in the case where the components of K are either quasiconformal arcs or closed Jordan domains bounded by a quasiconformal curve. The case where K is a uniformly perfect or a homogeneous subset of the real line is also of interest.

Recent developments: Details can be found in [3, 4, 5].

2.4 Multivariable approximations using radial basis functions by Martin Buhman

The talk focussed on multivariable approximations using radial basis functions, employing especially the Hardy multiquadric function $\phi(r) = \sqrt{r^2 + c^2}$, composed with the Eucidean norm, and its shifts by "centres". Other very suitable radial functions are Dagum and Bernstein functions, Gauss kernels, Poisson kernels, thin-plate splines etc. Among the methods using these radial basis functions, not only interpolation but also quasi-interpolation turns out again to be highly suitable, where not the pointwise agreement with the approximand (at the provided centeres) but more specifically the smoothing and the localness of the approximants (and their polynomial reproduction properties, so that approximation order results and convergence are obtainable) are central. In this talk, new approximation order results, no long requiring logarithmic terms in most instances, were presented, and a very general Ansatz for the quasi-interpolating approximants is used, not even requiring radial symmetry and Euclidean norms everywhere. Also, the theorems apply in almost all dimensions and to very general classes of approximands.

Recent developments: The most recent developments admit general operators in place of pointwise evaluations of approximands and allow from from very general Sobolev spaces. Different smoothness of approximant and approximand is allowed too, that is different L^p and L^q norms appear in the sought estimates. Moreover, many pointwise results are possible now, where before only uniform approximation results and approximations orders were offered; this became possible by employing the Hardy Littlewood maximal function.

2.5 Best onesided approximation and quadrature formulas by Jorge Bustamante

For $\theta \in (-1, 1)$, let $H_\theta(x)$, $x \in [-1, 1]$, be the Heaviside function with a jump at θ. Bustamante finds the explicit expression for all the polynomial of the best onesided approximation for $H_\theta(x)$ in the $L_1[-1, 1]$ norm. The solution require some facts related with quasi-orthogonal polynomials and some new quadrature formulas with a prefixed abscissa. The result are applied to construct some operators for algebraic onesided approximation. For the proof see [6, 8, 9].

Open Problems:

• For $\theta \in (-1,1)$ and $r \in N$, let $(x-\theta)_+^r$, $x \in [-1,1]$, be the truncated function with parameter θ. Find the polynomials of the best onesided approximation for $(x-\theta)_+^r$ in the $L_1[-1,1]$ norm.

2.6 One-parameter groups of operators and discrete Hilbert transforms by Laura De Carli

Let I be the union of disjoint intervals I_0, ... I_{N-1}, with $I_j = [M_j, M_j+1]$, with $j = 0, ..., N-1$ and $M_j + 1 \leq M_{j+1}$ for every $j \geq 0$. We construct exponential bases of $L^2(I)$ in the form of $\cup_{j=0}^{N-1}\mathcal{B}_j$, where $\mathcal{B}_j = \{e^{2\pi i(n+d_j)x}\}_{n\in\mathcal{Z}}$ with $d_j > 0$.

De Carli uses the properties of a family of operators $\{T_t\}_{t\in\mathbb{R}}$, initially defined in the space of complex-valued sequences with compact support as follows:

$$(T_t(\vec{a}))_m = \frac{\sin(\pi t)}{\pi}\sum_{n\in\mathcal{Z}}\frac{a_n}{m-n+t} \text{ if } t \notin \mathcal{Z}, \text{ and } T_t(\vec{a}) = (-1)^t a_{m+t} \text{ if } t \in \mathcal{Z}.$$

Some of the results presented at the conference are in collaboration with my student Shaikh Gohin Samad. The existence of exponential bases on the union of segments of $\mathbb{R}$ is proved in [17].
Presentation highlights:

Theorem 2.2 *The family $\{T_t\}_{t\geq 0}$ defined above is a strongly continuous group of isometry in ℓ^2; the discrete Hilbert transform H defined as*

$$(H(\vec{a}))_m = \frac{1}{\pi}\sum_{\substack{n\in\mathcal{Z}\\ n\neq m}}\frac{a_n}{m-n}. \tag{1}$$

is its infinitesimal generator.

Theorem 2.3 *Assume that*

$$\max_{\substack{0\leq j\neq j\leq N-1\\ 0\leq p\neq q\leq N-1}}\left\{\left|\sum_{p=0}^{N-1} e^{2\pi i(d_i-d_j)M_p}\right|, \quad \left|\sum_{j=0}^{N-1} e^{2\pi i d_j(M_q-M_p)}\right|\right\} < \frac{N}{N-1}. \tag{2}$$

Then, $\mathcal{B}$ defined above is a Riesz basis of $L^2(I)$.

Open Problems:

• Construct an explicit exponential basis for the union of families of segment of finite total length (For example: the union of segments in the form of $(a,\ a+2^{-m})$, with $m \in \mathbb{N}$)

2.7 Reverse Hölder's inequality for spherical harmonics by Han Feng

The sharp asymptotic order of the following reverse Hölder inequality for spherical harmonics Y_n of degree n on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ as $n \to \infty$:

$$\|Y_n\|_{L^q(\mathbb{S}^{d-1})} \leq Cn^{\alpha(p,q)}\|Y_n\|_{L^p(\mathbb{S}^{d-1})}, \quad 0 < p < q \leq \infty$$

is obtained. In many cases, these sharp estimates turn out to be significantly better than the corresponding estimates in the Nilkolskii inequality for spherical polynomials. This is a joint work with F. Dai and S. Tikhonov. Briefly, the obtained results can be represented in the following tables with $\lambda = \frac{d-2}{2}$
Open Problems:

- To complete the result table for $d > 3$
- To obtain the analogy results in a weighted setting

Table 1: the case: $d > 3$

p	q	$\alpha(p,q)$
$(0,1]$	$(p,\infty]$	$\lambda(\frac{1}{p}-\frac{1}{q})$
$[1,2]$	$(p,(1+\frac{1}{\lambda})p']$	$\lambda(\frac{1}{p}-\frac{1}{q})$
$[1,2]$	$[(1+\frac{1}{\lambda})p',\infty]$	$2\lambda(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}$
$[2,2+\frac{1}{\lambda})$	$(p,2+\frac{2}{\lambda})$	unknown
$[2,2+\frac{1}{\lambda})$	$[2+\frac{2}{\lambda},\infty]$	$2\lambda(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}$
$(2+\frac{1}{\lambda},\infty)$	$(p,\infty]$	$(2\lambda+1)(\frac{1}{p}-\frac{1}{q})$

Table 2: the case: $d = 3$

p	q	$\alpha(p,q)$
$(0,1]$	$(p,\infty]$	$\lambda(\frac{1}{p}-\frac{1}{q})$
$[1,2+\frac{1}{\lambda})$	$(p,(1+\frac{1}{\lambda})p')$	$\lambda(\frac{1}{p}-\frac{1}{q})$
$[1,2+\frac{1}{\lambda})$	$[(1+\frac{1}{\lambda})p',\infty]$	$2\lambda(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}$
$(2+\frac{1}{\lambda},\infty)$	$(p,\infty]$	$(2\lambda+1)(\frac{1}{p}-\frac{1}{q})$

2.8 Optimal Estimates on Robustness Property of Gaussian Random Matrices under Corruptions by Bin Han

This is joint work with Zhiqiang Xu. Johnson–Lindenstrauss Lemma is often used in dimensionality reduction and concerns low-distortion embedding of points from high-dimensional space into low-dimensional space. The existence of an ideal projection matrices in the Johnson–Lindenstrauss Lemma is often proved using Gaussian random matrices. Gaussian random matrices under corruptions also play a key role in the establishment of the robust restricted isometry property in compressed sensing.

Let $A = (a_{j,k})_{1\le j\le m, 1\le k\le n} \in \mathbb{R}^{m\times n}$ be a Gaussian random matrix such that each entry $a_{j,k} \sim \mathcal{N}(0,1)$ is an i.i.d. Gaussian random variable with zero mean and unit standard deviation. For $T \subseteq \{1,\dots,m\}$, $A_T \in \mathbb{R}^{|T|\times n}$ is the $|T|\times n$ sub-matrix of A by keeping the rows of A with row indices from T. Let $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 1$. For $\epsilon > 0$ and $0 \le \beta < 1$, define

$$\Omega_{\epsilon,\beta} := \Big\{ \Big| \tfrac{1}{|T|}\|A_T x_0\|^2 - 1 \Big| \le \epsilon \text{ for all } T \subseteq \{1,\dots,m\} \text{ satisfying } |T^c| \le \beta m \Big\},$$

where $T^c := \{1,\dots,m\}\backslash T$ and β is the erasure ratio. Similarly, we define $\mathring{\Omega}_{\epsilon,\beta}$ if the factor $\frac{1}{|T|}$ above is replaced by $\frac{1}{m}$. For $\epsilon > 0$ and $\alpha > 0$, we define

$$\beta^{\max}_{\epsilon,\alpha} := \sup\{0 \le \beta < 1 \ : \ \mathbb{P}(\Omega_{\epsilon,\beta}) \ge 1 - 3e^{-\alpha(\epsilon^2/4-\epsilon^3/6)m} \ \forall\, m \in \mathbb{N}\}$$

and

$$\mathring{\beta}^{\max}_{\epsilon,\alpha} := \sup\{0 \le \beta < 1 \ : \ \mathbb{P}(\mathring{\Omega}_{\epsilon,\beta}) \ge 1 - 3e^{-\alpha(\epsilon^2/4-\epsilon^3/6)m} \ \forall\, m \in \mathbb{N}\}$$

One of the main results in [14] is as follows:

Theorem 2.4 *For every* $0 < \alpha < 1$,

$$\left(\frac{1-\sqrt{\alpha}}{32}\right)\frac{\epsilon}{\ln\frac{1}{\epsilon}} < \beta^{\max}_{\epsilon,\alpha} < \left(\frac{2+2\epsilon_g}{c_g^2\epsilon_g}\right)\frac{\epsilon}{\ln\frac{1}{\epsilon}}$$

for all $0 < \epsilon < \min(\frac{1-\sqrt{\alpha}}{4}, \epsilon_g, 4\epsilon_g^2)$, *and*

$$\left(\frac{1-\sqrt{\alpha}}{32}\right)\frac{\epsilon}{\ln\frac{2}{\epsilon}} \le \mathring{\beta}^{\max}_{\epsilon,\alpha} \le \left(\frac{1}{4c_g^2}\right)\frac{\epsilon}{\ln\frac{1}{\epsilon}}$$

for all $0 < \epsilon < \min(\frac{1-\sqrt{\alpha}}{4}, 2e^{-1}c_g^2, \frac{1}{2c_g^2})$, *where* c_g *and* ϵ_g *are absolute positive constants.*

As a consequence of the above result, Han established the robustness property of Johnson–Lindenstrauss Lemma and the restricted isometry property against erasure.

Recent Developments:

• Currently, the authors are generalizing the results in [14] to subgaussian random matrices and other random matrices.

2.9 Polynomial approximation with doubling weights by Kirill A. Kopotun

A nonnegative function $w \in \mathbb{L}_1[-1,1]$ is called a doubling weight if there is a constant L such that $w(2I) \le Lw(I)$, for all intervals $I \subset [-1,1]$, where $2I$ denotes the interval having the same center as I and twice as large as I, and $w(I) := \int_I w(u)du$. Kopotun establishes direct and inverse results for weighted approximation by algebraic polynomials in the $\mathbb{L}_p$, $0 < p \le \infty$, (quasi)norm weighted by $w_n := \rho_n(x)^{-1} \int_{x-\rho_n(x)}^{x+\rho_n(x)} w(u)du$, where $\rho_n(x) := n^{-1}\sqrt{1-x^2} + n^{-2}$ and w is a doubling weight.

Among other things, he proves that, for a doubling weight w, $0 < p \le \infty$, $r \in \mathbb{N}_0$, and $0 < \alpha < r + 1 - 1/\lambda_p$, one has

$$E_n(f)_{p,w_n} = O(n^{-\alpha}) \iff \omega_\varphi^{r+1}(f, n^{-1})_{p,w_n} = O(n^{-\alpha}), \tag{$*$}$$

where $\lambda_p := p$ if $0 < p < \infty$, $\lambda_p := 1$ if $p = \infty$, $\|f\|_{p,w} := \left(\int_{-1}^1 |f(u)|^p w(u)du\right)^{1/p}$, $\|f\|_{\infty,w} :=$ ess $\sup_{u\in[-1,1]} (|f(u)|w(u))$, $\omega_\varphi^r(f,t)_{p,w} := \sup_{0<h\le t} \left\|\Delta^r_{h\varphi(\cdot)}(f,\cdot)\right\|_{p,w}$, $E_n(f)_{p,w} := \inf_{P_n\in\Pi_n} \|f - P_n\|_{p,w}$, and Π_n is the set of all algebraic polynomials of degree $\le n-1$.

Kopotun also introduces classes of doubling weights $\mathcal{W}^{\delta,\gamma}$ with parameters $\delta, \gamma \ge 0$ that are used to describe the behavior of $w_n(x)/w_m(x)$ for $m \le n$. It turns out that every class $\mathcal{W}^{\delta,\gamma}$ with

$$(\delta,\gamma) \in \Upsilon := \left\{(\delta,\gamma) \in \mathbb{R}^2 \;\middle|\; \delta \ge 1, \gamma \ge 0, \delta + \gamma \ge 2\right\}$$

contains all doubling weights w, and for each pair $(\delta,\gamma) \notin \Upsilon$, there is a doubling weight not in $\mathcal{W}^{\delta,\gamma}$. He establishes inverse theorems and equivalence results similar to ($*$) for doubling weights from classes $\mathcal{W}^{\delta,\gamma}$. Using the fact that $1 \in \mathcal{W}^{0,0}$, we get the well known inverse results and equivalences of type ($*$) for unweighted polynomial approximation as an immediate corollary.

Equivalence type results involving related K-functionals and realization type results (obtained as corollaries of our estimates) are also discussed.

Finally, the author mentions that ($*$) closes a gap left in the paper by G. Mastroianni and V. Totik [22], where ($*$) was established for $p = \infty$ and ω_φ^{r+2} instead of ω_φ^{r+1} (it was shown there that, in general, ($*$) is not valid for $p = \infty$ if ω_φ^{r+1} is replaced by ω_φ^r).

2.10 On multivariate "needle" polynomials and their application to norming sets and cubature formulas by András Kroó

Needle polynomials $0 \le p_n \le 1$ of degree n on the interval $[-1,1]$ attain value $p_n(x_0) = 1$ at some $x_0 \in [-1,1]$ and are "exponentially small" as we move away from this point

$$p_n(x) \le e^{-n\phi(h)}, \quad x \in [-1,1] \setminus [-h + x_0, h + x_0], 0 < h \le 1$$

with $\phi(h) \downarrow 0$ depending on h and the location of x_0. They resemble the behavior of the Dirac delta function and are widely used in different areas of analysis.

In the multivariate case the behavior of the needle polynomials at the boundary of convex bodies is closely related to the geometry of the boundary of the domain. For a compact $K \subset \mathbb{R}^d$ and $0 < \alpha \le 1$ we have α-needle polynomials at $\mathbf{x}_0 \in K$ if for any $0 < h < 1$ and $n \in \mathbb{N}$ there exist polynomials $p \in P_n^d$ such that $0 \le p(\mathbf{x}) \le 1, \mathbf{x} \in K, p(\mathbf{x}_0) = 1$ and

$$p(\mathbf{x}) \le e^{-cnh^\alpha}, \quad \mathbf{x} \in K \setminus B(\mathbf{x}_0, h)$$

with some $c > 0$ depending only on K and $\mathbf{x}_0$.

Recent developments: See [20].

Theorem 1. *Let $K \subset \mathbb{R}^d, d \ge 2$ be a convex body and $\mathbf{x}_0 \in \partial K$. Then K possesses $\frac{1}{2}$-needle polynomials at $\mathbf{x}_0$ if and only if $\mathbf{x}_0$ is a vertex.*

Theorem 2. *If K is C^β with $1 < \beta \le 2$ at $\mathbf{x} \in \partial K$ then there are no α-needle polynomials with $\alpha < \frac{\beta}{2}$ at this point. On the other hand α-needle polynomials exist for some $C^{2\alpha}$ domains, e.g. at vertices of $l_{2\alpha}$ balls in $\mathbb{R}^d$.*

Theorem 3. *Let $K \subset \mathbb{R}^d, d \geq 2$ be a $\mathbf{0}$ symmetric convex body. Then K possesses homogeneous 1-needle polynomials with $\varphi(h) \sim h$ at every $\boldsymbol{x}_0 \in \partial K$.*

Moreover, K possesses homogeneous logarithmic needle polynomials with $\varphi(h) \sim h \log \frac{1}{h}$ at $\boldsymbol{x}_0 \in \partial K$ if and only if $\boldsymbol{x}_0$ is a vertex.

2.11 Approximation in $\mathcal{H}(b)$ spaces by Javad Mashreghi

Collaboration with: O. ElFallah, E. Fricain, K. Kellay, T. Ransford

Hardy spaces H^p, the Dirichlet space $\mathcal{D}$, and Bergman spaces A^p are the most celebrated Banach spaces of analytic functions on the open unit disc $\mathbb{D}$. In these spaces, and several other cousin spaces, analytic polynomials are dense. A usual demonstration is as follows: we approximate f by its dilation f_r, where $f_r(z) = f(rz)$. The latter function is defined on a larger disc which contains the closed unit disc $\overline{\mathbb{D}}$. Hence, naively speaking, it should have ultra nice properties on $\overline{\mathbb{D}}$. For example, we can approximate f_r by the partial sums of its Taylor series. Hence, these partial sums, which are analytic polynomials, are close to the original function f too.

In his presentation, Mashreghi introduced the de Branges-Rovnyak space $\mathcal{H}(b)$, in which polynomials are dense, but the above approach dramatically fails.

Recent developments: He succeeded to find a symbol b and construct an explicit function $f \in \mathcal{H}(b)$ such that

$$\lim_{r \to 1} \|f_r\|_{\mathcal{H}(b)} = \infty. \tag{3}$$

Despite the above strange behavior, a constructive proof for the density of polynomials was presented.

Open Problems:

1. For which symbols b, the above strange behavior persist? More explicitly, for which b there is a function $f \in \mathcal{H}(b)$ whose dilates satisfy (3)?
2. Given a symbol b which falls in the above category, which functions in $\mathcal{H}(b)$ have exploding dilates?

2.12 Kolmogorov and linear n-Widths of Balls in Sobolev spaces on Manifolds by Isaac Pesenson

Pesenson determines upper asymptotic estimates of Kolmogorov n-width d_n and linear n-width δ_n of unit balls in Sobolev norms in L_p-spaces on smooth compact Riemannian manifolds. For compact homogeneous manifolds, he establishes estimates which are asymptotically exact, for the natural ranges of indices. The proofs heavily rely on our previous results [11], [25], such as: estimates for the near-diagonal localization of the kernels of elliptic operators, Plancherel-Polya inequalities on manifolds, cubature formulas with positive coefficients and uniform estimates on Clebsch-Gordon coefficients on general compact homogeneous manifolds.

One of the main results is the following.

Theorem 2.5 *Assume that $\mathbf{M}$ is a homogeneous compact manifold and $B_p^r(\mathbf{M})$ is the unit ball in a corresponding Sobolev space $W_p^r(\mathbf{M})$.*

1. *Suppose that $1 \leq p \leq 2 \leq q \leq \infty$. Then one has the following asymptotics*

$$d_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \asymp n^{-\frac{r}{s}+\frac{1}{p}-\frac{1}{2}} \qquad \text{if } r > s/p,$$

2. *Suppose that $2 \leq p \leq q \leq \infty$. Then one has the asymptotics*

$$d_n(B_p^r(\mathbf{M}), L_q(\mathbf{M})) \asymp n^{-\frac{r}{s}} \qquad \text{if } r > s/p.$$

Open Problems:

- To obtain exact asymptotics on general compact Riemannian manifolds.
- To obtain either upper, lower, or exact asymptotics on compact manifolds equipped with non-Riemannian metrics.

2.13 Fully discrete needlet approximation by Ian H. Sloan

In his talk Sloan reported on recently submitted work on a fully discrete needlet approximation scheme. He presented convergence results for the approximation of functions in certain Sobolev spaces, in which the rate of convergence, though losing something with respect to the continuous approximation, nevertheless seemed to be as good as we could hope for. The Chair of the session (Organizer Temlyakov) pointed out at the end of the lecture that the results were not optimal in the sense of optimal recovery.

Recent developments:

After the lecture another participant (Heping Wang) indicated how the result might be improved. This interaction led Sloan and Wang to decide to write a new paper giving the improved results. This is now in train.

2.14 Anisotropic approximation with shift-invariant subspaces by Moisés Soto-Bajo

Soto-Bajo considers approximation with shift-invariant subspaces in $L^2(\mathbb{R}^d)$ scaled by the dilation operator $\mathcal{D}_A f(\cdot) = |\det(A)|^{1/2} f(A\,\cdot)$ induced by a given dilation A on $\mathbb{R}^d$. He presents characterizations of different properties which measure the approximation power of the shift-invariant subspaces.

These results generalize several others recently appeared in the literature, dealing with general shift-invariant subspaces V (non necessarily finitely generated), with respect to general anisotropic dilations A, and in the framework of A-reducing spaces ($H = \mathcal{D}_A H$). All the provided conditions focus on the local behaviour at the origin of the spectral function of V, making use of the notion of anisotropic approximate continuity point.

In [26] several conditions equivalent to the completeness property ($\bigcup_{j\in\mathbb{Z}} \mathcal{D}_A^j V$ is dense in H) of an A-refinable ($V \subseteq \mathcal{D}_A V$) subspace V are given. In [10] a characterization of the anisotropic approximation and density orders of a shift-invariant subspace V is given. We say that V provides A-approximation order $\alpha > 0$ if there exists $C > 0$ such that

$$\|f - P_{\mathcal{D}_A^j V} f\|_2 \le C\,|\det(A)|^{-j\alpha/d}\,\|f\|_{A,\alpha} \qquad \forall\, f \in W_A^{\alpha,2},\, j \in \mathbb{Z}\,,$$

and we also say that V provides A-density order $\alpha \ge 0$ if

$$|\det(A)|^{j\alpha/d}\,\|f - P_{\mathcal{D}_A^j V} f\|_2 \xrightarrow[j\to\infty]{} 0 \qquad \forall\, f \in W_A^{\alpha,2}\,.$$

P_V denotes the orthogonal projection on V, $W_A^{\alpha,2}$ is the anisotropic Sobolev space given by the norm $\|f\|_{A,\alpha} = \|(1+\rho)^\alpha\,\widehat{f}\|_2$, $\widehat{f}$ is the Fourier transform of f, and ρ is a pseudo-norm for A^*, conjugate of A.

Open Problems:

- To construct explicit interesting examples of such a shift-invariant subspaces, specially scaling functions and low-pass filters.
- To characterize equivalence of dilations and pseudo-norms.

2.15 Weighted Bernstein inequality by S. Tikhonov

Tikhonov presented the recent results joint with A. Bondarenko [7] on Bernstein inequalities in the form $\|T_n'\|_{L_p(\omega)} \le Cn\|T_n\|_{L_p(\omega)}$, where $0 < p \le \infty$ and ω non-doubling weights. Sufficient and necessary conditions on ω for Bernstein's inequlaity to hold are discussed.

2.16 Exponential convergence-tractability of general linear problems by Guiqiao Xu

Xu studies d-variate general linear problems defined over Hilbert spaces in the average case setting. He considers algorithms that use finitely many evaluations of arbitrary linear functionals. The traditional tractability

of above problems have been well studied, see [23]. The author studies the corresponding EC-tractability in terms of the eigenvalues of the corresponding covariance operators.
Recent developments: In the worst case setting, A. Papageorgiou, I. Petras [24] obtained the necessary and sufficient conditions for Exponential Convergence-(Strong) Polynomial Tractability of general linear problems defined over Hilbert spaces. Afterwards, G.Q. Xu [28] studied the corresponding problems in the average case setting.
Open Problems:

• For the general linear problems defined over Hilbert spaces in the average case setting, how to characterize Exponential Convergence-Quasi Polynomial Tractability directly in terms of the eigenvalues of the corresponding covariance operators.

2.17 Polynomial Approximation in the Sobolev Space by Yuan Xu

Spectral approximation by polynomials on the unit ball is studied in the frame of the Sobolev spaces $W_p^s(\mathbb{B}^d)$, $1 < p < \infty$. The main results give sharp estimates on the order of approximation by polynomials in the Sobolev spaces and explicit construction of approximating polynomials. One major effort lies in understanding the structure of orthogonal polynomials with respect to an inner product of the Sobolev space $W_2^s(\mathbb{B}^d)$.
Recent developments: See [18]. For $s = 1, 2, \ldots$, we define a bilinear form on the space $W_2^s(\mathbb{B}^d)$ by

$$\langle f, g \rangle_{-s} := \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{k=0}^{\lceil \frac{\varrho}{2} \rceil - 1} \lambda_k \langle \Delta^k f, \Delta^k g \rangle_{\mathbb{S}^{d-1}}, \tag{4}$$

where λ_k, $k = 0, 1 \ldots, \lceil \frac{\varrho}{2} \rceil - 1$, are positive constants. Let $\mathcal{V}_n^d(\varpi_{-s})$ be the space of orthogonal polynomials with respect to the inner product $\langle \cdot, \cdot \rangle_{-s}$. Let $\operatorname{proj}_n^{-s} : W_2^s(\mathbb{B}^d) \mapsto \mathcal{V}_n^d(\varpi_{-s})$ be the orthogonal projection operator. Define define

$$S_n^{-s} f(x) := \sum_{k=0}^{n} \operatorname{proj}_k^{-s} f(x) \quad \text{and} \quad S_{n,\eta}^{-s} f(x) := \sum_{k=0}^{\infty} \eta\left(\frac{k}{n}\right) \operatorname{proj}_k^{-s} f(x), \tag{5}$$

where $\eta \in C^\infty[0, \infty)$ is an admissible cut-off function supported on $[0, 2]$.

Theorem 2.6 *Let $r, s \in \mathbb{N}$ and $r \geq s$. If $f \in W_p^r(\mathbb{B}^d)$ and $1 < p < \infty$, then, for $n \geq s$,*

$$\|f - S_{n,\eta}^{-s} f\|_{W_p^k(\mathbb{B}^d)} \leq c n^{-r+k} \|f\|_{W_p^r(\mathbb{B}^d)}, \quad k = 0, 1, \ldots, s, \tag{6}$$

where $S_{n,\eta}^{-s} f$ can be replaced by $S_n^{-s} f$ if $p = 2$.

Open Problems:

• Define the error of best approximation

$$E_n(f)_{W_p^s} := \inf_{p \in \Pi_n^d} \|f - p\|_{W_p^s}.$$

How can we characterize this quantity? The inverse estimate is usually established by Bernstein inequality. Is this still the case?

Open Problems

1. Extend the theory (as much as possible) to suitable compact subsets of R^n.

2. Establish that the quadrature weights discussed above are positive for quasiuniform data.

2.18 Entropy numbers of weighted Sobolev classes on the unit sphere with respect to Dunkl weight by Heping Wang

The author obtains the asymptotic orders of entropy numbers of weighted Sobolev spaces on the sphere with respect to Dunkl weight, which is invariant under a finite reflection group. He uses the discretion method to reduce the problem to the one of the entropy numbers of a finite-dimensional weight spaces, and obtain the upper estimates of the latter one. In order to obtain the upper estimates, the author proved the two key lemmas. In order to obtain the lower estimates, it is used the properties of the Dunkl transformation.

Recent developments:

• In the unweighted case, the exact orders of the entropy numbers of Sobolev classes BW_p^r on the sphere in L_q were obtained by Kushpel and Tozoni (2012) for $1 < p, q < \infty$ and H. Wang, K. Wang, J. Wang (2014) for the remaining case.

• In the weighted case $G = Z_2^d$, the Kolmogorov, linear, and Gelfand widths of the weighted Sobolev classed on the sphere in weighted L_q space were obtained in Huang and Wang (2011).

Open Problems:

• Find out the asymptotic orders of entropy numbers and various widths of weighted Besov spaces on the sphere with respect to the general product weights.

2.19 Local Bases on Spheres with Applications by J. D. Ward

The presentation focused on kernel interpolation and approximation in a fairly general setting. Given a set of N scattered sites, the standard basis when using positive definite or conditionally positive definite kernels utilizes N *globally* supported kernel; computing with this type of basis becomes unstable and prohibitively expensive for large N. Easily computible, well-localized bases with "small-footprint" basis elements, i.e., elements using only a small number of kernels, have been unavailable.

In the presentation, the theoretical development of small footprint bases that are well-localized spatially, for a variety of kernels was discussed. Another point of discussion was how to easily and efficiently compute these small footprint, robust (i.e., well-localized, L_p stable) bases for spaces associated with restricted surface-spline kernels on the sphere S^n and more general manifolds. While the bases discussed, local Lagrange functions, are not new, the number of points needed per Lagrange function, $\mathcal{O}(\log N^d)$ on S^d, to insure a stable, highly localized basis on S^d is predicated on the theoretical investigation. An offshoot of these results is a strategy for selecting centers for preconditioning that scales correctly with the total number of centers N.

Another offshoot of this work is a class of easy to compute quadrature formulas (dependent on the given kernel) for the sphere which are exact for the spaces spanned by the local bases. These quadrature formulas have implications for meshless methods.

The presentation was based on a series of papers with various authors including T. Hangelbroek, F.J. Narcowich, E. Fuselier, G. B. Wright and X. Sun.

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